Matrix Method of Structural Analysis
Prof. Amit Shaw
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 37
Introduction to Finite Element Method (Contd.)

Hello everyone. This is the second lecture of this week. In the first lecture we try to have a general definition of general expression of stiffness matrix. Today we use this definition and find out the stiffness matrix for, today and in subsequent classes for different structural components. And through that exercise we will try to understand the philosophy of finite element method, ok.

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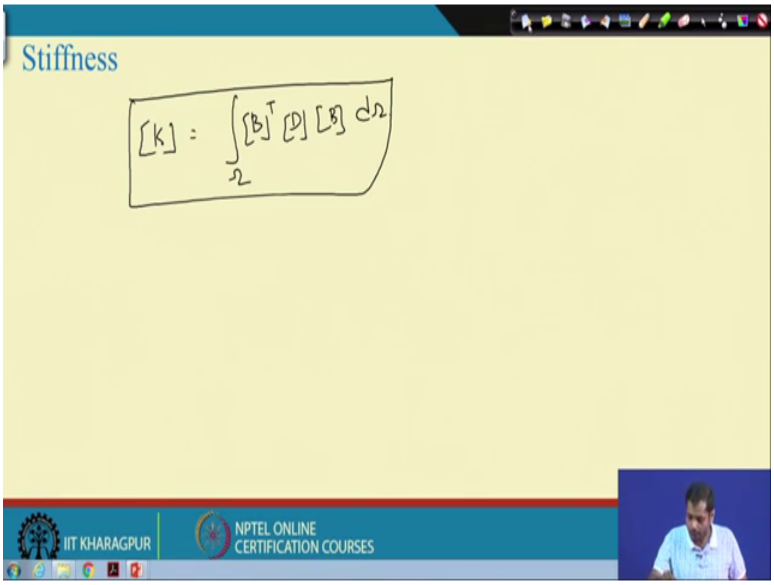

Now, let us we call if the stiffness matrix was last class we have seen that stiffness matrix is K is equal to, is equal to integration over the entire domain. And what is the domain and all? We have not yet defined that depends on what kind of elements, what kind of components of structural segments you have.

So, this is B transpose which is strain displacement relation, then constitutive relation and then again strain displacement relation that is integrated over the domain. So that was our relation, right. Let us use this relation.

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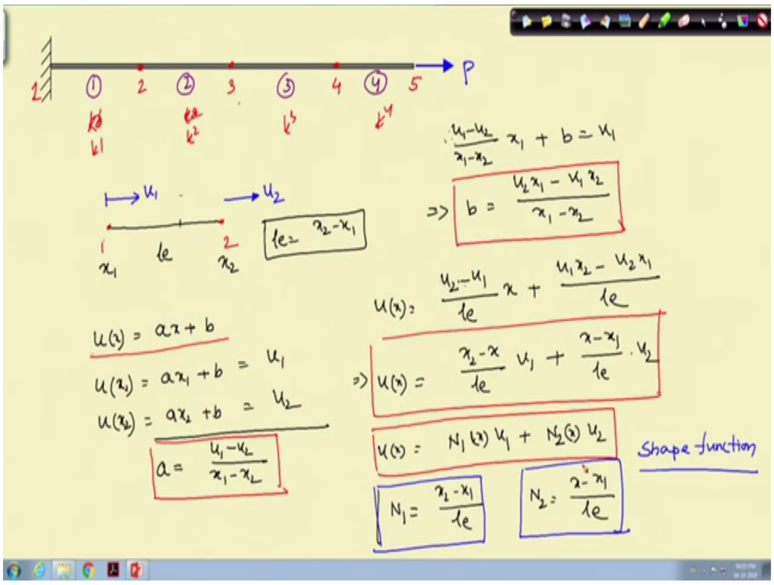

Now, let us take one example, ok. It is with the strength of material using strength of material and also using the direct stiffness matrix method you can solve this problem, ok.

