# Matrix Method of Structural Analysis

**Prof. Biswanath Banerjee**
**Department of Civil Engineering**
**Indian Institute of Technology, Kharagpur**

## Lecture – 26
## Matrix Method of Analysis: Frame (2D)

Welcome this is module 6 of Matrix Structural Analysis, Matrix Method of Structural Analysis. So, till module 5, module 4 and 5 you have learned truss and beam structure. So, trusses and beams are you have solved by stiffness method, and the matrix format or matrix analysis of matrix method of analysis you have learned by you may learn first truss and beams.

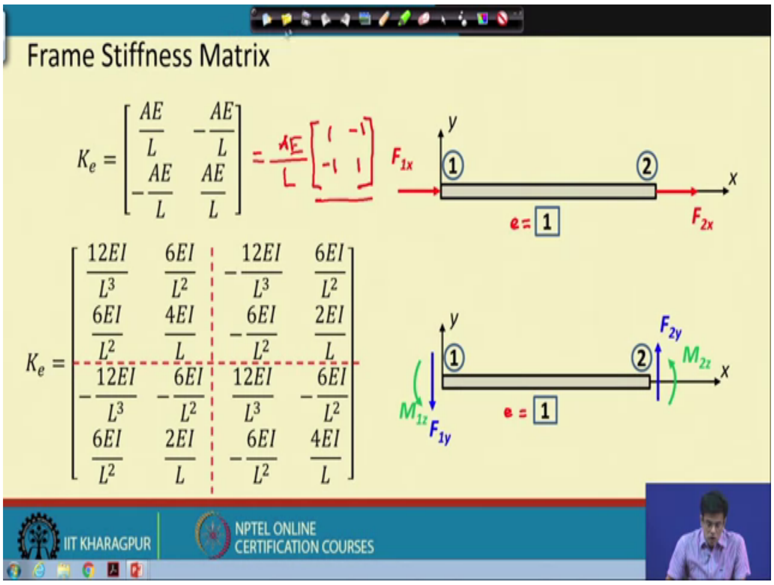
Now, so, you are enough matured with the procedure of matrix means matrix method how to solve a structural problem through a matrix method or specially the stiffness method. So, we are not talking about flexibility method here, but it is better to know that there is another method called flexibility method. But flexibility method is not in wider use because of certain restrictions of the flexible flexibility method, but we are not going into that detail, right now.

So, in this lecture what we are planned is to solve frame problems and specially the 2D frames, because you know the main objective behind the matrix method is that you really able to code these structures or the analysis in a computer through a software or a programming language. So, like programming languages like C, FORTRAN, Pascal or any other languages and other software's like GUI base software's like MATLAB or any other mathematic these things can be used to solve large scale structure.

What we mean by large scale structure is that where the degrees of freedom essentially is a large number. So, essentially your final equation that is k d equals to q or the stiffness multiplied by the displacement vector equals to force. These equations the dimension of the coefficient of the stiffness matrix is very large because it depends on the number of degrees of freedom in the structure you consider. So, when you essentially considered a large structure it is natural that you will have a large number of degrees of freedom and so by hand these calculations are not possible, so you use computer to solve such system.

So, naturally by hand we will solve very simple structure, but I will try to elaborate the procedure once again which you have learnt for beam and truss earlier. So, before we go into that procedure let us first find out what is the stiffness matrix of a 2D frame structure.

(Refer Slide Time: 03:51)




So, you have already learned the stiffness matrix of a truss. The truss essentially you have this axial member where essentially you have two nodes the node number 1 and node number 2, so you have F 1 x and F 2 x this is in the positive direction. So, direction actually does not matter because this quantity can be in terms of sign.

So, now you have already found out that stiffness matrix of this truss is we can write that if I take AE by L common 1 minus 1 minus 1 1. So, these stiffness matrix you have derived already or you know that. And for beam also we have where in the beam you have the shear force and bending moment and the kinetic degrees of freedom are essentially the displacement transverse displacement and rotation of the beam.

So, in this case also you have also found out how the stiffness matrices look like or specifically the element stiffness matrix E, here equals to 1, that hm here E equals to 1 so first element or second element or n number of element in a structure. So, this form of the stiffness matrix remains constant. So, that, but the value of EI L or a area these can change and so the stiffness matrix essentially changes. So, you know how to what is the meaning of these stiffness matrix especially we know from our definition that stiffness is unit force required for unit displacement.

So, essentially the first element of this means is that if I put one, if I have a 1 unit displacement of this node then what is the force required for this node and then do that force what will be the corresponding this thing displacement here and then corresponding stiffness. So, that that diagonal element means.

