Matrix Method of Structural Analysis
Prof. Amit Shaw
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 38
Introduction to Finite Element Method (Contd.)

Hello everyone. This is the third lecture of this week. Last class we discussed how to construct the stiffness matrix for bar element, same exercise we will do it for a beam today, ok. So, what we have now is if you recall we have consider a problem. This is take an example of this problem, this is a bar this is a beam which is subjected to some load say for instance concentrated load, right ok.

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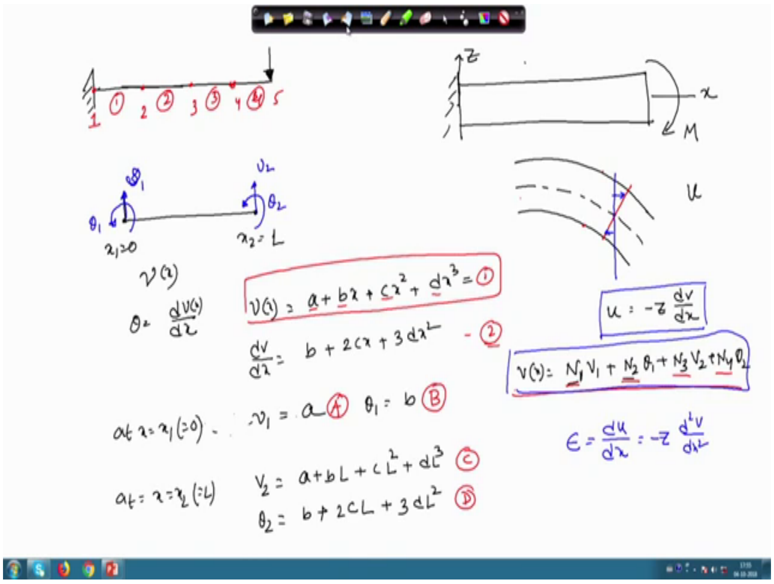

Now, we want to solve it. Now, please make note one thing I have mentioned in the last class as well. See there are many approaches you can construct you can arrive at this stiffness matrix final stiffness matrix, but it approach what we are following is the very simplest approach. And the reason is our objective here is to extend whatever information we have whatever knowledge we have on matrix method of structural analysis and also in solid mechanics and strength of material.

Our objective is to based on that knowledge we should just expand that the expand the application of that knowledge, and then see look beyond the, look beyond for possibilities of having a general methodology for solving structures at different continuum that is what we are doing. Now, similarly first we have to discretize it, we have to discretize into several segments. For instance let us discretize into several segments let us 4. Same as the matrix direct stiffness matrix where these are the node numbers and then corresponding these are the member numbers, right. And then for any representative element we have to construct what is the stiffness matrix, right.

Now, consider taken a representative element for instance this is a representative element, which is this is one node and this is another node. Now, if you in the bar element we had just one degrees of freedom and that degrees of freedom is the deformation along the longitudinal axis, but in beam you recall in direct stiffness matrix of the way we discuss per node we have two degrees of freedom one is transverse displacement, another one is the rotation. Now, our as per our sign convention the this is the sign convention that we are using is this is the this.

Now, this is transverse deflection in these direction we have a transverse deflection in this direction and then we have a rotation, anti clockwise rotation is positive anti clockwise rotation is positive. So, essentially we have say it is v 1, v 1 and this is theta 1 and this is v 2 and then theta 2,. So, these are 4 degrees of freedom we have.

Now, what is our objective? Is to construct the stiffness matrix such that this stiffness matrix relates these 4 degrees of freedom, and the forces associated with associated with these degrees of freedom, ok. Before that if you see consider at let us see how if we have a beam. For instance if we take a beam like this and it is subjected to some loading say for instance some loading m and this is the longitudinal axis of the beam, ok. And then what happens? That beam bends like this if I take any segments then this bend beams bends like this, right.

Now, then if we take any axis suppose this is the neutral axis of the beam this is the neutral axis of the beam, ok. And then if we take any axis for instance this axis if we take which was we all are using the Euler Bernoulli assumption that the for plane section normal to the neutral axis remain plane even in the deformed configuration, right.

