Matrix Method of Structural Analysis
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Lecture – 24
Matrix Method of Analysis: Beams (Contd.)

Hello everyone, this is the fourth lecture of this week. In the first three weeks, we try to understand the different steps involved in matrix method of structural analysis for beam. Now, today we will demonstrate those methods through an example, we will be considering two example in this week. Let us start with the example through which we demonstrated we understood, we discussed the different steps.

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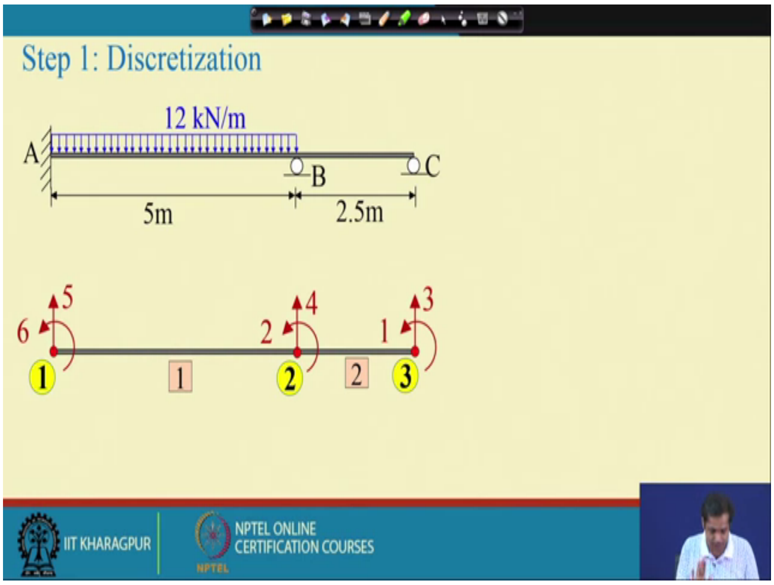

So, let us take this example, and if you recall in the first 3 weeks, when we discussed different steps that is we try to understand through this example. Now, let us actually solve these example, well so if you recall there are different steps in solving in solving any beam problem through this method, the first step is the discretization, means we have to understand we have to identify different members, number them; all the members and also number the all the joints and the associated degrees of freedom.

So, this has two member so, let us take the first step is discretization we have member number 1, these are the joints number 1, 2, 3. And then this is member number 1, and this is member number 2. And now here you see again just let me emphasis on the point that in the first 3 weeks, we use this example to understand the different steps, but that time we did not bother about; how the numbering is to be done, numbering of the degrees of freedom is to be done.

Now, you see look at this beam. You see at every joints we have 2 degrees of freedom, if we ignore the axial deformation at these joint, it all the degrees of freedoms is 0. At this joint we have there is a constraint in vertical translation at there also constraint in the vertical translation. And only degrees of freedom which are unknown are the rotation at B and rotation at C.

Now, when we do when we number them let us, give degrees of freedom at rotation at 3 is degrees of freedom 1; and rotation at 2 is degrees of freedom 2. So, this is u 1 and this is u 2, and then this is u 3 translation at node 3 translation in vertical direction is u 3. And translation in vertical direction at node 2 is u 4. And then we have u 5 and u 6.

Now, this is important; important in the sense, if you wish you may not do this, you can give the numbering the way we gave the numbering during the during discussion of the steps, but there is a precise an advantage of numbering at least for the context of this problem. There is an advantage of numbering like this and that advantage you will be cleared shortly, when we actually solve those equations the linear system of equation.

Let us see that, let us bear with, let us you, you bear with us bear with me for some time, to understand the advantage of these kind of numbering. Now, so this is the numbering of degrees of freedom nodes and the elements.