But the important property of this stiffness matrices is here these stiffness matrices are singular these we can validate very easily because we know this is linearly dependent columns. If you look the linearly dependent columns all if you take the determinant of this stiffness matrix this is going to be 0, because the physical reason of this because this is this does not have any constraint. So, it essentially means a rigid body motion. So, to have a deformable body motion you have, you need to have a proper constraint or the boundary condition that we call for the structure we know we know it is a boundary condition.

Now, this boundary condition when we plug into the stiffness matrix then stiffness matrix become invertible before that it is not invertible. So, this is true for this truss stiffness matrix as well as it is true for the beam stiffness matrix. Even if you take the determinant of these beam stiffness matrix it will go to the 0. So, with this knowledge we will proceed for the frame structure.

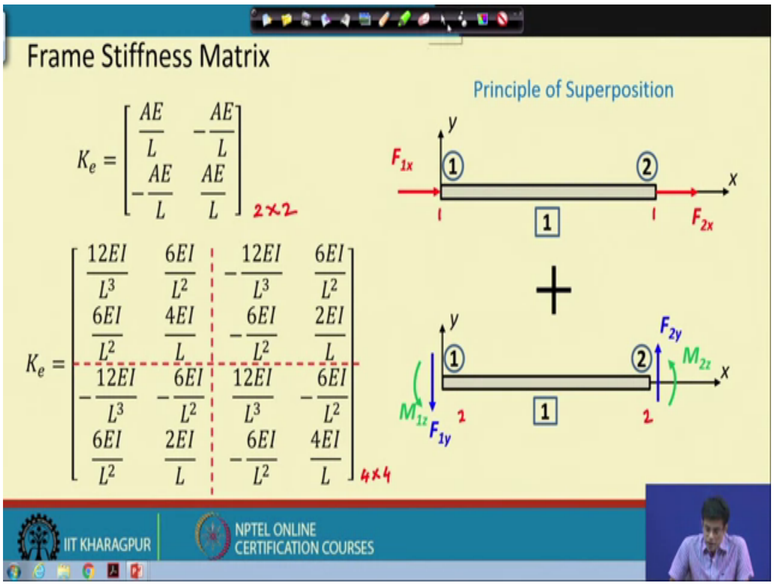
So, now there is an important principle that we have learned earlier that we will use. So, now, suppose there is a beam which are specifically beam element which first I give a axial load and then I give I find out the deformation and then the rotation or the moment and shear force I give. And then I find out the deformation then if the axial load moment and shear force I applied simultaneously then the deformation caused will be some of these two individual loading that is the axial force and moment shear force. So, this principle is known as the principle of superposition. That you have already studied in your this course or previous structure analysis course.

So, this principle superposition is valid for linear structure the linearity property of any function or the linear structure it is valid. So, we will not discuss in detail what is that principle of superposition, but we understand that for linear structure that principle of superposition is valid and this principle of superposition means that if I individually apply this load and find out these deformations that is the axial force deformation due to the axial force and deformation due to this kind of loading. And then simultaneously if I apply these two loads and then find out the deformation. So, the total deformation due to these two loads will be equals to the deformation on which we apply all load simultaneously.

So, in this based on this concept essentially we can derive the stiffness matrix of a frame. So, a frame is essentially your moment resisting members which can also carry the axial forces. So, likewise in a truss we have seen that truss is a purely axial member beam is a purely flexural member. So, generally we do not consider beam to carry an axial or we do not design beam to carry to axial force, we it generally do not use beam for resisting the axial force because beam is primarily a flexural member.

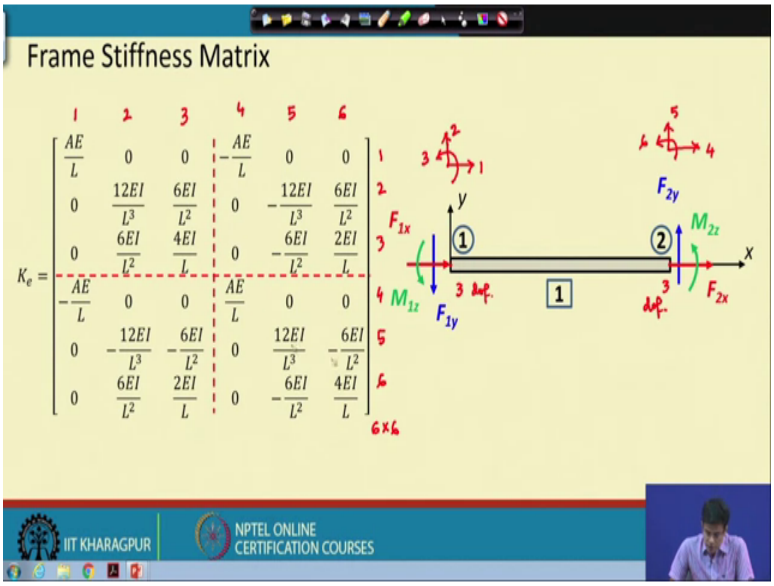
But frame where we allow to resist the axial force as well as bending moment and shear force, so beam also we design for resisting the transverse force. So, frame contains everything. So, now naturally at each point the number of degrees of freedom or each node the number of degrees of freedom will be 3 because in a frame structure the degrees of freedom for a truss structure the degrees of freedom is 1, and for each node for a beam structure the degrees of freedom will be 2, that is one is a transverse displacement and rotation about the xy plane.

