Matrix Method of Structural Analysis

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Lecture – 28
Matrix Method of Analysis: Frame (2d) (Contd.)

Welcome so we are in module 6 of Metric Structural Analysis where we have in the last lectures last few lectures we have defined or we have derived the stiffness matrix using the principle of superposition for 2d frames. And this with the 2d frames we also derived the transformation matrix, where essentially you know we have 6 cross 6 transformation matrix we will locate which relate the prime coordinate system with the non prime coordinate system. So this we have done in the last 2 lectures.

And then also we have a highlighted the procedure in a general sense and also I discussed about what are the meaning or what is the purpose of these lectures that is we are really not bothered about 2-3 degrees of freedom or the 2-3 element solution structures with 2-3 element solutions, 2-3 elements they have. So basically these methods are for large degrees of freedom so where essentially you need to code the problem and you need to solve a large degrees of freedom problem. So the basic objective here is to tell you the procedure by which you can really understand what are the minute or every details of the method where it can go wrong or where it can have problems.

So, when you use any software where whether you code it or you use any software you get some results. So as an engineer your objective is to interpret those result whether those results are correct or not so to do that unless you know the I do not know the methods properly or specially where the method can go wrong, then you really cannot find out what the errors in the problem. So to understand this thing more lucidly, we will solve some problems so or some examples for instance we will in the first example we will take a very simple structure and try to highlight what are the procedure for 2D frame analysis.

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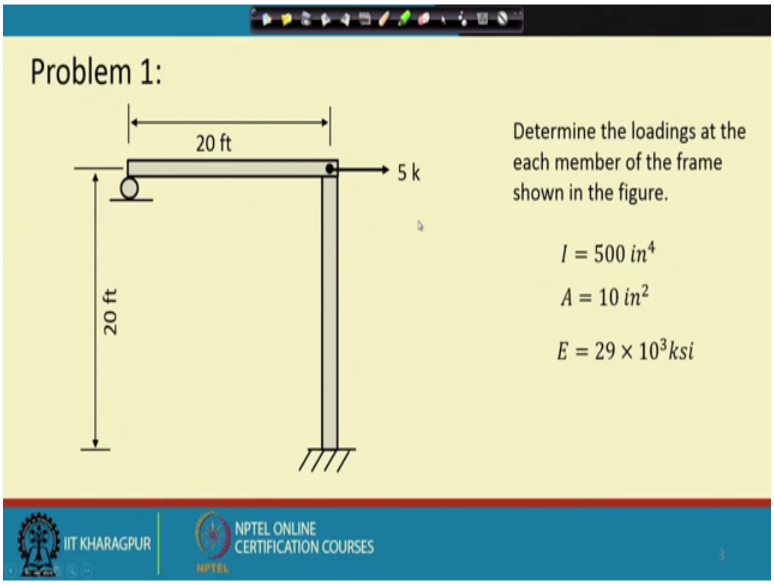

So, let us consider a simple frame which is a one shape frame, where one end of the frame is fixed so here the we know the fixed boundary condition and another part of the frame is a roller support and there is a 5 kilo pound load is there at the this joint. So, now, the length are very simple which is 20 feet for both the member, and for the horizontal and this is a vertical member so for all members the moment of inertia the high is 500 inches; inches to the power 4 and the area is 10 inches square for all members and the Young's modulus E is 29 into 10 cube kilo pound per square inches. So, now this data with this data it is required to find out the loadings and the each member of the frame shown in the figure so; that means, the what are the loads acting in this member so for each node and each joint specially.

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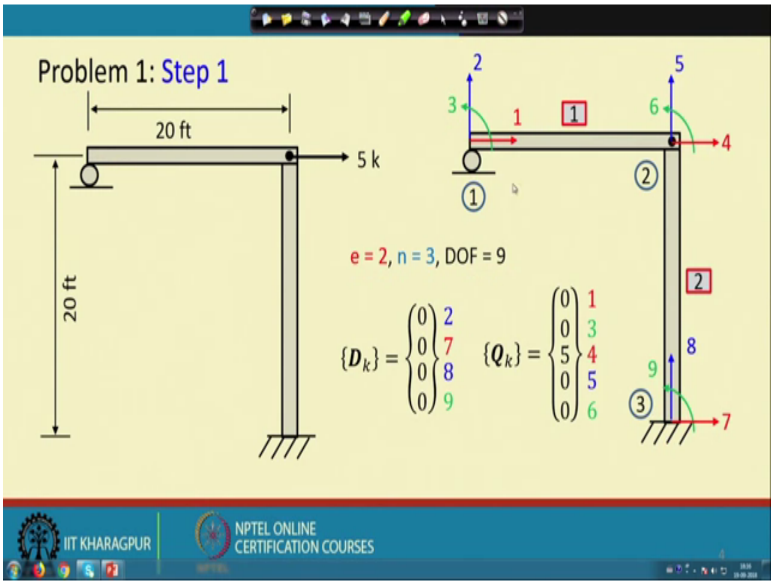

So, now to solve this problem through matrix method of stiffness method, the first step is to discretize the structure or to separate it out element wise. So to do that, let us consider the 3 nodes in the structure the first node I consider here the roller point, and the load point here is a second node and my third node I consider it here. So, now, naturally my first degrees of freedom is the axial direction in this direction, second is transverse third is the rotation, then I write 4 5 6 and then 7 8 and 9.

Now, I have naturally two element. This is my first element and this is my second element. So essentially in this structure I consider 9 degrees of freedom so 3 3 3 each node. So naturally we have 3 nodes and 2 elements. So I can increase the degrees of freedom and I can increase the number of node for instance; it is better to mention that, I can even point out a node here and find out a element here. This is my element and this is my element that is also possibility, but we do not do it here so that will be actually discussing in module 8.

