Matrix Method of Structural Analysis
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Lecture - 36
Introduction to Finite Element Method

Hello everyone. We are almost end of this course. This is the last week and now before we before we mention what we are going to do in this week let us look back and see the structure of this course.

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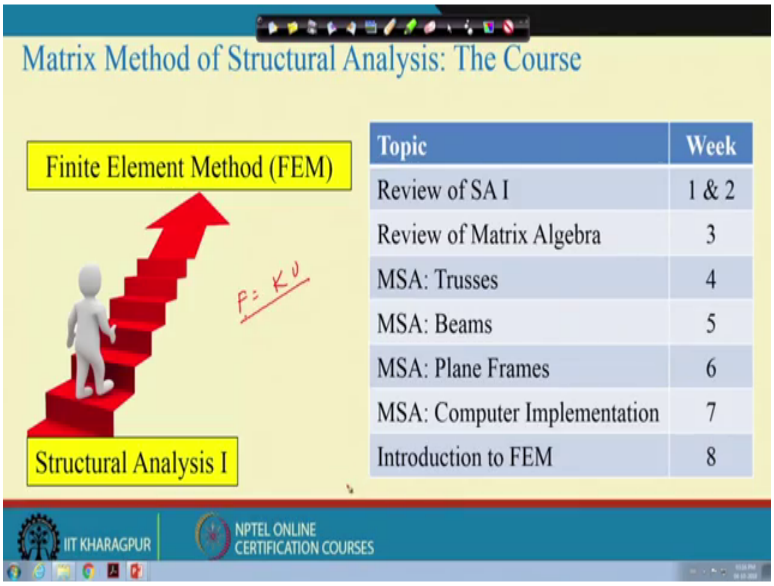

You see if you recall this was the first slide in very first lecture, this is the last slide and very first lecture this gives you over of the overview of this course. And in this course we have already discuss, we started with started with reviewing some of the basic concept of structural analysis one, we had a brief review on matrix algebra only that part which is which is essential for this methods to understand and this method to implement.

Then we also discuss how also formulated the matrix the direct stiffness matrix method for direct stiffness method for trusses beams and then plane frames. And we have done it for 3D trusses beam, 2D and 3D trusses beams and plane frames. We also discuss some of the issues related to implementation, computer implementation of the problem. And today is the last week where we will be having a brief introduction to FEM.

If you recall that time it was discussed the purpose of this course it is a course intermediate level course where the prerequisite was structural analysis one, and then it builds on structural analysis one. And whatever concept we learn through this journey at that will take us to another method which is more sophisticated method which is called finite element method.

Here we are going to we are going to briefly discuss what is finite element method. You see honestly speaking, honestly speaking the finite for finite element method to learn finite element method till there we need we need many courses to learn different aspects of finite element method, ok. We have linear finite element, non-linear finite element, we have say compute the compute the issues related to computations; then we have analysis of finite element and so on.

So, therefore, naturally we in few hours time our objective will not be to learn finite element method here. Here the objective is to open the door and letting you know that there is a beautiful world out world out there in the name of finite element method which deserve to be explored that is the our major objective in the next few hours time.

You see if you recall the underlying philosophy of direct stiffness method direct stiffness method or matrix method of analysis was you have a structure, then you break the structure into several segments, several members. And then write the equations for each member, when we write the equations that equations essentially equilibrium equations that equation was force us force is equal to force is equal to stiffness into displacement, right.

We write these write this equation for all the members and then we assemble this, we assemble this all these information all the equations for different members we assemble them we join them in such a way that the compatibility at different nodes are ensured, ok. And then we have a global system or equations and then we solve it. That is the underlying philosophy or matrix method of analysis or direct stiffness matrix.

Now, if you see the philosophy of finite element methods the finite element method is also based on the similar philosophy, but in FEM these exercise is done in a very general way, very sophisticated way and then let us see how it is to be done and what is that.