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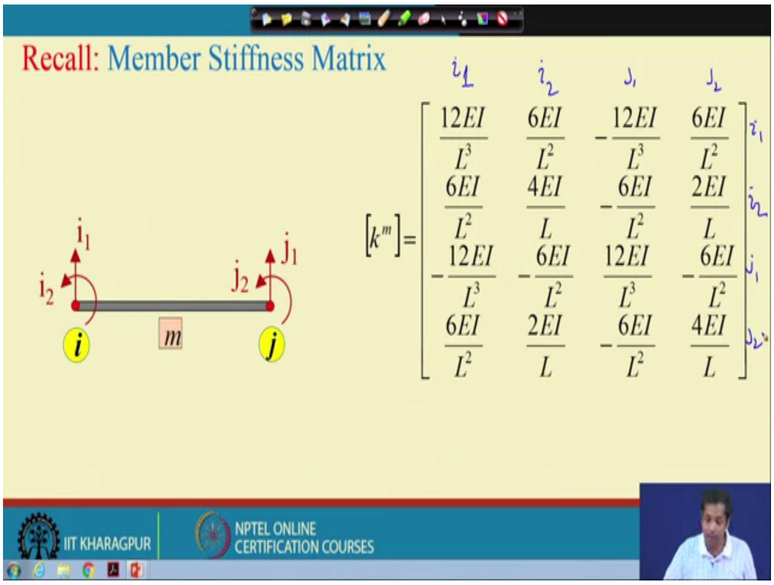

Now, once we have that so next step is calculation of member stiffness matrix I mean, if you recall for any arbitrary member say m, which is connected between node i and j; and their degrees of freedoms are i 1, i 2 and j 1, j 2 at i j node respectively. Then this is the member stiffness matrix.

Now, in these member stiffness matrix, if you recall this is associated with node number 1 degrees of freedom, degrees of freedom i 1 i 1, this is i 2 i 2, this is j 1 and this is j 2. Similarly, this is i 1 this is i 2 this is j 1 and this is j 2. Now, i j and consequently i 1 j 2 will be different for different elements.

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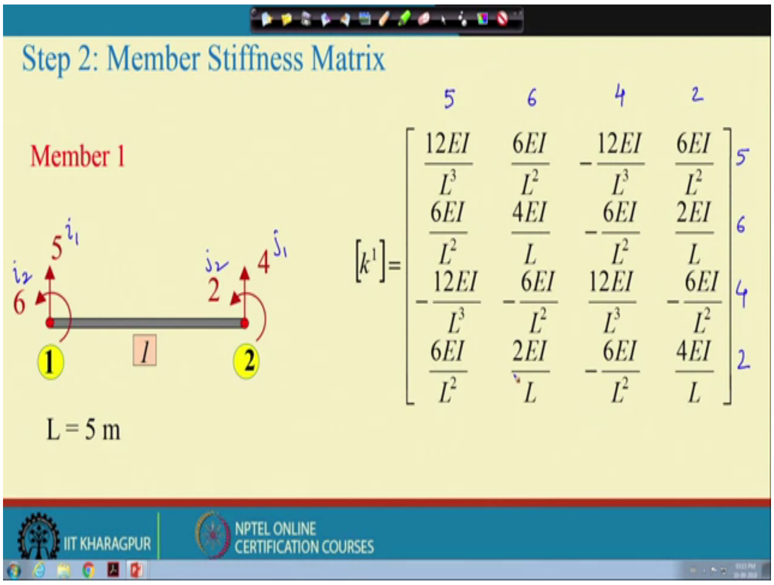

Now, let us first take the member number 1; this is the member number 1 is member number 1 is connected between node number 1 and node number 2. And node number 1, your degrees of freedom is 5 6 and node number 2 degrees of freedom is 4 2. So, essentially this is u, this is i 1 or just this is i 1, this is i 2, and this is j 1, and this is j 2.

Now, so this is the stiffness matrix for this member, for this member L is equal to 5 meter; E I for both the member the flexural rigidity E I, let us assume for both the members it is constant. Well in the second example, we will see if E I is not same for different segments then what is to be, how it is to be taken into account.

Now, then this will be for 5s, this is for 5, this is for 6, and this is for 4, and this is 2. And similarly, this is (Refer Time: 05:23) degrees of freedom 5, this is for 6, this is for 4, and this is for 2. Now, if we substitute values of L here, and take E I constant E I outside of the matrix, then your the expression of this will be like this.