It is a problem, we have a bar, remember it is not a beam it is just a bar which is subjected to say axial force one end is fixed and other end it is axial force is say P. So, what we are interested in? We are interested in what are the displacements at different points, what are the stresses at different points and strain at different points, ok.

Now, so as I said in the last class we will not discuss the theory of finite element method and all. We have the knowledge of matrix method of analysis, we know what is the philosophy of matrix method of analysis and we also have the knowledge of solid mechanics, and based on that we are trying to solve we can try to device a method which is more general, ok.

So, like finite and like matrix method of analysis let us divide the entire bar into several parts, ok. So, let us divide into safe you can divide in many number of parts, ok. Suppose divide in 4 parts, ok. So, this is node number 1, this is node number 2, node number 3, 4 and then 5. The same exercise we did for matrix method of analysis. And similarly we have element number 1, say this is element number 1, then element number 2, element number 3, member number 4 like this.

So, if we take any arbitrary element say element 3 element 3 it is connected between node number 3 and node number 4. In the same way we did in this method, ok. Now, then what we did in matrix method direct stiffness matrix? For a representative element for a representing structural component it could be truss it could beam or it could be frame or anything. For a representative in a structural component we derive the stiffness matrix. Let us do that exercise right now.

For representative element take any element for instance take an element like this, which is, ok. Take slightly take an representative element like this. Suppose this is connected between node number 1, node number 1 and node number 2, right. This is node number 1 and node number 2. Now since it is 1D element, just you have you have just 1 degrees of freedom means it is bar problem. So, you have degrees of freedom along the longitudinal axis of the bar.

So, this element at this point your displacement is safe u 2 that is our degrees of freedom and at this point displacement is u 1, right. Now, suppose this is x 1; x 1 is the this is x 1, x 1 is the coordinate of that point and x 2 is the coordinate of that point and suppose length of this member is l e, where le is equal to; le is equal to x 2 minus x 1, ok. That is the length of this member e stands for length of that particular element, ok. So, this is the thing we have.

Now, one of the first step required is that explicitly we have not done in case of direct stiffness matrix. You see what is u 1? U 1 is the displacement at a node 1 and u 2 is the displacement at node 2. Now, if I want to know what is the displacement at any intermediate point between node 1 and between node 1 and 2.

Suppose any arbitrary point I want to displacement at this point what will be the displacement because we do not know that displacement we know the displacement only at node 1 and node here node 2. So, what we need to interpolate this based on the what is the displacement at node 1 and node 2, we need to interpolate this value; we need to interpolate at this particular point, ok.

Now, for that assume that displacement between node 1 and node 2 varies linearly, ok. So, suppose displacement which is a function of x that varies linearly ax plus b, now a and b are the constant which we do not know, we have to find out that constant. But then we need some information to find out that constant and what information we have the information we have that u at x is equal to x 1 that is equal to ax 1 plus b that is equal to u 1, right. And then similarly u x is equal to x 2, which is x 2 plus b that is equal to u 2. So, we have 2 equations and 2 unknown therefore, we can solve for a and b. And that was the reason why we assumed this variation is linear we cannot take quadratic variation because we do not have sufficient information.

Now, if we solve them, if we solve them, then what we have is, we have that a is equal to u 1 minus u 2 divided by x 1 minus x 2 that is the expression of a, that is important. Now, once we have this expression for a we have to find out expression of b. Now, how do we find out we substitute in any equation if you substitute that let us do that. So, a will be, if we take this expression the first one, first one is a is equal to u 1 minus u 2, u 1 minus u 2 by x 1 minus x 2 into x 1 plus b is equal to u 1. So, from this expression if you solve that we get b is equal to, b is equal to u 2 x 1 minus u 1 x 2 divided by x 1 minus x 2 x 1 this is b, ok. So, this is important.