(Refer Slide Time: 10:55)



So, if I superpose these two things and then the, we will get the frame stiffener stiffness matrix. So, naturally the in the frame the each node the degrees of freedom will be 3. So, when we now find out the frame stiffness matrix naturally the frame stiffness matrix will be for an element it will be 6 cross 6. So, you see for a truss it is it was 2 cross 2, for a beam it is 4 cross 4. So, now, the degrees of freedom increases to 1 for a frame, if you compare in terms of with beam then the frame stiffness matrix will be 6 cross 6.

(Refer Slide Time: 12:05)




So, here if we now write the frame stiffness matrix with appropriate degrees of freedom this looks likes a frame stiffness matrix where it is 6 cross 6 stiffness matrix. If we look carefully that frame stiffness matrix that is at each node that is node 1 and node 2 here, so each node there are 3 degrees of freedom. So, degrees of freedom are 3 three degrees of freedom each node. So, naturally the stiffness matrix will be 6 cross 6 for an element which was 2 cross 2 for truss 3 cross 3, 4 cross 4 for the beam.

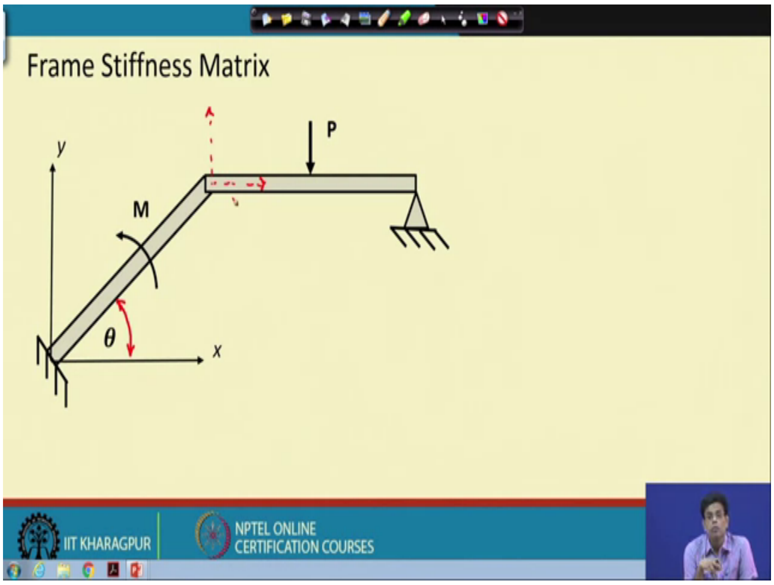
Now, for a beam what are those; for a frame what are those degrees of freedom? So, there are axial displacement along x displacement along y and displacement and rotation along xy plane and this is for a 2D frame. So, you can appropriately modify for the 3 difference which will not discuss in this lecture though. So, you will have 3 such displacement, 3 such rotations for 3 difference or the space frames we which we call popularity.

Now, similarly the forces which can act in a frame member or the 2D frame member or the axial force transverse force and the moment. So, the interesting thing is that, so we also know that this is the if we now write the degrees of freedom per node here. So, there is a first degrees of freedom, then this is a second degrees of freedom, then this is a third degrees of freedom and here corresponding to this node there is a fourth degrees of freedom, fifth degrees of freedom and the sixth degrees of freedom. So, these 6 these columns also represent corresponding to these degrees of freedom 3, 4, 5 and 6. So, the superposition actually we when we union the two stiffness matrices that is truss and beam we swell or we expand the stiffness matrix as per the degrees of freedom. So, this is important and this you must understand that this is not just an addition. So, it is a addition with appropriate degrees of freedom.

So, you see this first column represents corresponding to first degrees of freedom that is the axial force at node 1, similarly fourth represents the axial force at node 2. So, this 4, 4 is the stiffness coefficient when we apply when we have a unit displacement at node 4 node 2. So, this stiffness matrix finally, gives us elemental stiffness matrix or when I consider an element what will be the final form of the stiffness matrix.