Now, in that case you see though, so what happens is at the neutral axis your transverse the your longitudinal displacement is 0. Now, one assumption is whether you take this point or this point as far as transverse displacement is concerned, the transverse displacement will be same. And this transverse displacement is suppose the transverse displacement here this transverse displacement is v. So, v x give you the transverse displacement at any point, ok.

Now, then what will be the slope? Slope at any point theta will be dv dx, dv dx that will be slope. Now, so this transverse this displacement will be same for this point and this point. Now, but what happens to the longitudinal displacement? You see at the neutral axis your longitudinal displacement is 0 if the longitudinal displacement say is u, at the neutral axis your longitudinal displacement is 0. But, if you move away from the neutral axis this longitudinal displacement will be suppose this is z this is across the cross section this is z axis, suppose this is this is z axis

So, along the at the neutral axis this longitudinal displacement will be 0, but if you move away from the neutral axis suppose the longitudinal displacement this is the longitudinal displacement, ok. This is the longitudinal displacement. This longitudinal displacement can be written as u is equal to u is equal to minus z into dv dx, ok; dv dx this is the longitudinal displacement. Now, if we calculate v then you see the if we know v then we know: what is the longitudinal displacement from this relation. If we calculate v we also know: what is the expression for theta expression for slope, ok.

So, the major thing is our to start with the our objective is to calculate what is the v. Now, if you recall in the case of truss we started with an assumption, assumption of the displacement field. For instance in the in the case of truss, at one point if your deflection is u 1 another point your displacement is u 2 then we defined a function between point 1 and between point 2 this function gives us how displacement is how displacement varies along the length of that element, ok.

Now, in that function if we substitute x 1 then we will get u 1 and if we substitute x is equal to x 2 will get u 2 the nodal values same exercise we will be doing it here. Now, assume the first is suppose assume v x is equal to say a polynomial expression say a plus bx plus cx square plus dx cube.

Now, in the case of truss if you recall we just had the linear expression in the first two term, we did not use the a quad cubic expression. The reason was very, reason is very simple in the case of truss we already see when you use a polynomial use an approximation these approximations always associated with some constants we have to find out those constants. For instance here a b c d are the constants in the case of truss we had two constants. In order to find out those constants we need sufficient number of equations.

In the case of truss we had just two equations and those equations are. And what was the premise of that equation? Information about the nodal displacement; we had every node we had just one displacement, two nodes we had two displacement means two information that is why with that information we could only calculate two constant. That is why our approximation was linear.

But now, in this case we had 4 information. What are the 4 information? We know what is the transverse displacement this is v, we know what is the transverse displacement and rotation at node 1 and transverse displacement and rotation at node 2, 4 is 4 unknown that is the reason we considered 4 constants here, ok.

So, then in this so what will be dv dx? dv dx will be if we this will be b plus then 2 cx plus 3 dx square. So, this is, now let us substitute this equation from the first equation suppose this is equation number 1, this is equation number 1 and this is equation number 2 from the first equation we get two for first satisfy at every node we get two equation.

We satisfy equation this at both nodes we get another two equations. So, let us do that. So, it gives me say at x is equal to x 1. So, at x is equal to x 1 we had v of x 1 that is equal to then we have a plus a plus, then we have then x is equal to, ok. Now, let us let us do one simplification suppose instead of x 1 and x 2 suppose just change the coordinate system so that our computation will become easier. Suppose this x 1 is equal to 0 here and x 2 is equal to L here L is the length of this element, ok.

So, essentially x is equal to x 1 then what happens? Your, if you substitute x is equal to 0 then this becomes that v 1 is equal to. So, this also you can remove, ok. So, this v 1 becomes a. So, this is one expression and x is equal to x 1 this becomes theta 1. So, theta 1 becomes if you substitute x is equal to here so then theta 1 becomes b, ok.

So, this is two equation the and then at x is equal to x 2, at x is equal to x 2 means which is equal to L this is equal to 0. Then what we have is we have v 2 is equal to v 2 is equal to this becomes a plus bL plus cL square plus dL cube.