So, now with this set up, so I have 2 element I have 2 essentially 2 element 3 node and 9 degrees of freedom what are the known what are my known degrees of freedom for kinetic as well as force degrees of freedom, so let us see what are my known degrees of freedom.

So, the first naturally there for this case my boundary conditions are this. So here the vertical displacement has to be 0. So this is so second degrees of freedom my vertical degrees of freedom is my 0 and then naturally there is no boundary condition here. So there is a this is a fixed boundary condition. So this fixed boundary condition implied that the displacement along this displacement along this and displacement rotation of this plane is 0 so that is why my 7 8 and 9 degrees of freedom all of them are 0. So my all known degrees of freedom I know now which is 2 7 8 and 9. So, now, what are my force degrees of freedom that we need to we know.

So, first of all this force degrees of freedom naturally we know that those degrees of freedom we know the displacements we do not know the forces. So naturally my I know 1 3 4 5 and 6. So for instance, my first node there is first degrees of freedom there is no actual external axial force here. So naturally this will be one means my force degrees of freedom first force of the force degrees of freedom is 0.

And then naturally there is no moment acting so my third degrees of freedom will be also 0 so here, and then fourth naturally, there is a unit there is a 5 kilo pound load acting in x direction from this node 2. So this will be 4 and this acts in a positive direction, so if I take this is my axis of this, so this is my axis, so this is my x axis and this is my y axis suppose then this is in the positive direction, so this is 5 and fifth degrees of freedom there is no vertical force and 6 degrees of freedom there is no moment.

So, naturally along 2 work there will be a reaction force here, so there will be a reaction force here, there will be a reaction force here and there will be moment here so these things will be my unknown. And similarly the displacements along these there it can displace so these degrees of freedom is my unknown, these degrees of freedom so it can rotate these degrees of freedom is my unknown and then this naturally this will be unknown this will be unknown, so this will be unknown and these are my known degrees of freedom. So what are the dimension now? So my kinetic or the displacement degrees of freedom is my 5, displacement degrees of freedom which is unknown and 4 force degrees of freedom which is known. So if I so one of them is this reaction vertical reaction and the 3 of them are this vertical reaction horizontal reaction and the moment at the fixed end; so, with this, now if I now can proceed for the calculation of stiffness matrix.

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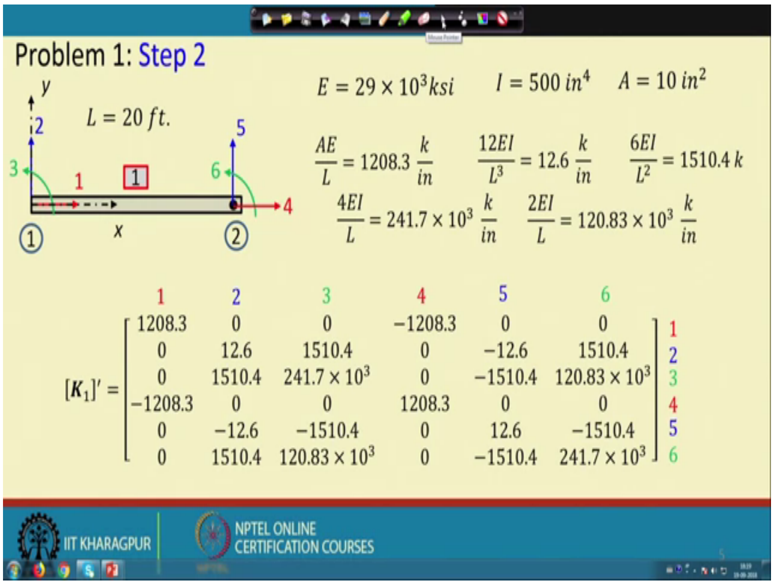

So, now, let us consider the first element, so we know what are the material properties for instance we know Young's modulus, we know I we know A. So this is given to you and this is constant over this element or this structure. Now the length also we know so we have 1 2 3 4 5 6; so we have 2 nodes one node and 2 node. So, now, I need to calculate so there is no force is acting in the member. So there is only one force acting at the, this thing so I really do not at the node 2. So I really do not need to have a fixed end moments here.

So, here now I based on that I calculate some of the quantities if you remember this quantity is a stiffness quantities for instance AE by L is axial stiffness, this is the bending stiffness and then all those things we know we calculate. So these are the terms in the stiffness matrix if you remember. So these terms I calculated earlier I calculate through this data. So this is my first step and this is as per unit also. So this is kilo pound and kilo pound per inch all those things we found out.

Now, so now, my first job is to calculate the local stiffness matrix. So the local stiffness matrix K 1 is naturally corresponds to the first degrees of freedom to 6 degrees of freedom. So these degrees of freedom is my local access. So corresponding 1 2 3 4 5 6 so here you see that this AE by L comes here and then minus AE by L and this is third degrees fourth row is fourth degrees of freedom is the axial force for the node 2. So these stiffness matrix we know, we can just find it out if we substitute we can find out so you see this stiffness matrix is symmetric because the, we know the stiffness matrix will be symmetry, so this is my 6 cross 6 stiffness matrix. So with this 6 cross 6 stiffness matrix my main links job is to transform this for the global coordinate system. So to transform into global coordinate system we need to know lambda x and lambda y.