Now, this is the coefficient of this stiffness matrix all of us know because we know beam stiffness matrix that is 12 EI by L cube, 6 EI by L square and so on. So, for a truss also we know what is AE by L and minus AE by L minus AE by L and AE by L. So, there is no stiffness extra stiffness coefficient that we do not know, but it is the only appropriate positions of that stiffness coefficient when we consider axial force with the beam stiffness matrix. So, this is the stiffness matrix of a frame.

(Refer Slide Time: 16:07)

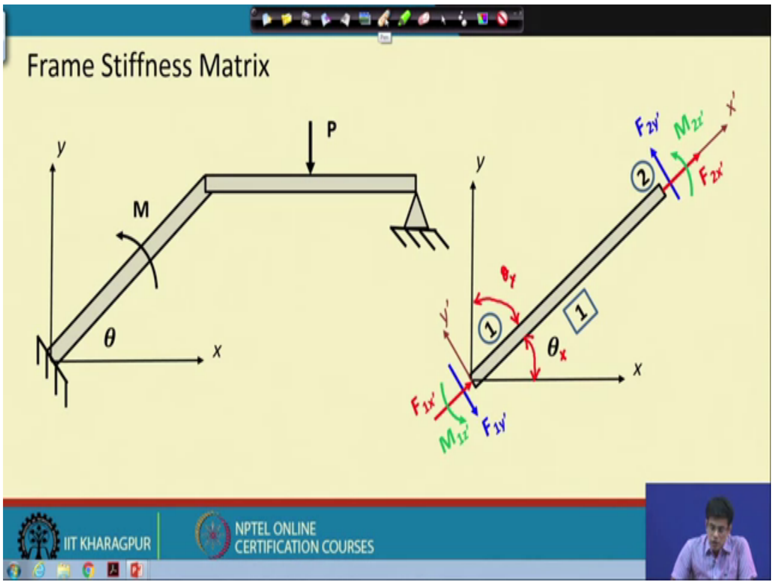



Now, this frame as we have seen in case of a truss, that suppose we want to solve this kind of problem. So, where these frames actually carries moments in case of a truss we were not when we considered at a these as a truss structure then there was no moments acting in the member because truss is a primarily axial force axial member.

So, now when we try to solve this kind of structure you see this structure is not just horizontal, likewise we have seen in case of a truss that truss members can be inclined also even beam can also be inclined. So, here if you see this member for instance, this member makes an angle theta with respect to the global x axis. While this member if you think this member this member does not make any so the angle is actually the 0 degree.

So, now as we have seen in case of a truss that we need to modify those stiffness matrix due to the inclination, and this modification we have also learned how to do it in terms of transformation matrix. So, that we will see what it looks if I have a transform if I had to transform the inclined members or find out the stiffness matrix is for the inclined member of a frame.

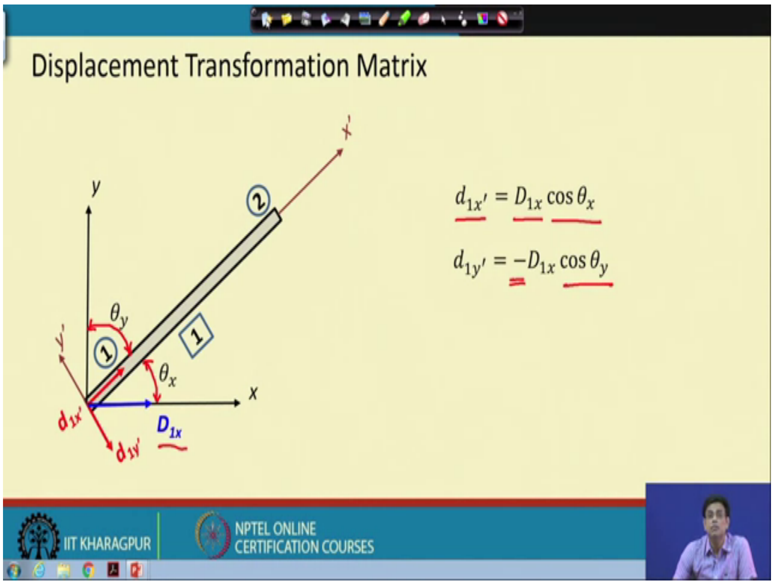
(Refer Slide Time: 17:45)




So, suppose this is a basically inclined frame where we have a again the angle it makes is theta with respect to x axis. So, I can say this is theta x and this is theta y. So, we know what is theta x and theta y if we know the coordinates of these two points. So, this is naturally 0 0 and this may be some coordinate say something x y. So, we know how to calculate theta x and theta y, and theta x theta x plus theta y is 90 degree. So, how to calculate theta x and theta y? We know, so we have to find out now what is the form of the transformation matrix that we are talking about.