Now, once we know a and b then what happens? Then if you substitute this relation in this expression and then we get a functional representation of u. So, u x will be then a, into a is this. So, if we substitute x 2 minus x 1 as le then it becomes u 1 x 2 no sorry not that this one this become u 2 minus u 1 divided by le. And then into x plus b, b is this one, b becomes u 1 x 2 minus u 2 x 1 divided by le that is the expression for u.

Now, if you write this expression from that what we get is just little bit of manipulation of this. What we get is x 2 minus x divided by le into u 1 this is very important; x minus x 1 by le into u 2, this is very important expression.

Now, let us write this expression as u x is equal to N 1 function of x u 1 plus N 2 which is also a function of x u 2, and where N 1 is equal to N 1 is equal to x 2 minus x 1 by le and N 2 is equal to x minus x 1 by le right this is important. These N 1 and N 2 are called shape function, ok.

Now, what is the advantage of writing this u in this expression? Because now if we want to find out u at any intermediate point you substitute corresponding coordinate in these expression and get this get the value at any intermediate point. If you substitute x is equal to x 1 here you will get u 1 if you substitute x is equal to x 2 here you will get u 2 and for any intermediate point you get a combination of u 1 and u 2.

Now, what is the basis of, what is the basis of arriving at this equation this expression for N 1 N 2? We will assume that the variation of u vary along the length of this element linearly. We can have quadratic fine it variation, we can q v variation for different kinds of element, ok. And will get shape function accordingly for that. These shape functions are important.

Before this shape function satisfies certain property and before coming to the property let us plot this shape function. Let us see if I plot it along the length of the member how the shape function look like. So, do this exercise, suppose.

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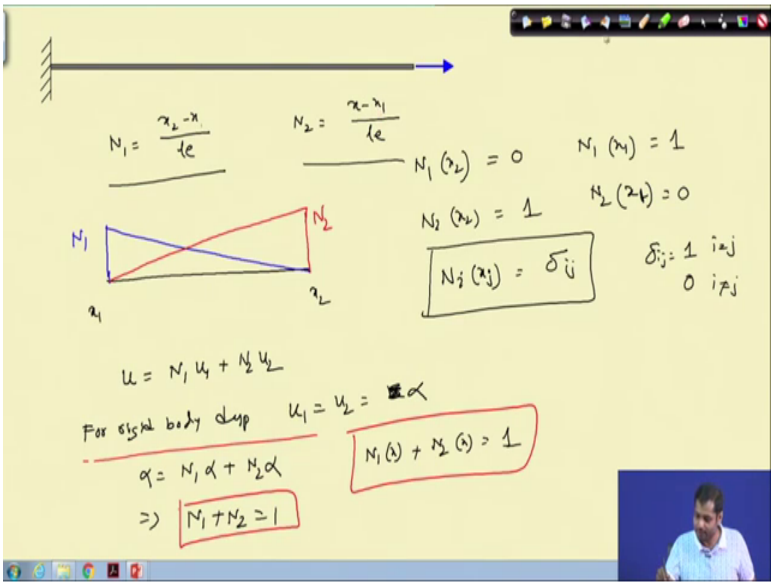

So, x 1 is if we plot x 1. So, x 2 if we N 1 was if we write once again N 1 was x 2 minus x naught x it is x 2 minus x 2 minus x divided by le, and N 2 was x minus x 1 by le, right.

Now, if I plot it suppose this is the member this is x 1 and this is x 2 and plot N 1, if I plot N 1 then the N 1 will be at x is equal to at x is equal to x 2 N 1 will be 0 and x is equal to x 1 N 1 will be 1 like this. And between this will be like this. Similarly if I plot N 2 N 2 plot will be this. So, this is for N 2 and this is for N 1 it means, ok.

Now, it means that now this property if you see N 1; if I calculate N 1 but x 2 that is is equal to 0 and N 1 at x 1 is equal to 1. Similarly N 2 at x 2 is equal to 1 and N 2 at x 2 N 2 at x 1 is equal to 0. So, this means that N i at x j that is equal to delta ij, delta ij is called

Kronecker delta. Delta ij means, delta ij is equal to 1, if i is equal to j otherwise j you look at this. So, this shape function satisfy Kronecker delta property right, this is one property important.