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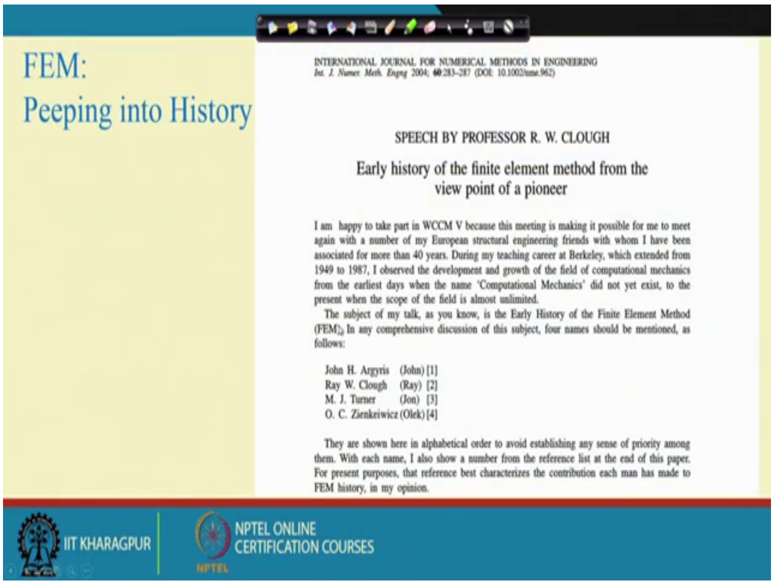

Now, before that some of the concept that you already, ok; this is an important read because you see this article it was published in 2004, this article gives you a very brief history of finite element method.

And then and whenever we start it is my suggestion to all student all of you when you are starting learning finite element method you all should read this article and you see how just a casual discussion can lead to a method which is so famous, so global, ok.

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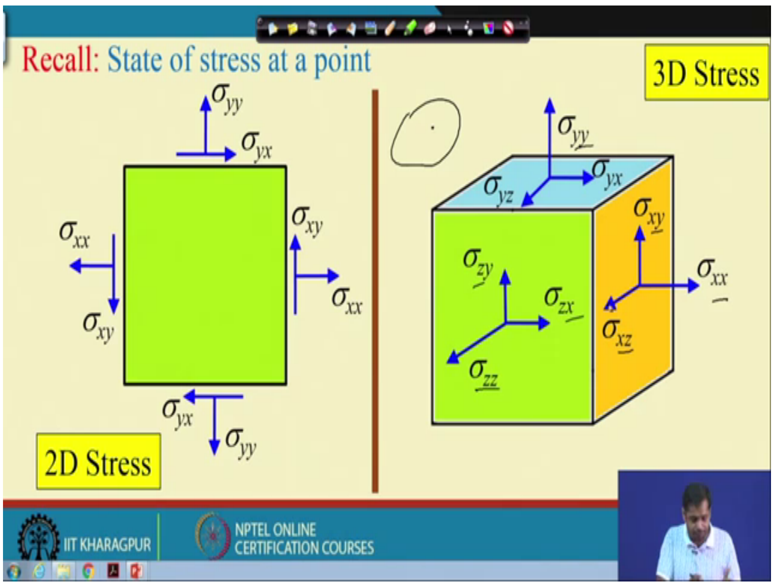

Now, before we come to finite element method let us understand some of the review some of the basic concept that you already had in solid mechanics course, ok. You see if we have this is the concept of stress and strain, stress at the point is if you recall if you have an arbitrary body anybody which is subjected to some external loading, then stress is essentially a response to that external loading, right. Stress and strain essentially at the response to that loading response to that environment where an material is subjected to.

Now, if we take a point in 3-dimensional space then we have 6 components of stresses or 9 components of stresses, out of 9 components we know that the 6 3; 3 the essentially 3 independent shear stresses we have. So, 6 stress components we have 6 are independent stress components among this 9 for a linear isotropic ah, for in a 3-dimensional space.

Now, if we take a point here any point and that point this is how we represent these stresses right, and similarly if we if the point if the point is in a 2-dimensional space 2-dimensional body then this is represented like this where the sigma xx, sigma yy, sigma zz are the normal stresses it is acting normal to the in the normal direction. And then sigma xz xy and all these sigma zx they are these shear stresses acting on different planes; similarly, for 2-dimensional case.