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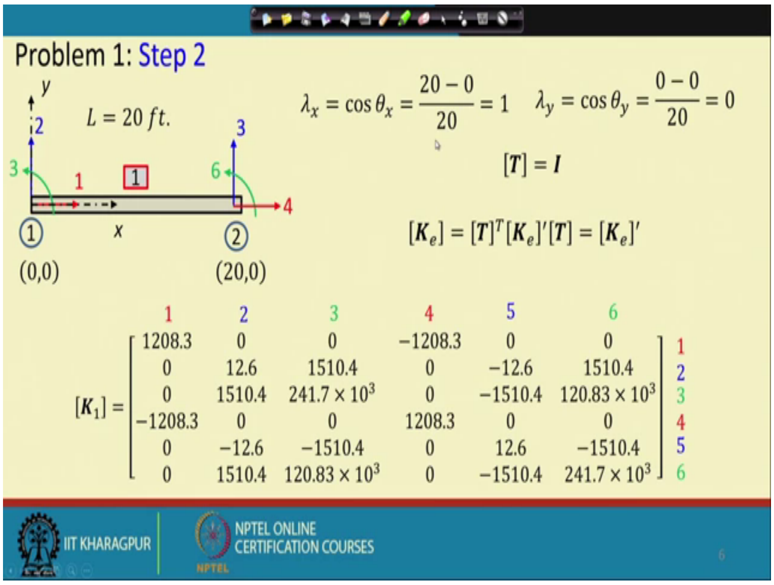

So, I can calculate if I know the coordinates of lambda x and lambda y for instance, this coordinate if I take my origin here and if I take x axis along this and y axis along this, so this is my structural axis essentially. So, now, if I know the coordinate of this point naturally first node is20 0 0 and second node is 20 0. So naturally the lambda x that is cos of theta x is 20 minus 0 by the length 20 so similarly cos y is 0 minus 0 by 20; so this is 0. So lambda x and lambda y we know so if lambda x is 1 and lambda y is 0, then t matrix is an identity matrix. Because if you remember the lambda x this is lambda x lambda y 0 and minus lambda y lambda x 0 and then 001 and then again repetition of this, so this becomes again another this matrix and its terms are 010. So you see if lambda y is 0 essentially this becomes an identity matrix, so this t becomes an identity matrix.

So, now if I have found out the prime coordinate system on the local coordinate system what is the stiffness matrix K e, and then if I transpose the transformation matrix be multiplied with the transpose of the transformation matrix and post multiply with the transformation matrix, I get the stiffness matrix in the global coordinate system. So that is xy coordinate system not in the prime coordinate system. You see since the prime coordinate system and the or the elemental or the local coordinate system matches with the global coordinate system exactly here, so that is why the transformation matrix is the identity matrix of identity transformation. So identity transformation essentially is multiplying with the identity matrix.

So, now this becomes so if this is identity so K e prime is actually K e so I can directly write now what is my K e. So, now, this becomes global degrees of freedom so here even though local and global degrees of freedom matches here so this is my global degrees of freedom; that means, in the non prime coordinate system and this is also my global degrees of freedom so in the non prime coordinate system.

So, now see the matrix since K prime and K e is same so K 1 remains so this e equals to 1 here so one here so these remain same. So, now, if you so this is done for the first element.

Now, let us go for the second element.

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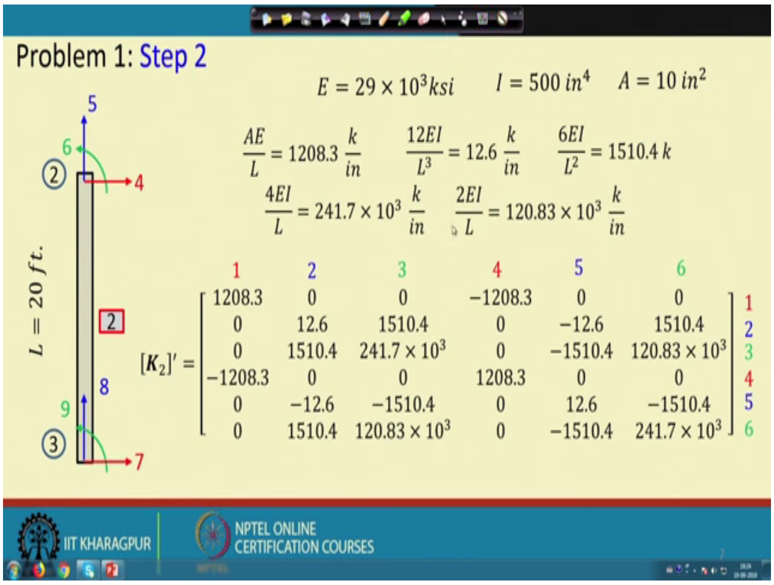

So, in the second element again in the same procedure, we will find it out so we know E, we know I we know A so these quantities we have already found out earlier. So, now, this since E I A does not change so local stiffness matrix does not change. Now you see this, my prime coordinate system is here, so my global coordinate system probably is here in the beam that is xy. So, now, here the prime coordinate system if I look. So, now, if I want to derive the prime right the prime coordinate system, for instance this is my prime coordinate system and this is 1 2 3 and similarly here. This is your 4 5 and 6 so this is that is why I have written 1 2 3 4 5 6, 1 2 3 4 5 6; so because this is this stiffness matrix in the prime coordinate system, but remember our local coordinate local degrees of freedom are not actually the global degrees of freedom here so which was there in the previous case so we need to transform it properly.

So, this transform is again dependent on the rotation matrix the formation matrix, which is essentially the t matrix here. So next our job is to find out the T matrix.

