Matrix Method of Structural Analysis
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Lecture - 33
Analysis of 3D Truss

Hello every one, this is the third lecture of this week. First two lectures what we have done is we have seen, how the theory the methods that we discussed in the last subsequent weeks, how that methods can be translated into computer code and a very basic code. And through that code, how we can large scale problems comparatively large scale problems, which is not otherwise possible using manual calculations can be solved ok. And we will be doing the similar exercise for three-dimensional truss today and the next class. We have not yet discussed, what is the theory how the how what are the stiffness matrices, how the three-dimensional truss is to be solved.

We have discussed for 2D plane, plane trusses. Now, today before we write 3D code for truss in the next class, let us today spend some time to understand what is the basic steps involved in analysis of truss in three-dimension. Steps are essentially same, only thing is because of the one additional dimension, your size of the stiffness matrices, formation of the stiffness matrices these will be different ok. Otherwise rest of the thing the essence of the steps will remain same. So, today we will discuss how to analyze truss in three-dimension using matrix method of structural analysis ok.

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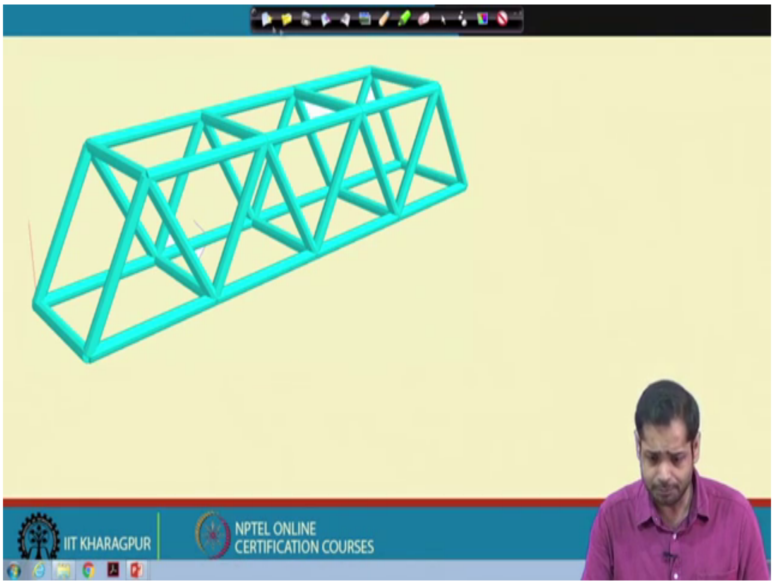

So, some example of 3D truss, these are some examples of 3D truss. In fact, these two examples I am showing, these two examples will be solved in the next class. Now, first is (Refer Time: 01:58) what we will do is we have done this exercise in detail for plane truss, some of the things we have we will take from the take for granted from the plane truss, because we have already discussed that. And we will then see, how that same methods can be extended to three-dimension, what are the modifications required for one additional dimension. So, sign convention as the same sign convention we will use let us now discuss this ok.

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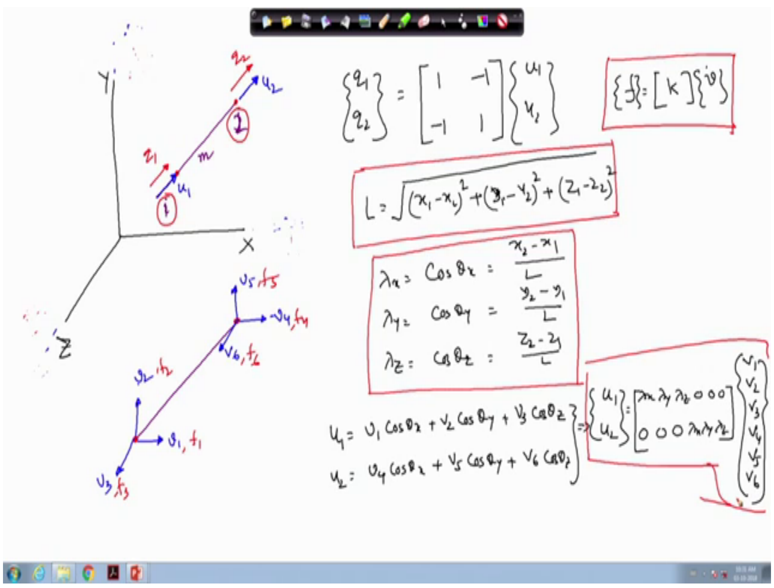

Suppose, let us fix our coordinate system, we have a coordinate system like this, we have a coordinate system this, this and this. Suppose this is X, this is this is this is Y and this is Z right. Now, we have a truss say this is a truss, this is the truss, this is a truss member, any truss member say m, truss member m. And this is ith point and this is jth point.