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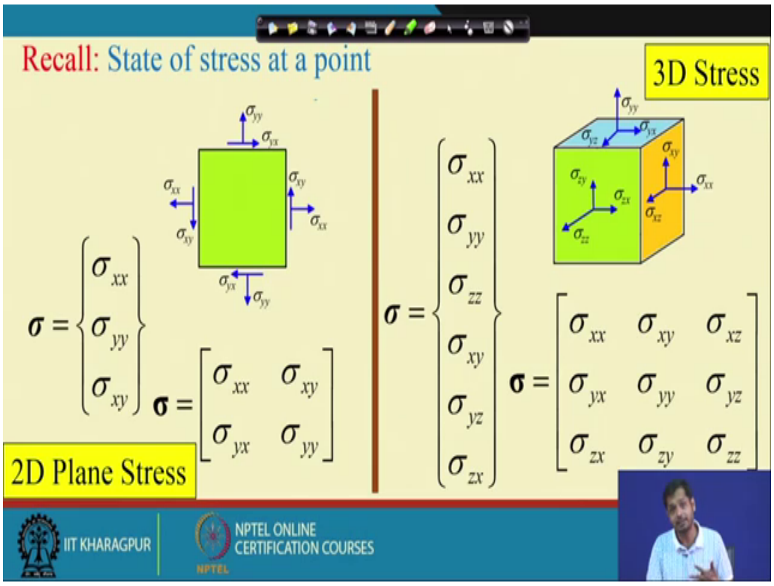

Now, this is the stress and now stress can be written as can be written in a in a tensorial form as this. So, this is a stress this is a stress written in a vector form and then these same stress can be written a tensorial form where it will essentially a 3 by 3 matrix.

Now, this is a symmetric tensor. Just now I said out of 9 6 shear stresses essentially we have 3 independent shear stresses and sigma xy and sigma yx and so essentially 6 independent components we have. Similarly for plane stress for 2-dimensional case we have these 3 independent 3 ah 3 stress components, ok. Now, this is one aspect of one aspect that is stress.

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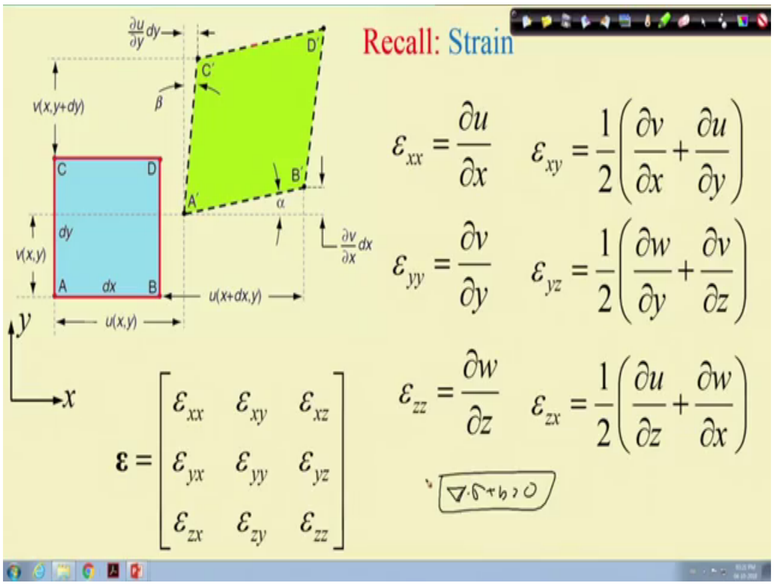

In addition to that we also have another important thing that is called strain. If you recall in we discuss in structural analysis that it has 3 important tenets, the 3 important pillar one is the equilibrium, another is compatibility and then another one is the relation between constitutive relation.

Now, the equilibrium when you talk about equilibrium, equilibrium essentially summation of forces if we look from structural analysis perspective. It is summation of forces; summation of moments in a given particular direction is 0. So, it gives it tells you how the different forces are related to each other.

Then the compatibility on the other hand it tells you how the different displacement components are related to each other. And then we have a relation between these force and the displacement which is called constitutive relation, ok. Now, similarly these the equilibrium conditions and compatibility condition can also be represented in terms of stress and strain respectively.