So, this is one equation and then another equation theta 2 in this expression if you substitute L b plus 2 c 2 cL plus 3 dL square. So, these are all 4 equations we have. So, this is equation number this is one equation, this is another equation, this is another equation and this is another equation. These 4 equations and we have just 4 unknowns. We have to solve these 4 equations for 4 unknown. And if we solved them then what will get is we get some expression for a b c d. And then what we have to do we have to substitute that expression in this equation, ok. At the same way we did it for truss.

If you substitute that a b c d in this expression and then you write this v in terms of in terms of in terms of v 1 theta 1, v 2 and theta 2 then what we have is v x is equal to if you write it say N 1 N 1 v 1 plus N 2 theta 1 plus N 3 N 3 v 2 plus N 4 theta 2. So, N are the N 1, N 2, N 3 are the shape functions, these are the shape function, right. Now, so if you substitute a b c d from these 4 equations and if you substitute them in this and just like manipulation you get an expression like this.

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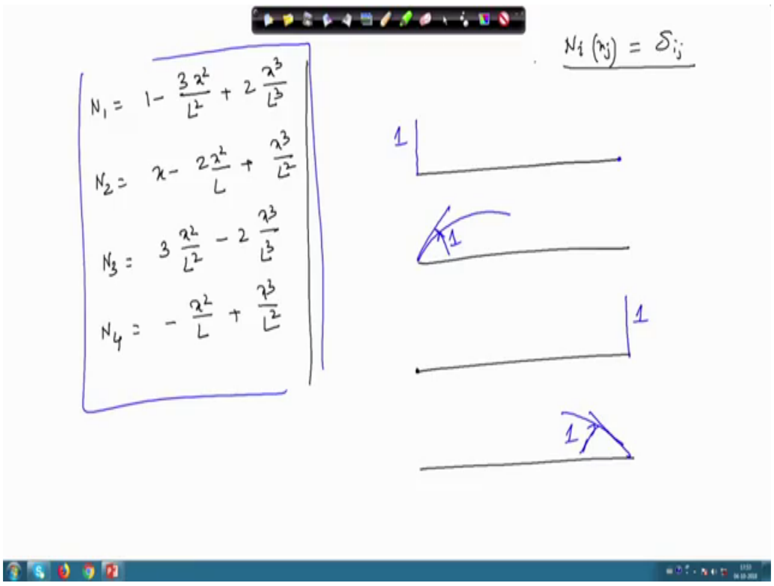

And in this expression let me give you the final expression for N 1, N 2, N 3 and N 4. And that is you that becomes N 1 become, N 1 in this become 1 minus 1 minus 3 x square, 3 x square by L square plus 2 x cube by L cube.

N 2 become x minus, then 2 x square by L square 2 x square by L, and then plus x cube by L square this is N 2. N 3 become 3 x square by L square minus 2 x cube by L cube and finally, N 4 you can check these values minus x square by L plus x cube by L square, ok. Now, we will see later that these we are we are doing in terms of x and L this can be done in terms of parametric coordinate system where it varies from 0 to 1 and those exercise will see later. So, these are the 4 shape functions.

Now, let us plot this 4 shape function if you have to plot it then the if you recall the properties of shape function the properties of shape function is that N i n N i at x j that has to be delta ij cubes. This satisfy Kronecker delta property, right. Now, so let us see that if we let us plot this shape function. Now, if I have to plot it say if I have to plot it plot N 1, N 1 if I sub at x is equal to 0 n will be 1 this and then x is equal to x is equal to x is equal to L, if you if you substitute x is equal to L here then this will be this will be this will be 0, ok.

So, if we substitute here x x this will be 0. Now, similarly if I plot now, I am not plotting what happens in between I leave it to you check the expression and then see: what is the x what is the plot in between. What I what I am showing here is that that the shape functions will this satisfy this Kronecker delta property and the other properties that we discussed.

Now, what is N 2? N 2 will be you see here it is important we will come to N 2 later let us talk about N 3 first. Now, N 3 is x is equal to 0 if we substitute N 3, N 3 is 0 and x is equal to L if you substitute L here N 3 will be 1 and in between it is cubic, ok. Now, and then finally, we have N 4 finally, N 4. Now, what is important here? You see N 2 is associated with if you look at, ok. Before that we also discuss if you recall last class, if we take the summation of all this shape function then it will be 1.