So, let us see what form it looks like.

(Refer Slide Time: 18:41)

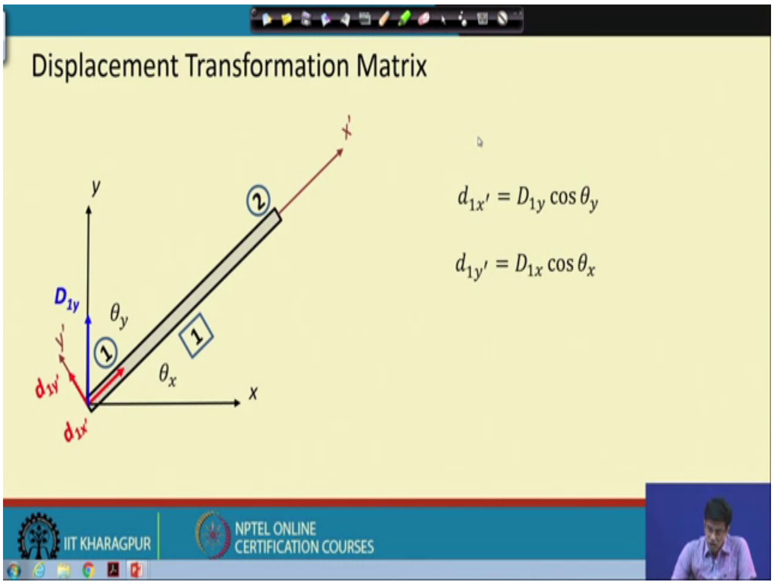



So, for instance if there is a x axis along x there is a this is first let us understand that this is a global x axis or the structural axis the structural axis and if this is the local axis the local axis is x prime y prime and the structural axis is x y. So, now, if I have a displacement along this structural axis which is capital D 1 x, the 1 represents the node number and x represents the direction at which the displacement is happening.

So, if I know now theta x and theta y then I can just like a vector component or the projections in the frame the elemental axis or the local system I can write the d 1 x is d small d 1 x that is in the local axis is D capital D 1 x cos theta x. Similarly in the y direction it will be in this side. So, I have considered positive D 1 x capital D 1 x, so in this it will be theta y. So, minus sign it will come because I have assumed this is positive direction and the y prime; so minus D 1 x cos of theta y. So, here we know cos of theta x and theta y we know from the geometry. So, now we can if this is for a displacement along the structural x axis.

Now, similarly we can do it for a displacement in the y axis.

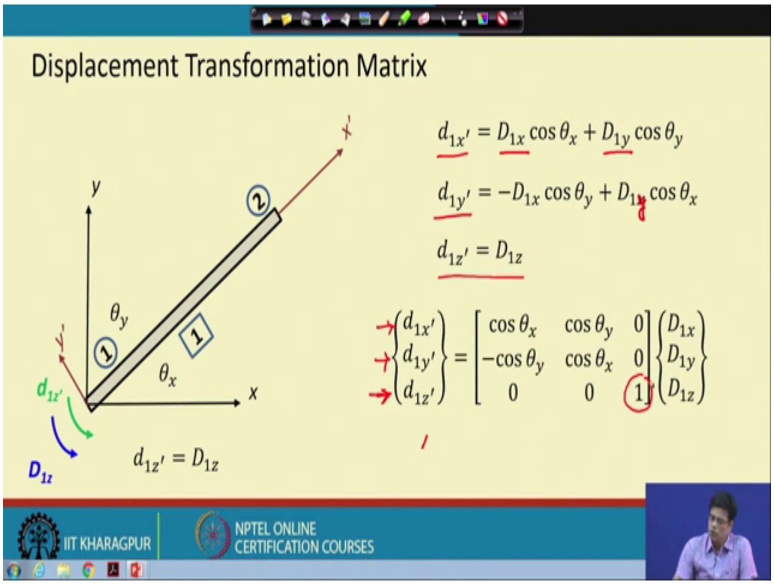
(Refer Slide Time: 20:37)




So, when we have a y axis this displacement I am representing in terms of d 1 y. So, due to this there will be a x component displacement in the material axis and the y component displacement in the or the frame axis, so or the elemental axis or the local system.

So, the local system x axis will be d 1 x prime small d and that will be equals to D 1 y cos theta by d 1 prime small d 1 y prime is D 1 y this will be y, so y D 1 cos theta x. So, that means, you have essentially if there is a displacement along structural axis in x direction you can convert it to the elemental axis or the local axis in the both direction.