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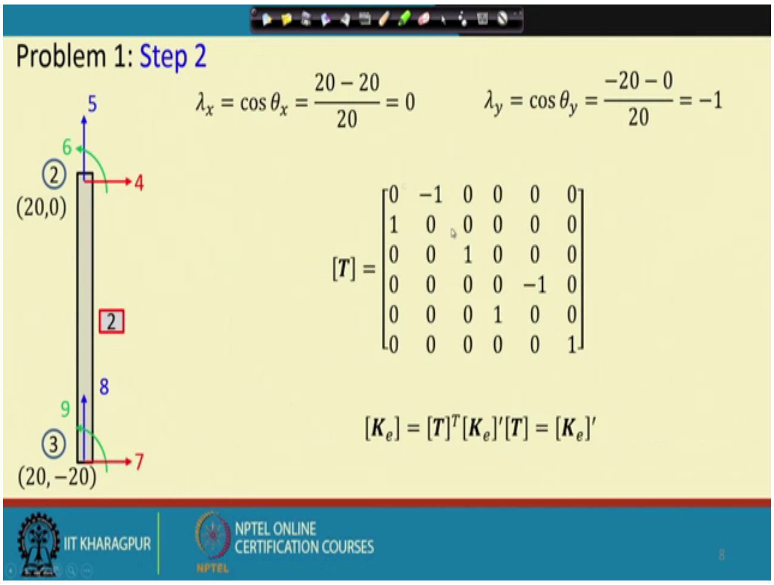

So, this t matrix here the lambda x and lambda y we can like in the previous case we can find out lambda x lambda x turns out to be 0, and lambda y turns out to be minus 1. So here with the coordinates if we know the coordinates, we can find out lambda x and lambda y.

So, now here this transformation matrix is essentially a diagonal all diagonal terms except the 3 and 6 are 0 because lambda x becomes 0, and this off diagonal terms these terms is will be minus 1 plus 1 then this is plus1 and this is 1 1. So, now, if I multiply its pre multiply with T transpose and post multiply T with the prime coordinate system, which was 1 2 3 4 5 6. Now I will get 4 5 6 and 7 8 9 for my global degrees of freedom.

So that we can multiply now very easily and then we can compute it for the, to find out the global stiffness matrix or global coordinate system.

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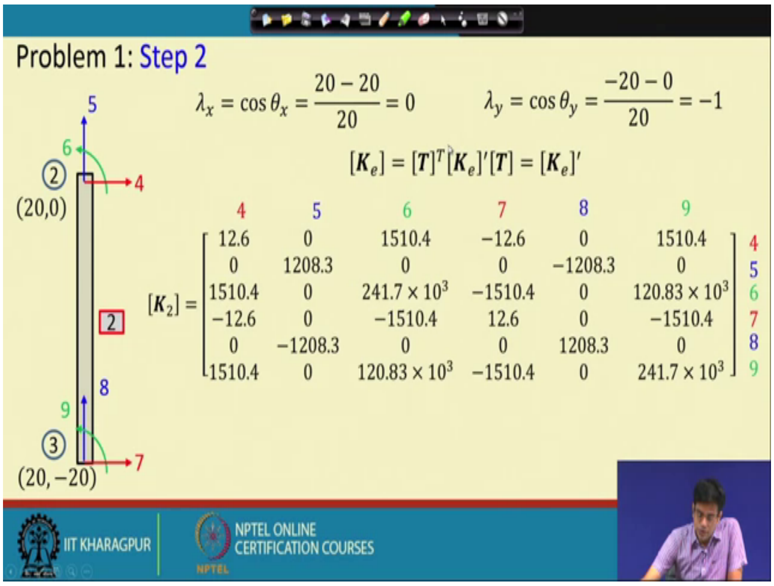

So, now if I do that, then naturally my degrees of freedom will be the global degrees of freedom so which is 4 5 6 7 8 9 4 5 6 7 8 9. So, now, I am equipped with my all matrices in the stiffness matrices in the global coordinate system. So the first step for this procedure is local forming of local stiffness matrix; once you know the local stiffness matrix you compute the transformation matrix; so once you know the transformation matrix you compute the global stiffness matrix or the global stiffness matrix global coordinate system, what is the elemental stiffness matrix.

Now, the next step is actually compute the stiffness matrix for the whole structure. So for all element once you have done, then the next step is to find out for the global case.

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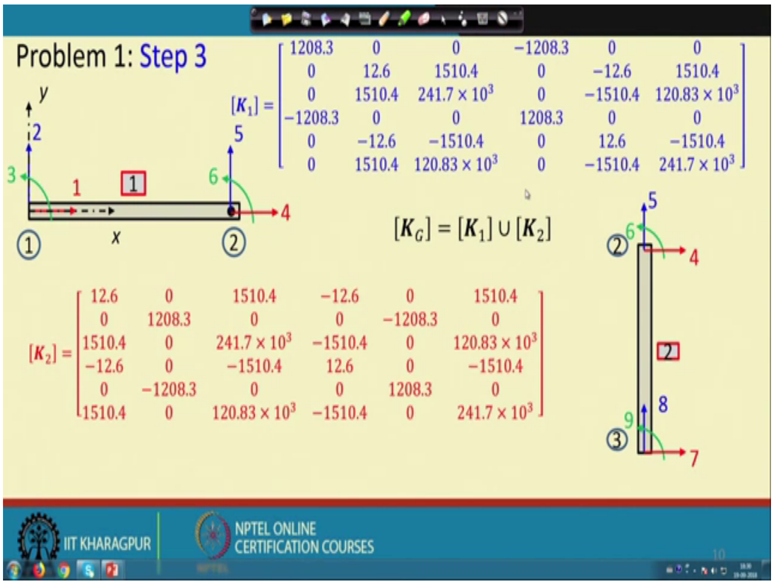

So for instance, here we know the stiffness matrices for instance this is my element 1 and this is my element 2. So this is my element 1 stiffness matrix and element 2 stiffness matrix, you see the K e prime for instance K 1 prime that is the for the global stiffness matrix for element 1 in the prime coordinate system, and K 2 prime for the stiffness matrix for the element 2 it was same, because there is the E A and I were seen for these 2 members E A and I and then length is also same so length is also same.

So, K 1 prime and K 2 prime are same, but it is noticeable now that K 2 and K 1 are not same. You see even though the entries or entries of the element the magnitudes of the elements the same, but it is position is different the position is different because of the rotation. So it was horizontal member, now it is now vertical members. So even though these 2 quantities were same because of the properties and the length are same. This globally in the global coordinate system the elemental stiffness matrix is not same so you see this the first element is this here while the first element is this.