So, equilibrium equation you might have done in solid mechanics the del dot sigma is equal to 0, del dot sigma plus b is equal to 0 that is the static equilibrium equation, right. So, equilibrium tells you how the different components of stresses are related to each other and then compatibility of the other hand how the different strains are related to each other, and then we have a relation between the stress and strain which is call constitutive relation.

So, these are the different components of strain if you recall. So, similar to stress we can have the 6 strain components independent state strain components. And then d are related to displacement like this, when if the relation is linear, if we say that we are still in the regime of linear strain displacement relation, ok, now as far as this course is concerned.

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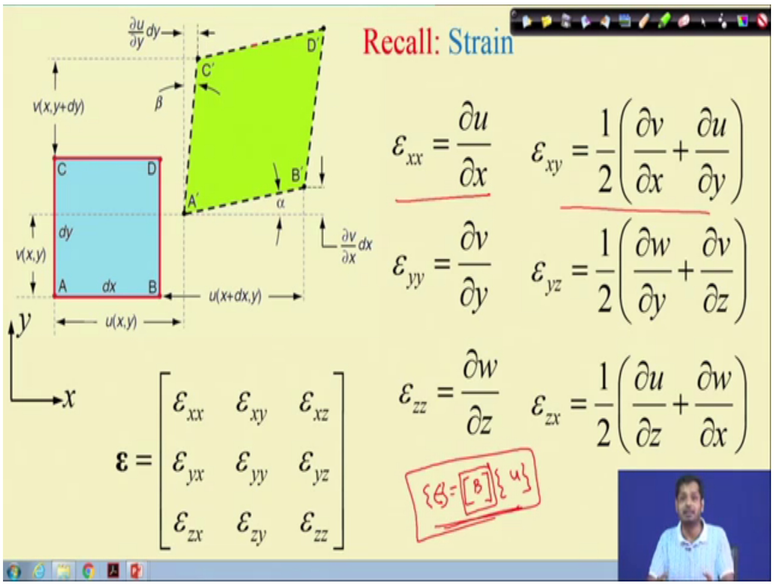

Now, you see, so these relations are called suppose for instance for instance if you take the first one it tells you how epsilon xx is related to u, u is the displacement in the x direction. Similarly it tells you how the shear stress on x y plane is related to other components of displacements. So, essentially this tells you how the strains are related to displacement this is called strain displacement relation right.

Now, strain displacement relation can be written into, can be written like this as well. For instance, if you have a strain these strain can be written as we will come to this point slowly this is and if this is u, ok. So, you look at this we can write the on that strain this suppose if we is written in a vector form.

So, all these strain components and then this is related to we take the displacement component put in a vector form and then we have a coefficient matrix something in terms of del x, del del y and so on. So, this is called strain displacement relation, right. So, this is how we can represent this relation, great.

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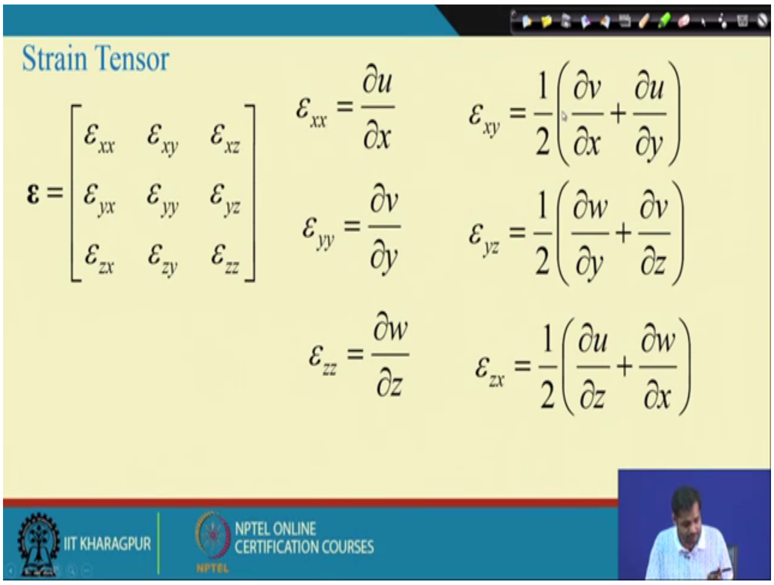

So, in addition to this strain displacement relation and these stresses yes these are the strain as a tensor we have just discussed now.