Now, the next is we have we just now have seen that u is equal to at any intermediate point u is equal to N 1 u 1 plus N 2 u 2 right. Now, consider a case, for instance this is the bar element consider a case at this point you have u 1 and at this point you have u 2. Now, suppose this is moving like this as a rigid body, if it moves like this then what happens throughout this length the displacement is this displacement is constant. So, u 1 will be equal to u 2 because it is moving entire thing as a whole as a rigid body. So, in that case u 1 will be u 2, right. So, u 1 will be u 2.

Now, if u 1 is u 2, u 1 is equal to u 2 what will be the intermediate value; intermediate u at in any intermediate point? This will be same as u 1 and u 2, ok. So, for rigid body for rigid body movement rigid body displacement what happens? Say u 1 is equal to u 2 suppose this is equal to u, ok. So, are some alpha, some constant, some alpha, ok. And then what happen? Alpha is equal to N 1 alpha plus N 2 alpha and this gives you very very important property N 1 plus N 2 is equal to 1, ok.

And if I more general way if I write that N 1 at any point plus N 2 at any point that is equal to 1, right. This is called partition of unity property, means if you take any point in the element and sum all these shape functions this shape function this summation will be equal to 1. If it does, if it is not one then this condition is not satisfy the rigid body motion your displacement and all the points are same this condition is not satisfied right. So, this is equal to these are the some properties of shape functions, ok.

Now, once we have the shape functions our objective here if you recall our objective is to find out the stiffness of this element right. Now, let us do that.

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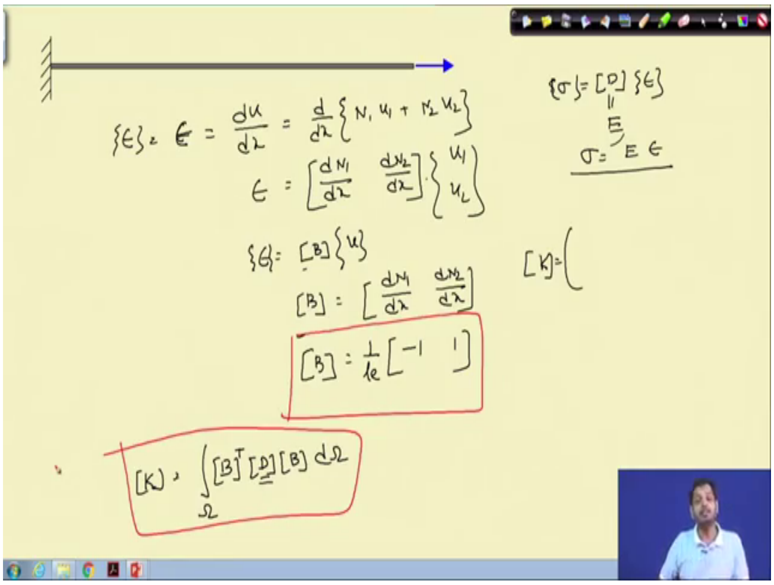

So, now next is once we have this; next is if you recall epsilon is equal to what? Epsilon is the definition of epsilon; epsilon is equal to the first derivative of displacement, ok.

Now, here it is just one displacement you have and one strain you have. So, this is equal to strain since it is one value. So, I am not writing in a vector form, strain is equal to if you du dx right this is this definition of strain. Since it is one component we can write it it is not partial derivative it is just the du dx.

Now, that is equal to d dx of now u we just now have u is equal to N 1 u 1 plus N 2 u 2, right. So, can we write this as this is equal to that d N 1 dx, d N 2 dx this and then we have u 1 u 2 and then dot product in between, ok. It is same as this. So, this is equal to what? This is strain.