Now, if you recall we had two coordinate system; one is local coordinate system, which takes truss is are the two force members. So, always we know that forces in the truss is along the longitudinal direction of the truss right. Now, that longitudinal direction is the local coordinate of the truss, and the global coordinates is XYZ is the global coordinate system.

Now, let us see u is equal to this is the longitudinal direction, suppose this is so in this direction, your displacement is in this direction, and the displacement is in this direction displacement in this direction. Suppose, this is u 1 u 1 and this is u 2 ok; so u 1 u 2 are the displacement in truss which is oriented in three dimension, u 1 is the ith point or say it is the point number 1; so that we do not have any confusion, this is point number 1 and this is point number 2, node number 1 and node number 2.

Then similarly, we have force say q 1, q 1 will be force in these direction and then q 2 will be force in this direction. So, this is q 1 and this is q 2 right. So, q 1 u 1 at the force and displacement at node 1; and q 2 u 2 are the force in displacement in node 2. So, total displacement the change in length of truss of this truss member m will be u 1 minus u 2 and the member forces. Member force in this truss will be q 1 minus q 2.

Now, if you recall. So, in local coordinate system if we write this q and u, q and u, if you find out the relation between q and u, if you recall the relation was, if you have say q, q 1 the force displacement relation, this was is equal to a stiffness matrix; and then u 1 and u 2. And if you recall, the stiffness matrix will be 1 minus 1 minus 1 and 1. Irrespective of whether it is three dimension, two dimension the local coordinate system your stiffness force displacement relation will remain same. This is the force displacement relation in local coordinate system.

And then what we did is, now this then we had to find out the force displacement relation in the global coordinate system. When you talk about global coordinate system, let us define what are the forces and what are the corresponding displacement with respect to global coordinate system. Now, global coordinate system suppose this is v 1 in this direction say this is v 1 in this direction it is say this is v 1; v 1 is the force in x 1, and then v 2 is the force in x 2 a y 2 or y direction, and then v 3, v 3 are the in z direction. Similarly, the forces are these will be force say this force is f 1, f 1; this force is similarly f 3, and these force is f 2.

Now, what we have to find out? We have to find out a relation between f and v. So, our objective is to get relation between f is equal to this force is equal to some stiffness, the stiffness is equal to k and then v ok. So, this is our final objective to get, we have done this exercise we have obtained this relation for two-dimensional for plane for plane truss, the same exercise for space truss. Now, if you recall what from local coordinate system to global coordinate system, we had to use one transformation, one transformation is required right. Now, we have to define a transformation matrix and through that transformation matrix, we transform the displacement as well as the force and then find out they find out this relation.

Let us do this exercise, now suppose length of this member will be for lengths will be, if we have length is equal to this will be x 1 minus x 2 square plus x plus y 1 minus y 2 square y 2 square plus z 1 minus z 2 square ok; so this is square root. So, every length we can obtain like this, length of any arbitrary member ok. Whether you write x 1 minus x 2 or x 2 minus x 1 both are fine, because we have square here, now this is the length of a member.

Now, you see in order to get the transformation matrix, what we need is, we need how these truss member in three-dimensional space, how what is the orientation of this truss member; and the orientation is defined by in plane truss this orientation was defined by two angles; how what is the angle with respect to x and what is the angle with respect to y.

And since it is orthogonal x and coordinate system. So, it is defined by just one angle how this truss is orient what is the angle between the truss and with respect to x or with respect to y axis. Similarly, here also in order to get the transformation matrix we need to see what is the angle, this truss member is making we different axis right. Now, suppose theta x, theta y and theta z are the are those axis with respect to x, y, z coordinate.

And then, if you recall with this we defined in 2D truss, we define two parameters; lambda x and lambda y; at the lambda x, lambda y is essentially lambda x was cos theta x, and lambda y was cos theta y. Similarly, here also we define three we define three parameters which is lambda x, lambda y and lambda z. So, the lambda x will be lambda x, which is essentially cos theta x, and if you recall this is x 2 minus x 1 by L right.