(Refer Slide Time: 21:57)



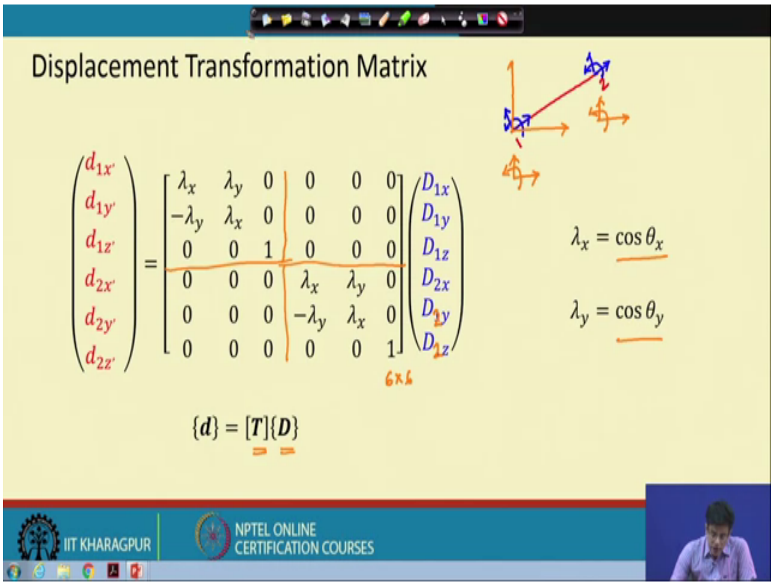

Now, suppose there is a moment or there is a rotation what will be that case? So, in case of a rotation this the rotation is actually the of the xy plane, and if you consider the xy plane is perpendicular to this board that means, even if you rotate the rotation does not change the theta x and so rotation does not change in the local or global axis or structural axis or the elemental axis. So, naturally the third degrees of freedom or that is z degrees of freedom. So, small d 1 z prime it will be equals to capital D 1 z. So, this has no transformation because rotation is along z direction or in a xy plane.

So, now finally, if we again use the principle of superposition then we have this form. So, essentially you when you have a this thing when you have when you apply d 1 x and when you apply d 1 y what will be the local coordinate system or the frame elemental coordinate system what is the x direction. So, along this x prime direction this is the displacement and so we add it because we consider linearity and then similarly, in the y prime direction, so these direction elemental y direction and then z direction. So, I just added.

So, now, if I write it in a matrix form, so this is simply this so which relates actually the prime coordinate to the non prime coordinate. So and how what is this? This is for this is the transformation matrix. So, essentially transformation of the displacement vector or the deformation vector, so in this deformation vector two are the axial displacement, these two are axial displacement and it is the rotation or the theta that we considered.

So, now this matrix is the transformation matrix. Now, this is for node 1. So, similarly I can write all those expression for node 2. And then what will be the elemental stiffness matrix? If you remember in case of a truss that you are you found out similar kind of rotation matrix, but there was no one here, so this rotation matrix probably you have learned in the truss. So, now, for two such elements when I consider two such element, then what will be my elemental transformation matrix?

(Refer Slide Time: 24:43)



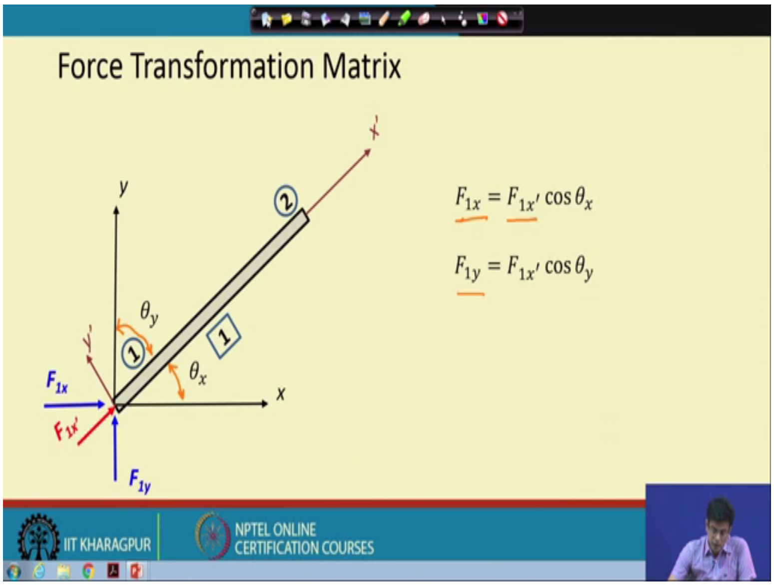

So, for instance in this case I consider two such element and these elements are of these forms. So, that means, I have this node number 1, node number 2 I have degrees of freedom what are these degrees of freedom are like this. So, this is my degrees of freedom, and this is my degrees of freedom in the local coordinate axis or the local system or the elemental system.