Now, you see this is essentially what? This is essentially strain displacement relation. So, this is essential is epsilon is equal to B into u, epsilon is equal to B into u, where B is equal to, in this case B is equal to strain displacement relation d 1 dx and d N 2 dx. Now, expression for N 1 N 2 already we have, this is N 1 and this is N 2, ok.

Now, if I differentiate it with respect to x we get something if I differentiate it with respect to x what we get. If we differentiate with respect to x we will get d 1 d N 1 dx is equal to minus 1 by le and this will get plus 1 by le, right. So, what we get is minus this is equal to minus 1 by le if you take if we take common, minus 1, this will get 1. So, this is equal to B right. This is the expression for B we have strain displacement relation great.

So, once we have that, so what is the expression for expression for D? Expression for D is we have the stress is equal to stress is equal to D into that was our expression that D into epsilon, right. Now, in this case since it is just a 1D bar problem it is your D is essentially Young's modulus E, right. So, here we have sigma is equal to D or E into epsilon, right. So, D is essentially E.

So, we have find out what we have found out what is the expression for D, what is expression for B. Let us now found out, let us now find out what is the expression for the stiffness matrix. And that stiffness matrix expression we use the general definition of stiffness matrix and that definition if you recall that was this the stiffness is equal to integration over the entire domain of this into B transpose D into B and that integration is over the entire domain.

Now, in this case entire domain means, if you see this problem in this case entire domain is the length of the length of the member, ok. So, it is essentially either you can integrate from x 1 to x 2 or you can integrate from 0 to le, ok. Now, if you do that let us do this exercise.

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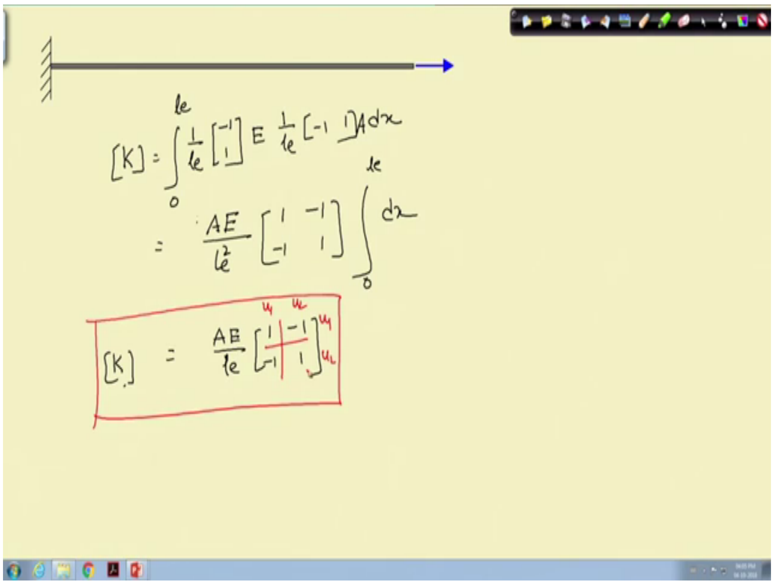

So, this expression will be, so K will be let us write it here. So, K will be, use a different, ok. So, K will be integration over 0 to le length of this length of this element then B, B transpose will be this. So, transpose of this transpose of this will be by 1 by le. So, it becomes minus 1 minus 1 1 or you can, now then we have D, this D, this D is essentially E. So, we can write it as E and then again B, B is 1 by le and minus 1 1 and then over dx, because it is integration over the length.

So, we can take these outside we can have E by, ok. Now, one more thing since it is integration this integration has to be over the entire this integration has to be over the entire domain. So, entire domain will be what? Entire domain means in this case entire volume entire volume means, means cross sectional area multiplied by the length, ok. So, cross section we have to multiply it by cross section as well a into dx, a into dx.