So, now again we also know this is corresponding to first degrees of freedom second degrees of freedom third degrees of freedom 4 degrees of freedom 5 and 6. Similarly this is for 6 7 8 sorry this is 5 5 6 7 8 4; 4 5 5 6 7 8 and 9 so this is corresponding to 4 5 6 7 8 9. So, now, the interesting thing is here that when you take the union when so with these 2 stiffness matrix you have to have a global stiffness matrix, and you see all the elemental stiffness matrix in the global coordinate system 6 cross 6 this is also 6 cross 6.

But in the global coordinate system it is in the global coordinate system, but when you have when you are going to find out the stiffness matrix in the whole structure that will be union of K 1 and K 2. What this union means this union means that you have to add these stiffness matrix to elemental stiffness matrix with appropriate degrees of freedom, the addition will be a common degrees of freedom.

For instance see 4 5 6 is here common with this node number 2; that is the node number 2 is common with the, for both the members. So member one will have is having node number 2 member 2 is also having node number 2 so 4 5 6 degrees of freedom is common. So this part of the matrix so this part of the matrix x and this part of the matrix is this part of the matrix is essentially common it needs to be added not this part this part this part; so this will give me a proper global stiffness matrix.

So, let me see how we can do it.

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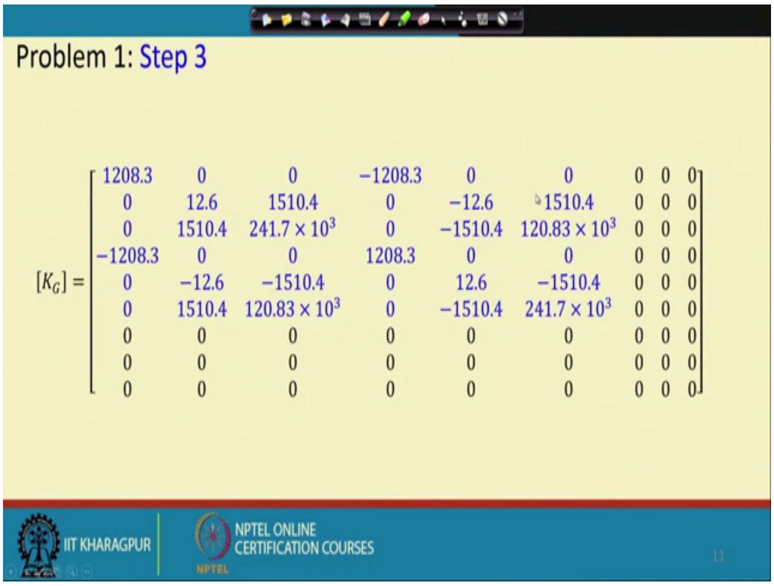

For instance I just write and the number of degrees of freedom in the global stiffness matrix, also I know because my structure contains 9 degrees of freedom my structure contains 9 degrees of freedom. So total 9 degrees of freedom. So my total global stiffness matrix for the whole structure will be 9 cross 9, so I write a 9 cross 9 0 matrix, and then first input the local the element 1 the stiffness matrix for the element 1 as per the global degrees of freedom. So this is my first degrees of freedom, this is my second degrees of freedom this is my 3rd 4th 5th 6th 7 8 9 so similarly this is one 2-3 4 5 6 7 8 9.

So, now you see these part it is left 0 because the first element if you remember that is contains only 6 degrees of freedom for and it is to more so only 1 2 3 4 5 6 was there so this part then rest of the part is blend. So, now, again I will consider now the second element so in case of a second element.

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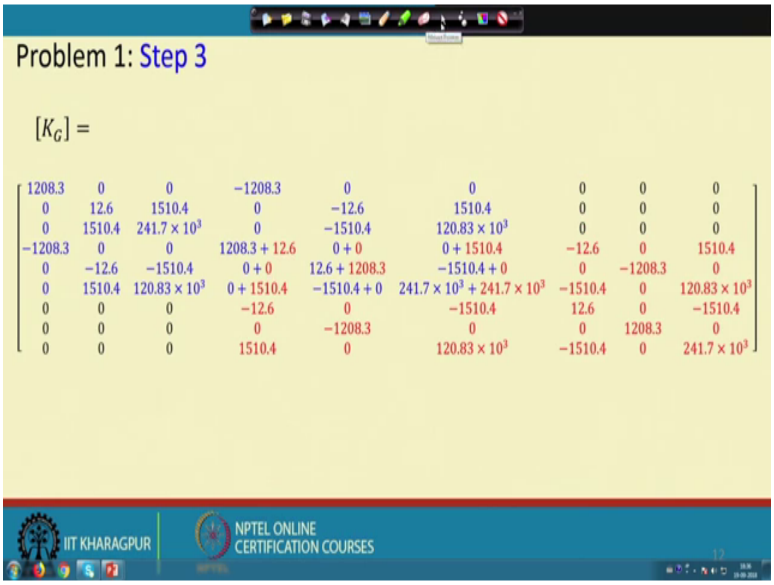

ah So let us because of the size of the matrix let us we do this so total degrees of freedom is 9 so this matrix will be 9 cross 9, so this is my 123 4 5 6 7 8 9 and then 1 2 3 4 5 6 7 8 and 9 so now here you see in the for the element 1 all the blues are for the element 1 and now see for the element 2, 4 5 6 7 8 9 is the global degrees of freedom.

So, as per the global degrees of freedom the fourth element and the 4 4 4 5, 4 6 and then 5 4 5 5 6 5 6 and then so on these 4 6, 5 6 and then 6 6 these has to be added with the global stiffness matrix.