Now, you take summation of all N 1, N 2, N 3, N 4 any arbitrary value of x suppose x is equal to 0 or x is equal to 1 or any point check whether they are 1 it or not. Now, what we will see is that N 1 these shape functions in the case of truss we are two shape function one is N 1 and another one N 2 and both shape functions were associated with similar kind of displacement, right, but here we have two kinds of displacement one is the transverse displacement another is rotation

Now, N 1 and N 3 are associated with that displacement transverse displacement whereas, N 2 and N 4 is associated with rotation. So, when I say that at x is equal to, at x is equal to when I say that N i x j satisfy delta ij it means that for N 1 for N 2 it means that at x is equal to at x at that particular point where that N 2 is N 2 is define the rotation what you get is that rotation should be 1, right. Means it should be the rotation should be here 1. In this case it is 1, in this case it is 1 what says that this rotation should be 1, this value should be 1. Similarly here it is seeing this rotation should be 1, mean these value is 1, right.

Now, I am not plotting it in between what will happen you follow these expression and then plot it and another exercise you have to do you take any arbitrary point at x any value and then see sum them whether you get a 1 it or not. If you get 1 then it satisfies the property, but if you do not get 1 does that mean that it does not satisfy the Kronecker the partition of unity of property. And if you do not get 1 try to find out why you are not getting it 1, ok. So, these are the expression for shape function.

Now, next is once we have this shape function if you recall in the case of, in the case of in the case of in the case of truss we have to find out what is the strain displacement. So, these shape functions give you how the, this from this shape function we can obtain the strain displacement relation, right. Now, before that let us now discuss what kind of strain we have we need to incorporate in this formulation. If you if you look at just now, we discuss here that epsilon u transverse your a these displacement is u is equal to minus z into del v del x, so then longitudinal strain will be.

What will be the longitudinal strain? Then longitudinal strain will be epsilon is equal to du dx du dx, which essentially becomes if we substitute that minus z into d 2 v dx 2, right this is the longitudinal strain. Now, let us now, we already know expression of v as a function of N 1 N as a function of the nodal coordinates. Then substitute these expression or v in this equation and if you do that let us see what we get.

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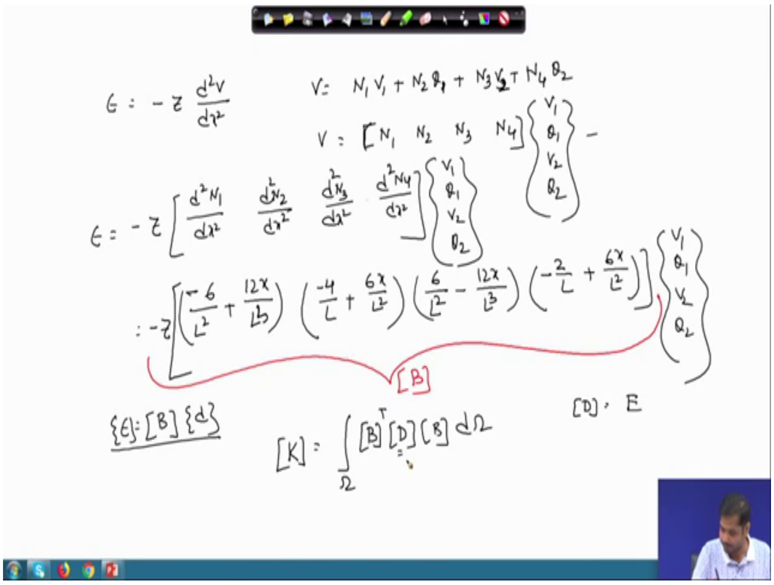

So, we have epsilon is equal to say epsilon is equal to minus z into d 2 v and then dx 2, and then v we have N 1 v 1 plus N 2 theta 2 theta 1 plus N 3 v 3 plus N 3 v 2 and N 4 theta 2. This is the expression. Now, if we substitute that in this expect. Now, this can be written as like this we have say N 1, then N 2, N 3 and N 4 these are the shape function. And then here we have the degrees of freedom which is v theta v 1 theta 1 v 2 and then theta 2, ok.