Similarly, lambda y will be cos theta y this will be y 2 minus y 2 minus y 1 by L. And lambda z was cos theta z which is z 2 minus z 1 by L ok. And 1 and 2 depends on how the members, how you this how you are describe the connectivity ok, here this is one this is very important, this is this is your this is, this member this point is 1, this point is 1 and this point is 2 ok; this point is 2 and this point is 1.

So, how we define the connectivity depending on that is x 1, x 2, y 1 you have described. So, this is the same extension of lambda x and lambda y with respect to in 3D. Now, then what we did next is we did, then we have to find out a relation between now we have displacement u u 2 and, u 1 and u 2 with respect to local coordinate system.

And then we have displacement v 1, v 2, v 3 with respect to with respect to ok, one thing we missed here is with respect to global coordinate system. We have actually 6 6 coordinate 6 forces, and that we have ok. Suppose, at this point at this point also here three displacement. Let us these are not v 1, v 2, v 3. Let us define in a different way, define v 3 defined let us defined different way.

And similarly, force also let us defined slightly different way, because then it will be easier for us ok. So, what is required is suppose at this point, if I draw the truss once again if we draw the truss, this is the truss and this point is 1, and this point is 2. And then in this point, we have t 3 displacement one is along X X direction, one is along Y direction, and one is along Z directions right. So, suppose this is v 1, this is v 2, and this is v 3. So, v 1, v 2, v 3 are the three displacement at node 1.

Similarly, here also we have displacement along Z X direction, and displacement along Y direction, and displacement along Z direction, and suppose this is v 4, v 5, and v 6. Similarly, we have forces will be if in this direction it is f 1, then it is f 2, f 3. And similarly it is f 4, and then f 5 and f 6 ok. So, what we this relation is essentially this v 1, v 2, v 3 and this f 1, f 2 these v's and this f. So, in a given member we have total 6 degrees of freedom. So, three displacement at node 1, and 3 displacement at node 2 ok.

Now, so what we have to find out u 1, u 2 are the local displacement, this is a displacement with respect to local coordinate system, u and v is the displacement with respect to global coordinate system, you to find a relation between this u and v. And, if you recall, the relation we obtained just by taking the projection of v 1, v 2, v 3 along the along the longitudinal axis of the truss.

So, if we project it v 1, v 2, v 3 along the longitudinal axis, we get see this u 1. And if you project v 4, v 5 and v 6 along the longitudinal axis, we get u 2. And this projection will be u 1 will be then v 1 cos theta x plus v 2 cos theta y plus v 3 cos theta z right. Similarly, u 2 will be u v 4 cos theta x plus v 5 cos theta y plus v six cos theta z. So, this is the relation.

So, once we then now this relation can be written as if we just this relation can be written as if you have u 1 and u 2, this will be then equal to we have we have this is equal to we have say a matrix, and then this will be v 1, v 2, v 3, v 4, v 5, and v 6 ok. And this matrix will be, if you x cos theta x is lambda x cos theta y lambda and cos theta z lambda z, this will be lambda x lambda y and lambda z 0 0 0, and then 0 0 0 lambda x lambda y and lambda z.

Similar, expression we had in case of 2D in case of 2D frame ok. So, this is the relation. So, this relation is the relation between displacement in local coordinate system and the displacement in global coordinate system. Now, if we have to do same exercise for forces as well, the forces in local coordinate system is q 1 and q 2, and forces in global coordinate system is f 1, f 2, f 3, f 4, f 5 and f 6. We have to find out the relation between this q and f. What we have to do is we have to take we have to take the component of q 1 in X, Y and Z direction, and we get correspondingly f 1, f 2, f 3. If we take component of q 2 in X, Y, Z direction, we will get respectively f 4, f 5 and f 6.