Now, next is stress strain relation. So, how the stress and strain is related? What we are doing is we are just these you these are the concepts that you have studied in solid mechanics we are just revisiting them because when we when we try to see what is finite element method these concepts will be useful along with the concept of matrix method of analysis, ok.

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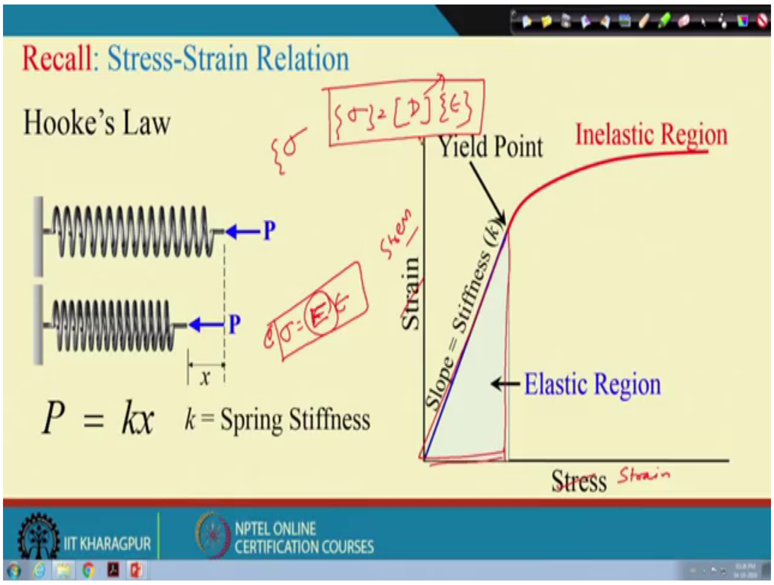

Stress strain relation is what if you recall Hooke's law, then if you have a spring which is subjected to say some axial load P. And then the displacement is suppose x then we know that relation between displacement and the force is P into k x, and where the k is called spring stiffness right.

Now, if we this is not the, for this is not stress strain diagram for this is a general stress strain diagram you see we have a stress in the x axis and then strain along the y axis. And then we beyond this up to this if you see the blue line your strain and stress is linear, ok, so beyond in this region your Hooke's law valid. So, Hooke's law says that this stress and strain is linear. And beyond that you can have non-linear zone you can have in elastic zone, but for this course for the entire course and for the discussion to follow in few hours tau hours time we are also we are only interested in this part, only this part, this is the linear region the elastic region we are interested in, ok.

So, in this case then how the if the stiffness is k, then how this stress and stress is related to each other if stress is strain is equal to if you recall the stress is equal to k into or e into displacement, u into epsilon this is Hooke's law, where e is the Young's modulus right.

Now, you see in this case we had just one component of stress and that stress is the normal stress. And therefore, one component of strength in normal direction and this if we plot the normal stress and almost strain it should be strain, it should be strain, please correct it stress, ok.

If you loop if you plot normal strain and then versus normal stress the slope what you get it that is essentially Young's modulus. But if you we are not now dealing with just one component of stress, we have several components of stress for instance in 3-dimensional space we have 6 text components, in 2-dimensional we have for plane stress problem where 3 stress components, right. So, this relation will not be like this, ok.

Now, how we can write this relation suppose that relation is sigma is equal to say sigma is equal to sum matrix d into epsilon, and for 1D case this D essentially becomes the Young's modulus. And for high dimension case D, D is a function of material property Young's modulus and Poisson's ratio for linear isotropic material, ok. So, this is a general relation or we can see the general generalized Hooke's law, right. So, this gives you strain how the stress is related to strain. So, now we have this D is called constitutive matrix right or constitutive relation this relation for constitutive relation.