So, see this part these red parts are added with the global degree the previous stiffness matrix. So the blues so this will be red only so this and this quantities these things does not play any role because the first element does not contain any 7 8 9 degrees of freedom, similarly second element does not contain 1 2 3 degrees of freedom so that is why these 2 part will be 0, but these stiffness these part the second element contains 4 5 6 and then 7 8 9 so that is why this stiffness matrix. You see there the important thing is that the stiffness matrix is very much banded so there is a band structure in the stiffness matrix. So, that means, this band is essentially very important for solution because when you really solve a large scale system.

Essentially if you have a band and stiffness matrix there are specially 2 advantage; First advantage is your you can store the matrix in a band storage format which is essentially for instance this 9 cross 9 matrix if you want to store into a MATLAB or any computer through MATLAB or any programming language. So you see there are 81 such elements and each element records some bit memory.

So, now, if you know the band structure of this matrix, if what will be the band structure of the matrix because you can predict the band structure of the matrix in the review of matrix this (Refer Time: 27:46) I have discussed this. Now this 7 8 9 you see these 3 cross 3 that is 9 such element will be 0. So why should I store these 9 plus 3? These zeros in the matrix, so there I can reduce my storage space required for the matrix so I can have a band storage form.

So, band storage form will looks like this form for instance this kind of only, this band this is the band and that there is a bandwidth and this I can store it efficiently. So this 9 and 9 I can I do not need to store even so because I know this 0. And second issue is that when you know the bands and banded matrix the factorization for instance the decomposition part of the solution if you follow the direct method, then there is an advantage in the band storage form so this is very important.

Now, once you have this global stiffness matrix ready, then the next step is actually you have to find out the partitioning and solution of that system.

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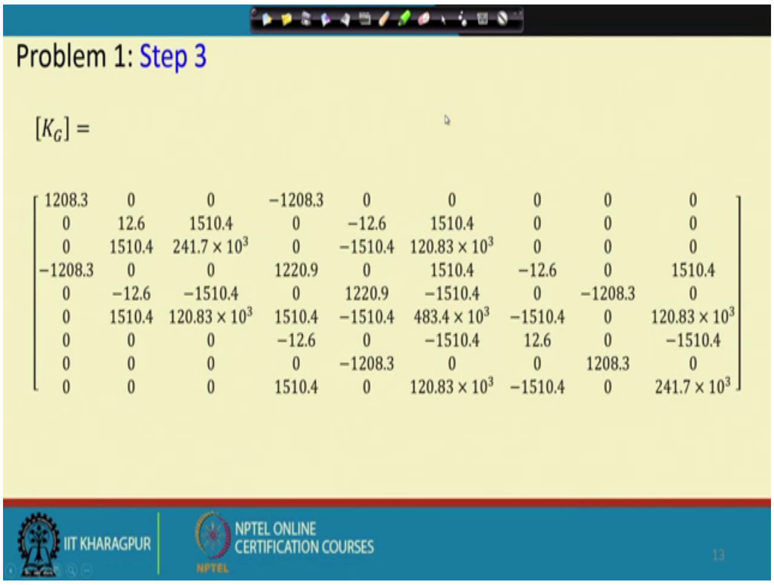

For instance this is my final so these are change the colour because final what will be the final format. S0 say again this is 4 1 2 3 4 5 6 7 8 9 and then 1 2 3 4 5 6 7 8 9 so here you see the, this is my global stiffness matrix for the structure which is a 9 cross 9.

Now, if you look carefully this structure is symmetric again. So this is my symmetric form of the structure a symmetric stiffness matrix. Now you see this is symmetric so this is this and so on so let us now try to solve this, what is the unknown degrees of freedom.

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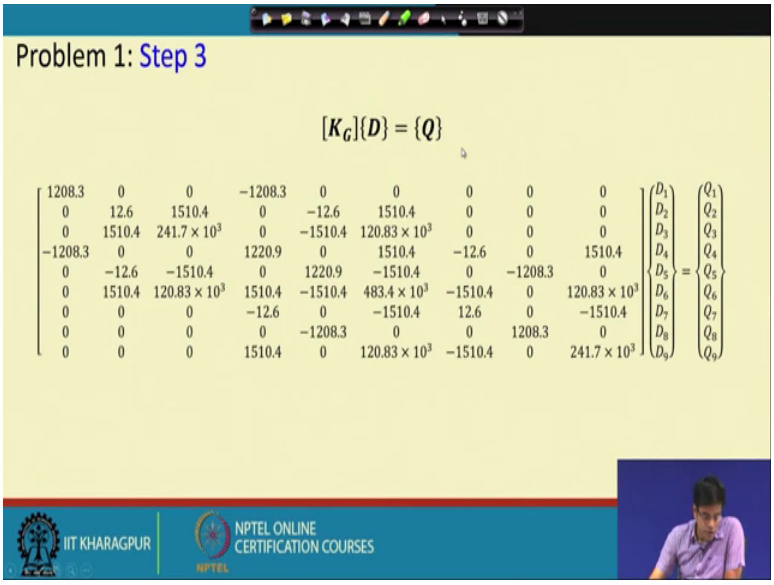

So, if I write now my global equilibrium equation; that means, the K G the global structural stiffness matrix which is 9 plus 9 matrix, my all degrees of freedom 9 cross 1 and 9 Q the force vector is 9 cross 1. So this is D 1, D 2, D 3 D 4 D 5 D 6 D 7 D 8 D 9 and Q 1 to Q 9. So, now, if you remember some of the d ones D is we know and some of the Q is also we know so we can now invoke those quantities.