Now, I have similarly global system in the global system I have this is my global system. So, in the global system I have similarly for these I have these, these axial and this moment and similarly at this point I have these, these and these. So, these two quantities this blue degrees of freedom and this orange degrees of freedom are related with this matrix. So, this is we know, so the lambda x and lambda y we assume cos theta x and cos theta y. So, it looks like this. So, this part is for node 1 and similarly this part will be for node 2.

So, now this is the, this will be 2, this will be 2. So, this will be for the node 2. So, now, these matrix we say this is a transformation matrix of T. So, in case of a truss where you have probably you see this transformation matrix here it is a 6 cross 6 transformation matrix, but in case of a truss you have also found of this transformation matrix. So, now based on that truss degrees of freedom, which is 2 per node; because it can have y and x.

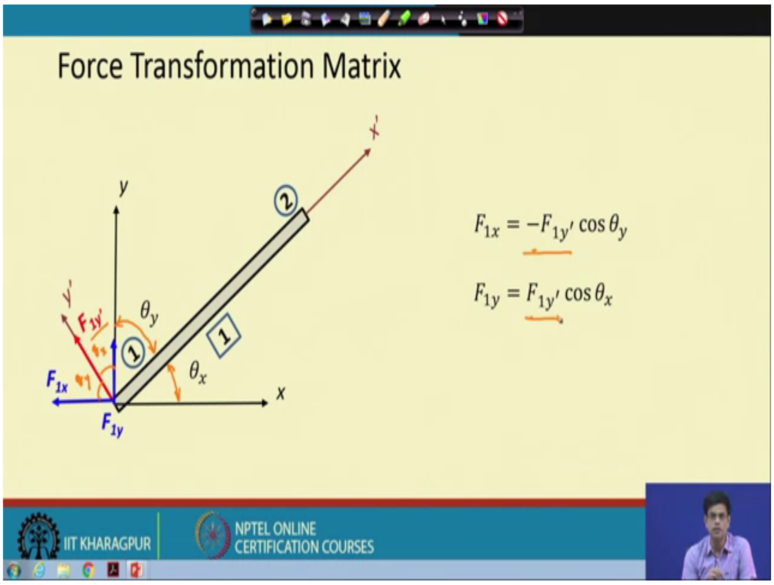
So, here we know; so what is outcome? Outcome is that that if we know the global or the structural displacements we can efficiently convert it to the local or the elemental displacements. So, this is done through the transformation matrix 3 or similarly I can do the inverse of that. So, if I have if I know small d's and then I can find out capital D by taking the inverse. So, this procedure we know.

(Refer Slide Time: 27:29)




But what happens if I have a force. So, if suppose we have a force which is applied in the local stiffness local elemental axis. So, that is suppose this F 1 x prime is applied along this. So, similar to the previous case if I know theta x and theta y; so, theta x and theta y if I know then I can simply project these in the structural x axis and or the global x axis and global y axis and these looks in this form. So, if due to this prime x direction force axial force I have the x component in the global direction and y component in the global direction; so, due to the only x component in the prime coordinate system.

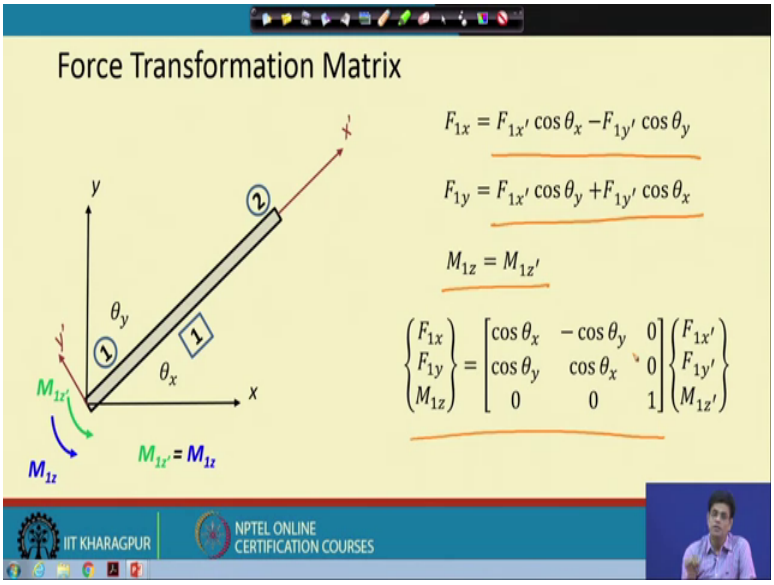
(Refer Slide Time: 28:27)