Now, we just revisit what is stress what are the different components of stresses, we also seen what are the different components of strain and how this strain is related to displacement and how the stress strain displacement relation can be represented. Then here we also have seen that how the stress is related to strain through a matrix, through a constitutive relation that is these matrixes D, ok.

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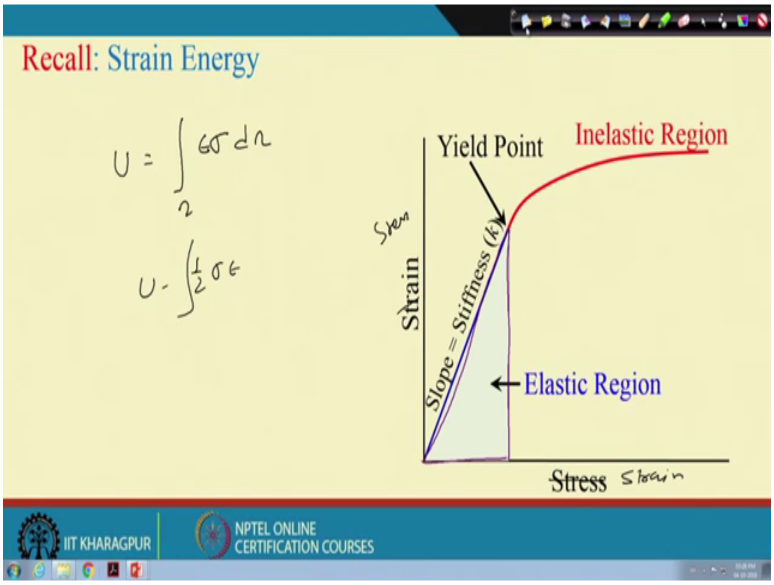

Now, next is strain energy. Now, what is strain energy? For instance, if I take if I take this figure only now we concentrate only on the elastic region. What is strain energy?

Strain energy if you if you know that if you recall the strain energy is essentially the area of this gives you the strain energy right, elastic strain energy.

Now, what is this? If I have to write it this if I write is as U these will be your half of a this will be integration of integration of strain, and then stress over this entire region over this then you have ah. So, essentially if you do that integration d omega, so you get half of epsilon half of half of strain into stress, ok. So, here you get half of sigma into epsilon integrated over the entire domain, ok. That you get this will give you these, the energy density and if you if you integrate it over the entire domain this gives these total strain energy, right.

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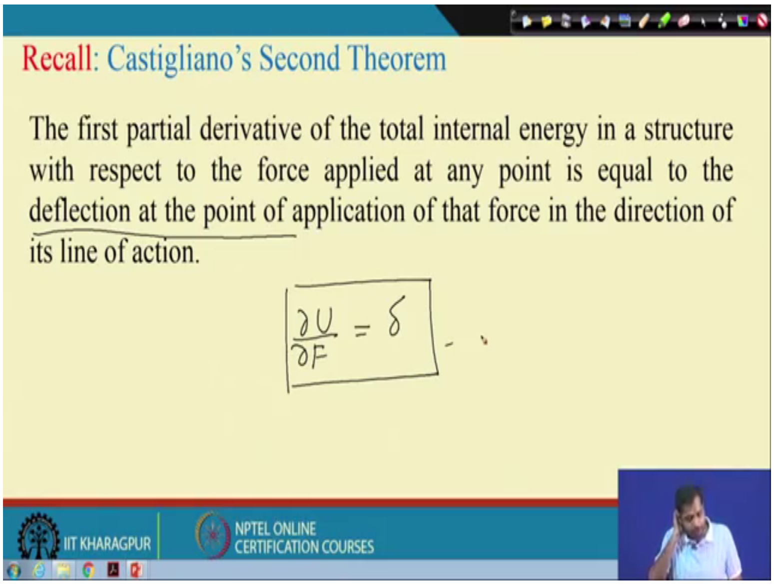

Now, why this strain energy is important? Now, if you look at, now this is an important theorem that you already you already studied in solid mechanics also in structural analysis if you recall the energy method of structural analysis you studied Castigliano's theorem, Castigliano's first and second both theorem.