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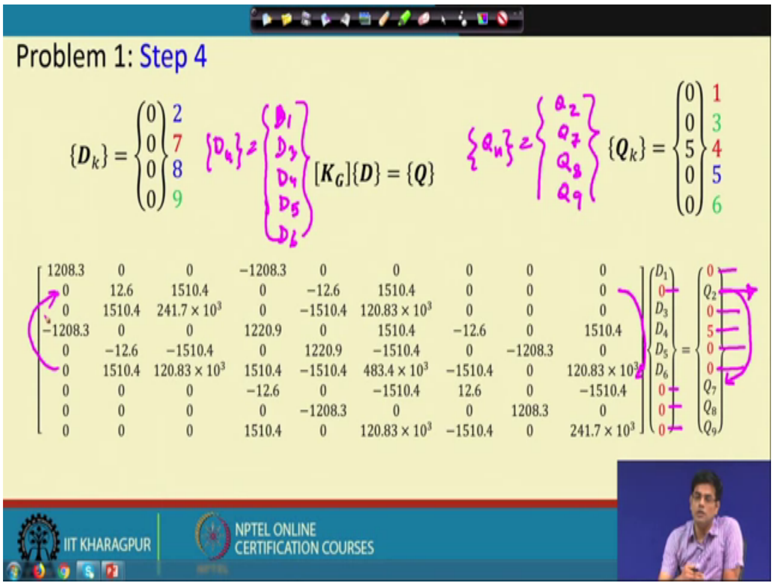

So in the step 4 if you look that D k these known degrees of freedom what are the 2 2 7 8 0. So I can just plug this thing 2 7 8 9, so this is my known degrees of freedom 7 also we know this 8 also we know 9 also we know.

Now, similarly Q known degrees of freedom 1 3 4 5 6; so 1 we know 3 we know 5 and this is 4 we know 6 5 we know 6 we know. So this is my unknown so my Q u that is unknown force vector contains only this Q 2 Q 7 Q 7 Q 8 and Q 9 and similarly my D unknowns D unknowns, will be your all these degrees of freedom Q 1 sorry D 1 D 2 D 3 D 4 D 4 D 5 and D 6. So if you remember those degrees of position of the degrees of freedom, then you will see there is D 1 D 3 D 4 D 5 and D 6 is my unknown displacement matrix.

So, now to solve this equation we have to partition this how to partition it? So you see that each row of this matrix vector equation represents one equation. So, now, you see partitioning means you have to have a known part and unknown part really, but you see its jumbled up here. So to solve these jumbling things suppose I want to bring Q 2 here and this part I want to move up.

So, what it means essentially? It means essentially you write second equation; that means, this second this is this represents my second equation so this second equation I will write later rather I will write these known force vector first. So; that means, first equation third equation fourth equation fifth equation and sixth equation I will write first, and then I will write 2 7 8 9 so it is just shifting the rows.

So, shifting the rows of the stiffness matrix so what it means? I will write this I will shift this thing to the suppose I I will shift it to 6; 6 node and I will this I will put it in the here so this will not change my displacement vector. So the position of the displacement degrees of freedom will not change. So it will not I will just write 2 different deviation for instance if there is a 3 equation first 1 2 3. So if I need to write the third equation first, so 3 2 1 so that does not mean my these displacement degrees of freedom or my vector the unknown vector would notchange its position so this you can verify actually.

So, then once I did that so this will look like this so what happens is that I just change Q 2 from here to here.

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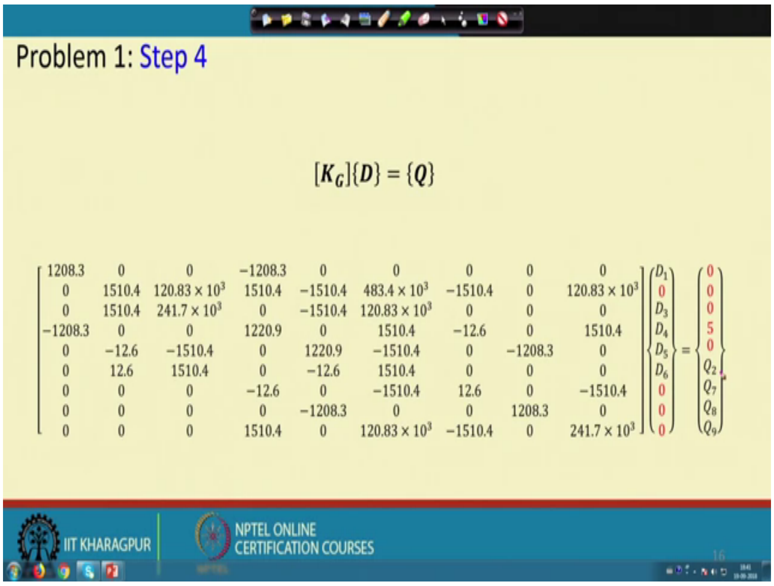

So similarly I change this thing this was earlier this row that is this row earlier, so this row it was earlier here and this row which was here earlier is replaced here so this becomes 0. Now you see this does not affect or this does not change my this thing so this is my changing the stiffness matrix as well as the force vector; so row row exchange procedure so, but I did not change this thing. So the again if you see we have got it nice the force vector, but this one is not again is jumbled up so I want to move these degrees of freedom or essentially D 2 which is one degrees of freedom to me is here and D 6 I want to put in place of this D 2.

So, that to do that essentially I have to shift the columns, so this is columns corresponding to second degrees of freedom and this is columns corresponding to 6 degrees of freedom. So, now, if I interchange it so I can change this thing.