So, now, we can take this out the entire thing out, if you do that we are almost by now you might have understood that what expression we are going to get 1 AE by le square and then this becomes 1 minus 1 minus 1 1 and then integration 0 to le dx, ok. This essentially gives you AE by le 1 minus 1 minus 1 1; K. Now, please see whether you have seen this or not. What is this? We have done it for cross analysis this is the stiffness for truss member in a local coordinate system. And truss member means what? Truss member is it is we have force only in axial direction, right.

There the same element we are talking about here. We have a bar element which has force only in longitudinal direction and this is the stiffness matrix. The same stiffness matrix we used in dx stiffness method, but that time we derived this we arrived at this stiffness matrix in a different form right in a different way. But now we have arrived this stiffness matrix in a more general way the reason is, now you may ask that once we at the end if you are going to get the same stiffness matrix then what is the point of all this exercise.

The reason is this will allow us to derive stiffness matrices for different kinds of element, it is just not for truss but demonstration we are using truss we will be using beam and other 2-dimensional structure as well but this will allow us to (Refer Time: 22:58) which would allow us to calculate the construct the stiffness matrix for a general class of segments which is a which was otherwise not possible the way we derived for in a matrix method of structural analysis, ok.

Now, if you recall this beam was divided into 4 parts, now similarly we can for every individual part we can calculate the stiffness matrix for every individual part. So, for this part we can calculate what is K 1, then K 2, K 2 sorry yeah K 1, K 2, K 1 then we can have K 3, K 4 and so on. So, 4 stiffness matrices the same way we can have for stiffness matrices. But only difference for different stiffness matrices your x 1, x your this coordinates will be different and the eventually our length will be different you get this stiffness matrix.

Now, once we have this. So, this stiffness matrix is essentially it corresponding to, so if you recall this is for u 1, this is u 2 this is for u 1 this is for u 2, ok. Similarly we can have stiffness matrix for each bar segment and once we have these bar segments we can once we have the stiffness matrix for all the bar segments we have to assemble them exactly the same way we assemble stiffness matrix in direct stiffness method, right. So, this was the assembling part.

The next is the nodal load vector. Now, nodal load vector is if you look at this structure, the nodal load vector will be how many what would the size of the stiffness matrix? Size of the stiffness matrix will be we have 4 nodes. So, total global stiffness matrix size would be 5 by 5. Similarly nodal load vector the size of the nodal load vector will be is with a 5 elements, is a vector with 5 elements and all elements will be 0 only thing is for a 5 for P 5 this will be P, right. So, this essentially will be nodal load vector.

Once we have the nodal load vector then we can apply the boundary conditions exactly the same way we apply boundary condition in matrix method, and then once we apply the boundary condition we can partition the matrices stiffness matrix. And then calculate the unknown displacement exactly the same way we did it for dx stiffness method, once we have the unknown displacement the rest of the process is exactly same as the method.

So, only difference between the methods that we have discussed in the last 7 weeks, 7 weeks and the approach that we have just now discussed lies how we are calculating the stiffness matrix member stiffness matrix. And that is not a very small difference that will subsequently see that that is a very huge difference, ok. And that makes our life, that that allows us to calculate stiffness this expression to calculate to construct stiffness for a very very general class of elements, ok.

Now this was the bar element, similarly we can have we can define the stiffness matrix for beam element, stiffness matrix for any other any other kinds of element we will see subsequently. But the beauty is irrespective of the element it is whether it is a bar element, beam element or any other element, this expression will remain same, this expression will remain same. Depending on the elements we have different degrees of freedom and therefore, we have different strain displacement relations, and depending on the dimensions of the problem we have expressions for d but the general expression for stiffness matrix this will remain same, ok.

So, next class what we do is, next class we do the similar exercise but for a beam element, where the degrees of freedom will be this; the degrees of freedom will be translation vertical transverse displacement and the rotation. And then see what and then see what is the corresponding stiffness matrix and we will compare that stiffness matrix to the stiffness matrix we already know. So, with this I stop here today. See you in the next class.

Thank you.