Now, what this theorem says? This theorem says that the partial derivative of total internal energy if the internal energy is U, if the internal energy is U, U. Then its partial derivative with respect to the force, so its partial derivative with respect to the force then applied in any particular direction that gives you the application of that that gives you the deflection in that particular direction. So, this gives you deflection say delta, ok. So, this is an important theorem we will be using this theorem, shortly.

Similarly if you recall the first theorem this gives this tells you if you have the internal energy and if you differentiate partial derivative of internal energy with respect to displacement gives you force and that you have used in energy method of structural analysis, right. So, this is important.

Now, so what where do we stand now is we know the we know the, we know the concept of direct stiffness matrix and we also know the some of the concept of solid mechanics just now we have reviewed, that is the information we have right now right. So, what we do is we try to understand or try to understand how this information, whatever we have the direct stiffness method and the and the and the solid mechanics whatever information we have, whatever knowledge we have let us try to understand.

Let us try to do that exercise with this knowledge can we extend or can we can we can we have can we extend the theory the methods that we have learned in this course. Let us do that extension. Can we have a general for instance one of the very important concepts in direct stiffness matrix the name itself suggesting the stiffness. Once we divide the entire structure into small segments we calculate member stiffness matrices right, then we assemble them. So, stiffness plays, calculation of stiffness plays a very important role with the knowledge of solid mechanics this and with the knowledge of what we have matrix method can we have a more general definition of stiffness let us see that, ok.

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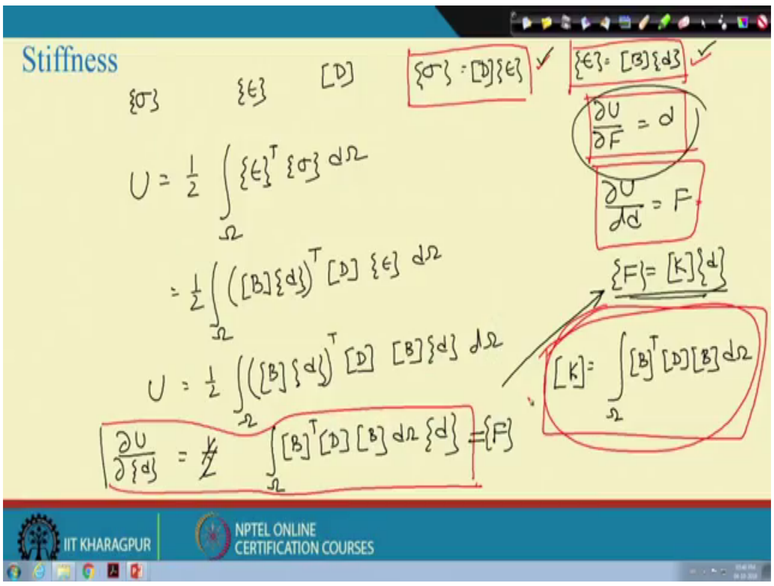

So, consider this case, we have just now we had is say internal energy suppose, now we have seen that the stress is can be written in a vector let us write them stress and strain in a vector form, ok.

Then we have epsilon and then we have a stiffness constitutive matrix D which gives you sigma is equal to D into epsilon right this information we have. And then we also have that strain is equal to is equal to B into displacement say displacement is D, or D will be using U for displacement but since U, your capital U you are using for the internal energy that is why I have to using here. So, this is the thing this is the relations we have here.

We also know that internal energy that is equal to half then integration over the entire domain or entire volume then this gives you epsilon transpose and then sigma. Then integration over the entire volume d omega this is the internal energy. And then we also know Castigliano’s theorem the del U del F, del F that gives you displacement D. So, these are the 3 information we have stress strain relation, strain displacement relation and also this theorem, ok. And this is the U of internal energy we have, ok.

Now, let us substitute from this what is epsilon here and let us substitute what is say from these two relations relation one and this is relation let us substitute them in this equation. And if we do that exercise what will be this? This will be half of integration over omega.