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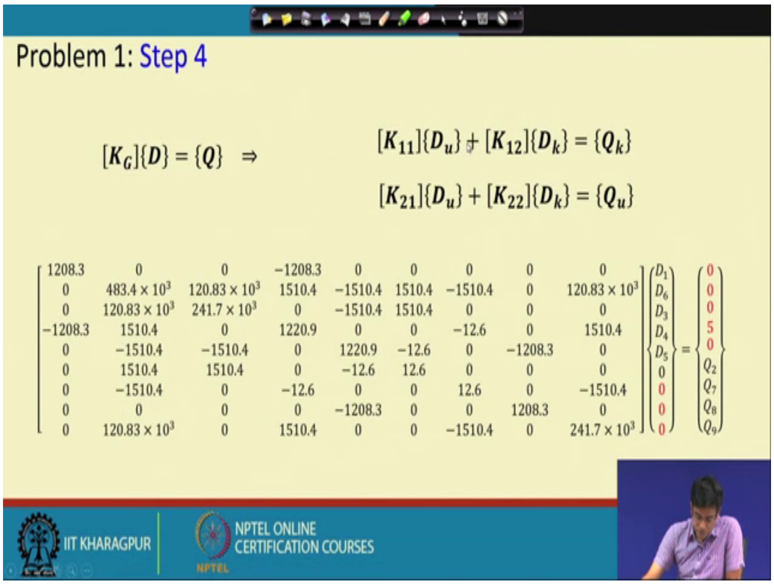

So suppose I do that and again I put D 6 here and D so this will be red this 0, will be red so this is known to me so essentially I change this thing D 5 by till D 5 is (Refer Time: 35:56) unknown. So, now, you see my after this operation so what I did? First I moved this row of the stiffness matrix to the next and this thing with the sixth row of the stiffness matrix, and then I change the second column of the stiffness matrix with the sixth column of the change stiffness matrix. So this procedure is known as the partitioning of the matrix.

So, you see this partitioning is very important because when you try to this when you try to solve it essentially you need to have this form where D u. So, now, my D unknown is my this part of the displacement vector, and Q known is my this part of this displacement vector, and Q unknown is my this part and Q known d known is my this part. So this is my D known, this is my Q known and this is my D unknown D u and this is my Q known sorry this will be Q unknown.

Now, which is a K 1 1? So naturally the size of the K 1 1 will be 1 2 3 4 5 so till here 5 cross 5 so this is my K 1 1 so this is my partitioning so this is my partitioning. So, now, what is K 1 2? K 1 2 corresponding to the known so this is my K 1 2 so and this is my naturally K 2 2 and this is my K 2 1. So if I write now now what is the size of this K 1 1? So K 1 one size is 5 cross 5 so and this D unknown; D unknown is also a 5 cross 1matrix 5 cross 1 matrix, and K 1 2 K 1 2 is my 5 cross 4 so 5 cross 4 and then D known is my 4 cross one so and what is my Q known? Q known is my 5 cross 1.

And similarly K 2 1 is my 4 cross 5 4 cross 5 and this K u unknown is 5 cross 5 cross 1. So K 2 2 is my 4 cross 4 and this K D known is my 4 cross 1 and Q unknown is my 4 cross 1. So finally, this I know this thing. So, now, again I will use this equation the first equation because I know D k which is 0, here because D k is 0. So if you multiply 0 with K 1 2 this matrix, this part of the matrix, then if you multiply with the 0 vector it will be a 0 vector so when it goes to it does not change. So this is my K 1 1 part and solve this thing.

And then you see noticeably this D 1 this is not D 2 or this is not D 3 D 4 D 5. So you can also do you can in the increasing order you can arrange the degrees of freedom, but that will required again a row operation or a column operation. So essentially you have to shift all the columns to the columns and then you have to write so that is not required actually.

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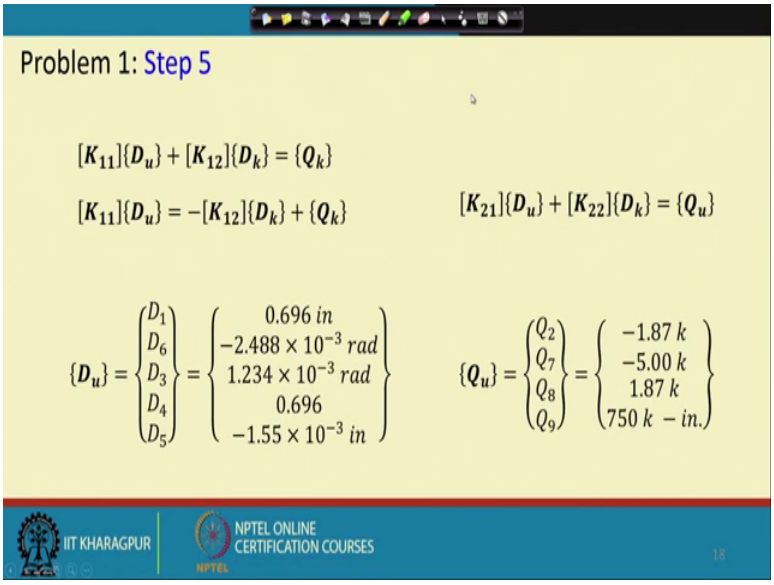

So, now, once I know this I can solve this problem so this solution is through the first equation I can find out. So essentially K 1 1 I know D u I do not know D k I know which is a 0 vector, and so if I multiply 0 vector and Q known I know so which is only one axial force and then I solve this. So this is a fifth 5 cross 5 equation; if I solve this I get this solution.

o I know D 1 D 6 D 3 D 4 D 5 now once we know D u here. And then with this D u if I now d known is again sorry this equation the second equation, so if I know D u now and D k is 0 vector, so D does not contribute so this will go to 0 and so if I substitute this. So I know K 2 1 from my previous slide. So once I multiply this I get the unknown forces at each unknown location or where the displacement degrees of freedom is specified.

So, this is in a nutshell the total procedure so you understand where you can go wrong actually this matrix partitioning you can go wrong then matrix assembly you can go wrong. So these things are very essential when you code for the structure. So in the next lecture we will also try to solve some other problem some another problem, to tell you what are the process involved in it so I stop here.

Thank you.