Now, let us substitute this v. Let us substitute v the from this to this expression. And if you do that epsilon will be minus z and then this will be d 2 N 1 dx 2, just the second derivative of this d 2 N 2 dx 2 dx 2 and then u yes. And then d 2 N 3 dx 2 and then finally, we have d 2 N 4 dx 2 this is for the this part and then we have these degrees of freedom v 1 theta 1 v 2 theta 2, ok, right. So, this is the expression.

Now, we already know the expression for N 1, N 2, N 3, N 4. Now, if we substitute that expression in this equation the equation what we get is minus z this is the this, ok. Let me just write the expression here what we get is from N 1 if we if we if we this N 1 we will get minus 6 x by L square, minus 6 x by L square sorry minus 6 by L square and this gives you 12 this gives you this x cube this gives you 12 x by L cube, right

So, this becomes minus 6 by L square and plus this become 12 x by L cube. So, L cube this is for this is for first one N 1. Similarly we can have for N 2, N 2 is this part will be 0 this part will becomes minus 4 by L and this part become 6 x by L square. So, this becomes minus 4 by L and then 6 x by L square we have plus, this is for this. Similarly for the third one, this becomes this becomes 6 by L square and this becomes minus 12 x by L cube. So, this becomes 6 by L square and then minus 12 x by L cube L cube, right.

And finally, this is this becomes minus 2 by L and this becomes 6 x by L square. So, minus 2 by L plus 6 x by L square, ok. So, this is and then finally, here we have v 1 theta 1, v 2 and theta 2 this is epsilon, right. Now, so, this is you recall that epsilon is equal to epsilon is equal to some matrix B into displacement that was strain displacement relation and then what is here B matrix is this part entire part is B, right. So, we have B.

Now, then what will be D? Now, once we have B, now if you recall the stiffness matrix the general form of the stiffness matrix was what K is equal to the element stiffness matrix was integration over the entire domain entire volume then we have B transpose then D the constitutive relation and then B and then integration over the entire domain, right that was our stiffness matrix.

Now, we have obtained B already this. So, B we can get from this what is D. Now, you see what we are only talking about we are only considering this longitudinal strain because this longitudinal strain has all the information about the deformation, right. So, this longitudinal strain then we have just considered is one component of strain along the longitudinal directions.

Then what will be the corresponding stress? Corresponding stress will be just along the longitudinal direction and what will be the this stress and strain relation these stress and strain is related through Young's modulus because it is just one longitudinal strain we are talking about and then integrate it over the entire volume, right. Now, so if this is then our in this case that B will be D will be is equal to Young's modulus E. Now, then finally, what we have to do is we have to substitute this expression here and let us do that. Let us substitute that.

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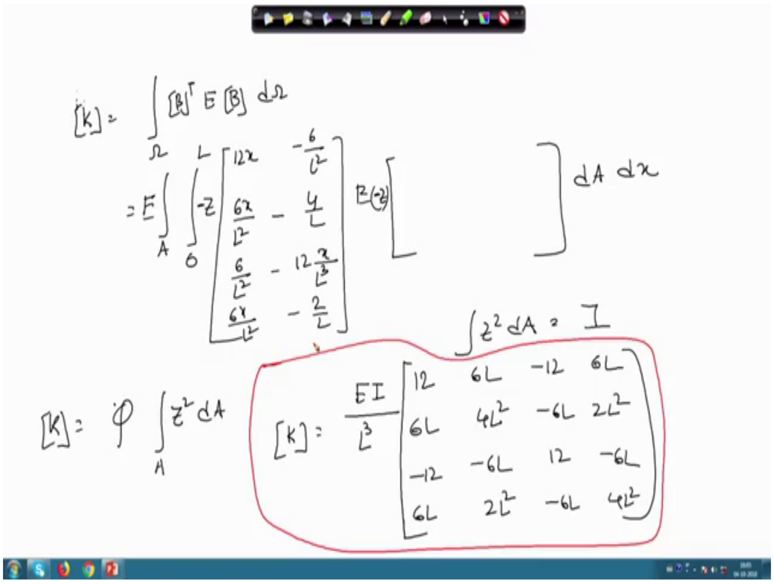

So, then K will be same K. So, K will be integration of integration over the entire volume and this will be, this will be that D transpose it has not writing it once again you know that and this Young's modulus and then B right and d omega. Now, this we how we are integrating this? We are integrating this once again let me come to this beam. Please note we what we are essentially doing, we are taking a longitudinal fiber along the longitudinal direction considering only the transverse considering only the longitudinal strain because, that longitudinal strain has information about the transverse displacement and the slope.