Now, epsilon will be B epsilon will d, this one B into B into d this, and then first let us write sigma is equal to sigma is equal to let us write epsilon first this gives you B and then d, ok. And this entire transpose, and then we have this is for epsilon then we have again sigma is equal to D into epsilon, we have D and then epsilon, D omega.

Now, then again from this half omega now B d transpose the entire transpose then D then epsilon can be again written epsilon can be expressed as this, ok. This gives you again B and then d, d this is this is d omega. So, this is internal energy right. Now, so, we have used this expression and we have used this expression.

Let us now use this expression what this tells you that ah that that the derivative ah, this is this was actually Castigliano's first theorem. And the second theorem was del U del d del d is equal to force that is the second theorem, ok.

Now, we will be using this theorem. So, partial derivative of displacement partial derivative of internal energy with respect to displacement is equal to d force, ok. Now, this is the internal energy now let us find out let us differentiate it partially with respect to with respect to d. So, if you do that del U and then we have del d is also a vector, ok.

Now, if you do that you what you get here is from this expression you get there is a square term here to because of this square term this half term will go there will be no half term here. So, what you get is integration of omega then from this you get B transpose and then you get D. And then you get another B here and then d omega another d. This expression you will get this is important, now this is important expression, ok, right.

So, if it is, so this theorem says that this gives you force. So, this has to be equal to force this equal to force, ok. Now, we already know that force is related to displacement as force is equal to force is equal to this will be force, force is equal to stiffness into displacement right into displacement that is the force displacement direction.

Now, let us compare this expression with these expression. So, this is force and this is force. So, here can we say that this expression also gives me the same expression if K is equal to, K is equal to integration B transpose D and then B this is integration over omega.

And what is K here? K is the stiffness matrix right. Now, what is the size of the stiffness matrix? The size of the stiffness matrix depends on what are your degrees of freedom, how many degrees of freedom you have, how many stress components you have. For instance, if you if you are in if you are in if you are if you are; if you just you have 1 degrees of freedom for instance in the case of spring, if it is 1 degrees of freedom then B is essentially the size B is a scalar because you have just one strain one displacement, so B is a scalar. And D will be what? D is essentially Young's modulus D is Young's modulus B is a scalar. So, this relation gives you just a scalar for 1D, which scalar relates the force in one direction and the displacement in that particular direction.

Similarly, if you have a 2 degrees of freedom system so where you have just 2 degrees of freedom and if it is 2 degrees of freedom then similarly your D will be you have your stiffness matrix will be 2 by 2. So, depend on the depend on the how many degrees of freedom you have depending on that you have the size of B, size of D and the size of B and D will give you what is the size of the stiffness matrix, ok.

You see we will see in the next class that these stiffness matrix, these definition of stiffness matrix through the definition of stiffness matrix we will have the through and one example obviously. We will arrive we can arrive at the same stiffness that we have for a given problem we will see same stiffness that we derive that we used in direct stiffness matrix, ok. But while calculating stiffness matrices in our matrix method of analysis we have not followed this approach, we have just directly used the relation between force and degrees of freedom and the force and use that relation directly in the stiffness matrices that was that method.

Now, here one point let me very clearly tell you there are different approaches through which you can there a different way you can view a finite element method ah. For instance, there is an engineer's approach, there is mathematician's approach, you can view finite element method as a as a as a numerical technique for solving partial differential equation or different ways. Similarly, different way you can arrived at this at this expression.

But remember one thing here our objective is not to study the theory of finite element method, right we are not going to do that. Here our objective is to with the knowledge of matrix method of structural analysis and knowledge of solid mechanics can we device a method which is more general as compared to the (Refer Time: 27:10) stiffness matrix and that is our objective. And towards that we followed a very simple way to have the expression for this matrix, a very simple interpretation of these or the definite of this of this stiffness matrix here.

So, next class what we do? Next class we use this stiffness matrix these definition of the stiffness matrix, take one example and then say and then see how stiffness matrix next few classes how for different structural components using this relation of stiffness, how for different structural components stiffness matrix can be obtained. And then we compare that stiffness matrix with the stiffness matrices we derived for direct stiffness method, ok. With that I stopped here today. See you in the next class.

Thank you.