So, similarly I can have a prime coordinate system if I have y prime coordinate system in a x y direction that is F 1, F 1 y prime. So, I can again you know by knowing the theta y and theta x I can find out the F 1 y and F 1 x. So, that is means due to this F 1 y minus because F 1 x is opposite side because this if we know, so this and this angle we know and this angle is essentially theta y. So, this is again 90 degree. So, this is theta x and it is theta y. So, F 1 y prime cos of theta y so this direction and this is minus since it is the opposite positive direction. So, similarly F 1 y that is in this direction which is cos of theta x; so, F 1 y prime cos of theta x.

Now, instead of a displacement or the axial forces or the shear forces if I apply a moment now this moment does not change, because moment is along z the perpendicular direction or that is acting in a plane that is xy plane. So, it does not change any magnitude because of rotation of the axis. So, this transformation remains same. So, this is identity transformation we can say, so likewise in the previous case for a rotation.

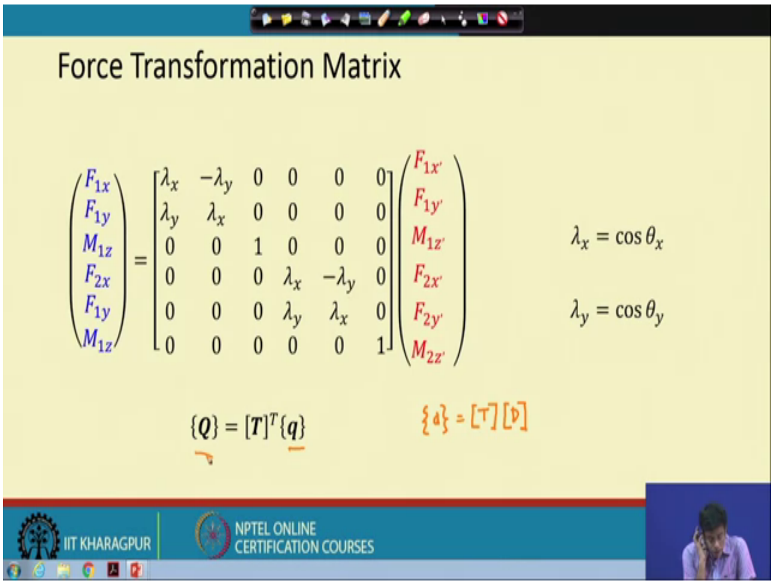
(Refer Slide Time: 29:43)




So, now similar to the previous case we can find out how using the superposition when the axial force, shear force a moment is applied simultaneously then what will be the transformation; so, these leads to this quantity that we have seen. So, addition and these are addition and subtraction due to applying the superposition I mean if I write the matrix form it looks like the same matrix only the sin flips. So, this thing we have what we what this mean meaning is that if I know in the prime coordinate system, then I can get the values at the non prime coordinate system or the global coordinate system and this is the transformation matrix and this transformation matrix is for node 1.

Now, similarly as in the previous case we can have the node 2 transformation matrix. So, the node 2 transformation matrix and node 1 transformation matrix; if I now finally, write in this form.

(Refer Slide Time: 31:17)




So, the same thing only changes is that this science where earlier here in case of a displacement transformation and remember our displacement transformation was d small d is capital T into capital D. So, it transforms the, this T transforms the global to local and T transpose is naturally transforms here, so here from local to global. So, essentially we can comment that the, this is a, so this matrix is essentially the skew symmetric matrix if you look carefully.

Now, once we know this we can find out the stiffness matrix as we have done for the truss structure. In case of a inclined truss members we can find out the inclined frame members what will be the stiffness matrix for the inclined frame members. So, but here today I will stop here and so what we have learned is that is we use the principle of superposition to find out the stiffness matrix of a frame which is of 3 degrees of freedom per node because it can carry axial force shear force and bending moment. So, the kinematic degrees of freedom are displacement in two directions that is x direction, y direction and then the rotation of the xy plane.

So, we have 3 degrees of freedom. So, the dimension of the stiffness matrix will be 6 cross 6 for a two noded element of a frame. Now, if the frame is rotated or the inclined frame members; we know how to transform displacements and forces that is also a 6 cross 6 transformation matrix. With this knowledge in the next lecture we will derive first what is the stiffness matrix for a inclined frame members.

Thank you.