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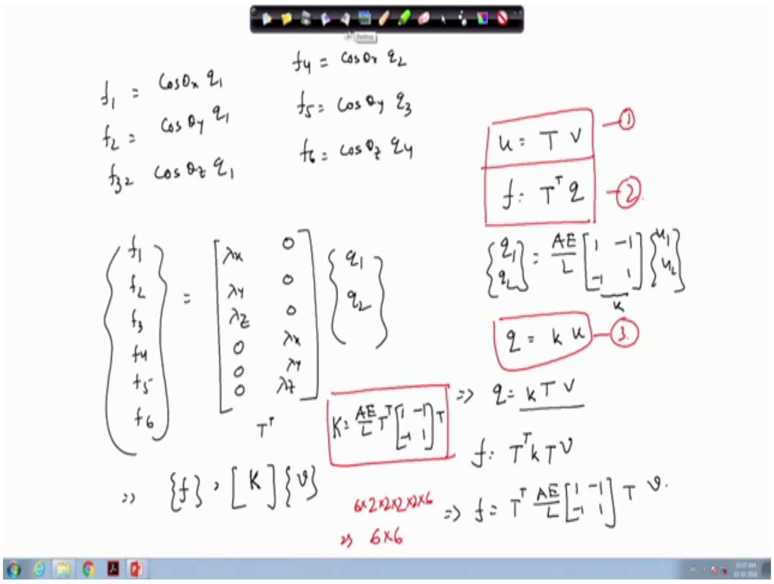

And if we do that exercise, then this will be so let us take one more so this will be this will be say f 1 will be f 1 is f 1 if we take the component if we take this component q 1 in x direction, we get f 1. So, f 1 will be cos theta x into q 1, and then f 2 will be f 2 will be cos theta y into q 1, and f 3 will be cos theta z into q into q 1 right. Now, similarly f 4 will be cos theta x into q 2, f 5 will be cos theta y into q 3, and f 6 will be cos theta z into q 4. Now, if we combine this, then we have f 1, f 2, f 3, then f 4, f 5, f 6 all the forces in global coordinate system is equal to matrix like this, and then we have q 1, q 1 and q 2. And this is if we write lambda x this is lambda y lambda z, and this is 0 0 0, and this will be 0 0 0, lambda x lambda y and lambda z ok.

Now, if I if I say that this is equal to the entire thing this entire thing is equal to this entire thing, if I say this transformation as T, if we say that then what we have is then, we have this expression. We have u is equal to u is equal to T into v, t is the transformation. And then from these becomes in T transpose, this is equal to f is equal to T transpose into q. These two relation we have a similar relation, we also we derived for in the case of 2D, 2D truss.

The exercise is essentially the (Refer Time: 17:49) the steps is essentially are same. So, this is the expression between now once we have this, we are almost done. Now, we already have a relation between q 1 and u 1. Let us write that relation, we had a relation is q 1 q 2 is equal to A E by L A E by L. This is the with respect to local coordinate system minus 1 minus 1, and then u 1 and u 2 right. Now, say if this is small k if this is small k, then this is essentially q is equal to small k into u ok.

Now, this is expression this expression this is 1, this is 2, and this is 3 this is 3. Now, with this expression what we do is we first substitute expression 3 in expression 2 and then expression 1 in expression 2. And then what we have is q is equal to so q is in this expression. If we substitute say u is equal to u is equal to T into v, then what we have is that q is equal to k into T into v ok. So, I am not writing please note that k v they are the v at the vector, and K is a matrix. So, when you write this you please refer to their corresponding definition, how they are defined. So, K into v.

Now, again we have a relation between we have a relation between this between between ok. So, now this is done now then if we substitute the entire thing this entire expression, these entire expression into expression 2, then what we have is we have then f is equal to T transpose T transpose and this expression in q small k, and then T into v ok. Now, this is equal to so if I this is equal to f is equal to T transpose T transpose small k is equal to A E by A E by L A E by L into 1 minus 1 minus 1 1, and then again T into v.

So, these we can write finally these we can write f is equal to some matrix K it is capital K into in to v right into v, where what is capital K capital K is equal to capital K is equal to T A E by L we can take out A E by L, and then T transpose T transpose, then 1 minus 1 minus 1 1, and then T. This is the stiffness matrix this is the stiffness matrix ok. Now, let us see what is the dimension of the stiffness matrix, A E by L is a constant, T transpose will be T transpose dimension of T transpose will be 6 cross 2. Then this is 2 cross 2 2 cross 2, and this dimension of T will be 2 cross 6 2 cross 6. So, essentially this becomes 6 cross 6. So, this stiffness matrix will be 6 cross 6 stiffness matrix ok.