So, we are considering the longitudinal direction, and then that strain is that strain is here epsilon and this is and that is why we have just only one D is equal to a scalar a Young's modulus. Now, when we integrate it that integration has to be done not only along the length that integration has to be done along the cross section as well, right. Because along the cross section your longitudinal strain will be different in, this strain depends on z which depends on what fiber what longitude fiber we are considering. So, in this case this integration will be first along the cross section and then along the length, along the length means 0 to L in this case.

So, this becomes let us write it B transpose is I am writing it because then 12 x and then minus 6 by L square, then we have 6 x by L square and then minus 4 by L. Please check these values 6 by L square minus 12 x by L cube and then finally, 6 x by L square and then minus 2 by L. So, this is for B transpose E and then another we have the same thing as this, ok. This integration will be over dA and then over dx, ok.

Now, so this is the stiffness matrix. Now, let us and then there will be a z also, then it will be it will be E we can take outside, E is constant and there will be a minus z term, ok. Now, if we perform this integration over x over the length as well as across the cross section means over z from minus say minus h 2 to h depends on how you represent the depth of the beam. And if you do that then what do you get? But remember one thing in this expression we will get an expression like we will get an expression like this integration over, ok. So, we have we have minus z for this and then another minus z for this, another minus z for this. So, this is for B transpose this is for B.

So, you see look at this term you will get an expression something like this z square dA. If you integrate once the integration along the length is performed then all you can take outside and then we are left with something. So, K will be something say some phi into this, this phi is the integration of the entire thing along the length.

Now, what is this Z square dA? Z square dA is essentially if you recall Z square dA is second moment of area, right. Now, if you substitute second moment of area here, and then do this integration once again then this stiffness matrix the expression for stiffness matrix will get E I, E from this and I because Z square dA is equal to I, E I and L L u get from this expression E I by L cube and then the x the term will be it will be 12 and then 6 L. By now you have understood: what is the form that we are going to get because you have seen this form many times, right 4 L square minus 6 L 2 L square.

And this becomes minus 12 and then minus 6 L 12 once again minus 6 L and finally, 6 L, 2 L square and then minus 6 L and 4 L square. Yes, this as the stiffness matrix for you have obtained the stiffness matrix, you are familiar with the stiffness matrix because we have seen it in case of in direct stiffness matrix method for beam member, right.

Now, but at that time we derived this matrix based on the directly based on the based on the load displacement relation, based on the how the moments and curvature is related and the load and displacement is related with each other, based on that we directly obtain this matrix. We did not follow this approach, ok. But as I said in the last class this approach has as a flexibility to determine construct determine stiffness matrix construct stiffness matrix for a more general class of elements. These are the some elements we have we are considering just to establish the fact that the final outcome, the stiffness matrix will be the same as the stiffness matrix we had for direct stiffness matrix in the direct stiffness matrix method.

And another reason why you are following this approach because as I said it is in extension because you already had the knowledge of direct stiffness method, you had the knowledge of strength of material and solid mechanics. It is just based on the knowledge we are just trying to see whether a more general perspective, more general formulation of stiffness matrix can be obtained or not which is applicable for a more general class of elements. So, this is for beam element.

Now, next class what we do is next class for truss element calculating nodal load vector was not it was very straightforward. But recall in the case of beam element for in the direction stiffness matrix method for beam element how we calculated the equivalent joint load. First you had to find out the fixed end moments, and then you apply the equilibrium at different joints to get the equivalent joint load.

We will see here next class using the again the same a general or the general form how these nodal load vector can be constructed. And then we then that will be for beam and then from the subsequent classes we will see, we will apply this concept to consider stiffness matrix for a more general class of elements for 2-dimensional elements, ok.

With this I stop here today. See you the next class.

Thank you.