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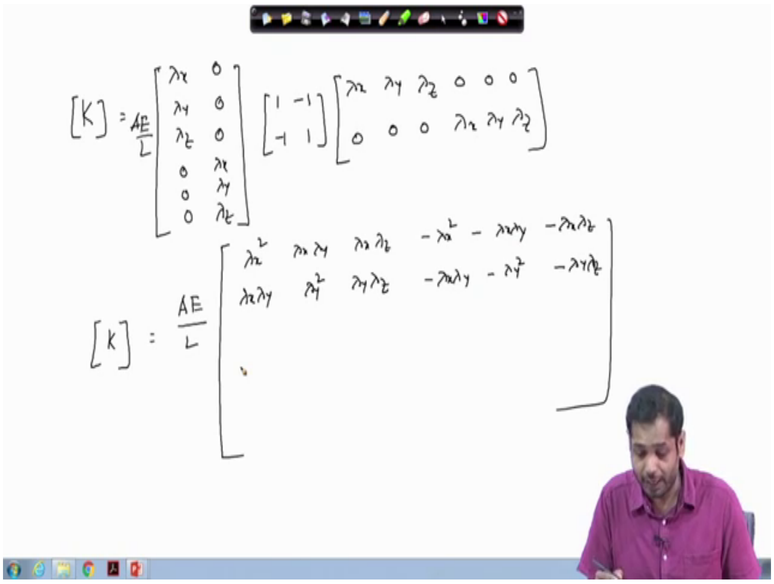

And then let us write the expression for that matrix, the expression for that matrix will be finally, if we have say K, K is the stiffness matrix. So, K is equal to we have we have T transpose, which is lambda x lambda y lambda z 0 0 0, and then this is 0 0 0 lambda x lambda y lambda z, and then we have 1 minus 1 minus 1 1, and then here we have lambda x lambda y lambda z then 0 0 0 0 0 0 lambda x lambda y lambda z this is the matrix. We have A E by L outside A E by L outside. So, this is the entire expression.

Now, if we just do this manipulation, final expression very similar to very similar to the plane truss A E by L, these matrix this will be finally, if you recall, lambda x square and then lambda x lambda y and then lambda x lambda z ok. And similarly, negative of this minus lambda x square minus lambda x lambda y lambda y and then minus lambda x lambda z. So, and then similarly, this will be lambda x, since it is similar it is symmetric, lambda x lambda y. This is lambda y square and this is lambda y lambda z. And then this is negative of this minus lambda x lambda y then minus lambda y square minus lambda y lambda z and so on. So, this is how you can do this and get the inter stiffness matrix.

And if you compare this with the stiffness matrix in 2D truss, what you do is, you remove the corresponding column. Say for instance z, this is z-axis the elements associated with z-axis corresponding column and corresponding row. If you remove them, and then and then see the stiffness matrix that you get that is very similar to the plane truss or not. So, so from 3D, you can you can get the 2D version of this plane truss.

Now, once you have this, then rest of the things (Refer Time: 24:03) this is the stiffness matrix. Once we have the stiffness matrix, again you also check whether the stiffness matrix is symmetric or not. You also check these stiffness, what is the inverse of the symmetry stiffness matrix, the way that the determinant of the stiffness matrix whether it is 0 or not. You should be getting the determinant 0 because of the obvious fact that no boundary condition information have so far been enforced in this in these stiffness matrix ok.

Now, next is your rest of the thing is once we have stiffness matrix, rest is we have to assemble this stiffness matrix. When at the time of assembling at the time of assembling, all these steps that we followed in the case of 2D plane truss same thing we have to do the process will be entirely is entire the same.

And the thing is once you get the assemble stiffness matrix, then the process of partitioning the stiffness matrix into depending on the known displacement field, and the known reaction known force field, partition the stiffness matrix, and get the and an then you and get the solution for unknown displacement. And once we have the unknown displacement, rest of the things like your calculation of support reactions, and then member forces ok, so there will be same. So, up to the calculation of support reactions (Refer Time: 25:31) same. Now, let us see how to calculate the member forces.

Now, member forces if you recall the member forces, the force will be since all the force will be essentially q 1 minus q 2 will be the force. Now, the relation between q 1 and q 2 we have. If you have if you see this, this is q 1 and q 2 right. Now, how q 1 and q 2 is related to each other, q 1 and q 2 is related to each other, q 1 and q 2 is related to each other with this. And this is how this q 1 and q 2 is related to each other with velocity with for with displacement these expression. Let us use this expression to get the member forces.

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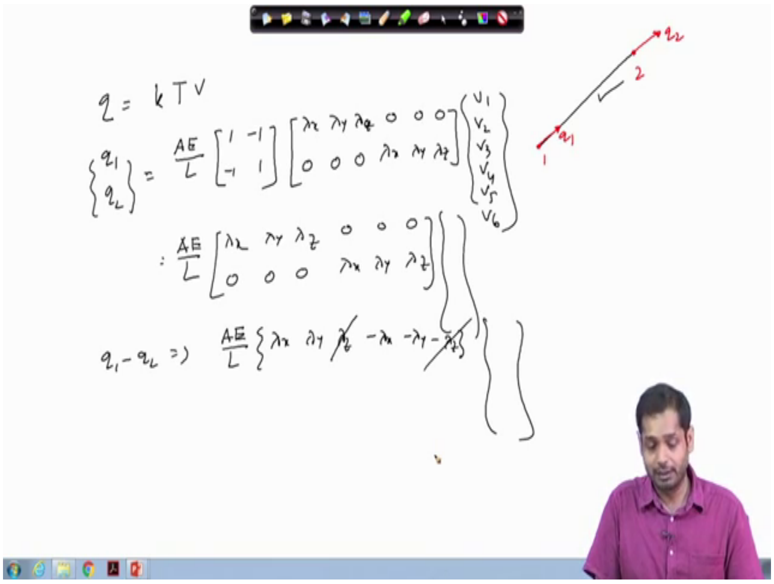

So, what we have is q is equal to q is equal to k, k into T into v right. Now, if we have this is the member in the cross at this point you your force is q, these points it is 1, this point is 2, here you have q 1, and here you have q 2, q 2. So, the forces in this member will be q 2 minus q 1. So, say let us let us calculate, what is q 1 and q 2, only thing is only thing is you have solved for the unknown displacement global displacement right.

So, q 1 and q 2 will be will be your k, which is which is A E by L A E by A E by L into 1 minus 1 minus 1 1, and then finally your T, T is we have this is equal to T, which is lambda x lambda y lambda z 0 0 0 and 0 0 0 lambda x lambda y lambda z and then corresponding your v 1 v 2 v 3 v 4 v 5 v 6. Now, v 1 v 2 v 3 already we know, we have already determined that value.

Now, so what will be then this, from this expression what we have to find out is we have to find out what is the you have to express this thing you have to from this, we have to express what is the value of q 1 in terms of lambda x lambda y lambda 1. And from this expression what is the expression for q 2 in terms of lambda x lambda y and lambda 1 ok.

So, for instance, so this will this one will be A E by L. If we if we take A E by L, and if we multiply this, then this expression will be this expression will be your so, this expression will be your this will be lambda x lambda x, and then this will be lambda y. And then let us let us let us this will be lambda x lambda y lambda z and then z 0 0 0, and then you will get 0 0 0 lambda, lambda x lambda y and lambda z ok, and then you your this vector.

So, your force will be q 1 minus q 2 if you take the force, and this will be this will give you essentially A E by L A E by L and then lambda x lambda y lambda z and then minus lambda x minus lambda y minus lambda z into the force into the velocity. So, this will give you the member force in a particular member force in this member ok.

So, you have when you implement it, please be consistent with the sign convention. So, here maybe sign conventions are mixed up, so you have to be very careful about the sign convention that you use. This is all you can calculate the member forces. And you remember when in the case of 2D truss, we if you remove lambda z, if we remove this term, and this was the this was the thing that we had in case of 2D truss. So, this is how we can calculate the member forces, the rest of the thing is exactly same.

Now, you see the idea of this lecture has been to has been to see how the concept that we learned in 2D can be extended to 3D. And this is a very brief discussion of that extension. You can have the detailed derivations everything you can do on your own, but rests at the end of the day you will be getting say this gives this kind of expression.

Now, next is we have to translate this entire thing, the way we did in for did for 2D truss, the same thing we have to do it for 3D truss, and then see and then see how some complex structure relatively larger structure can be used can be solved easily with this method. So, computer implementation of this of this of 3D truss will be discussed in the next class. So, I stop here today; see you in the next class.

Thank you.