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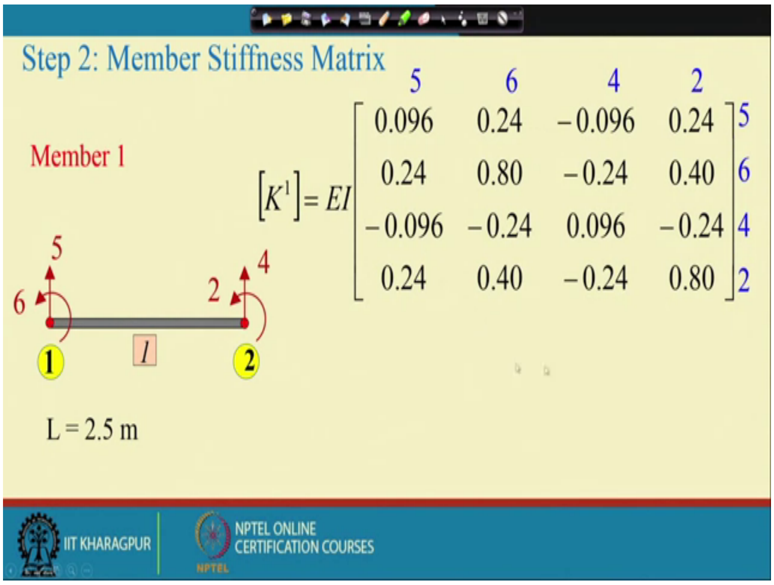

This is 5, 6, 4, 2; 5, 6, 4, 2. So, this is the stiffness matrix for member 1.

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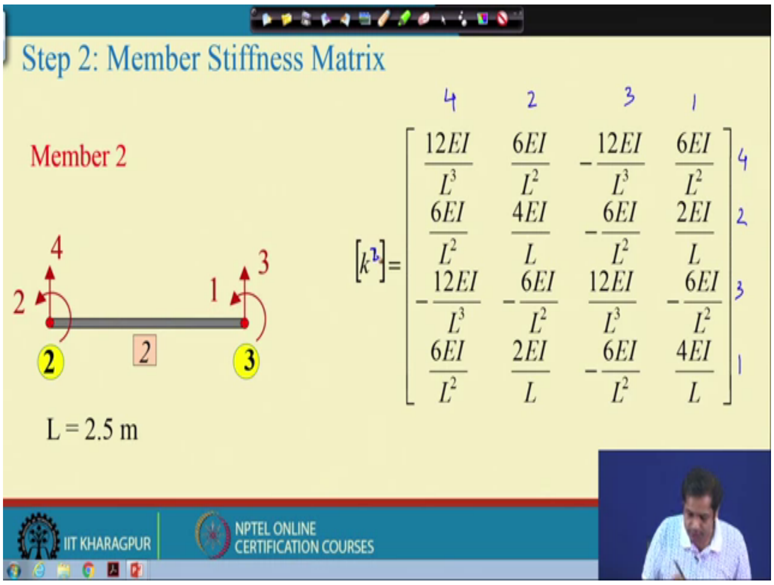

Now, step 2 is now stiffness matrix for member 2, please correct it, it should 2.5 meter for member 2, and this should be 2. And, if you recall the discretization member 2 is connected between node number it should be node number 2 and 3; it should be node number 2, and it should be node number 3 please correct it.

And the corresponding degrees of freedom is 2 4 a 4 2 and 3 1. So, this is essentially i 1, this is i 2, this is j 1 and this is j2 for member 2. So, this is the stiffness matrix now again, if we can substitute L is equal to 2.5 here take E I outside and get the corresponding member stiffness matrix for member 2. So, in this case so this will be 4, this will be 4 and this will be 2, this will be 3 and this will be 1. So, 4, 2, 3, 1 this is how this stiffness matrix is defined.

Now, if we substitute that then the corresponding stiffness matrix will be, if we Shiva, [FL] Hello.

Sir [FL].

[FL].

[FL].

[FL] 1 2 3 2 3 2 [FL] 4 2 3 1; 4 2 3 1[FL]. Now, you can substitute the values of L here, take E I outside the matrix, and we get the corresponding matrix for element number 2 stiffness matrix for element 2.

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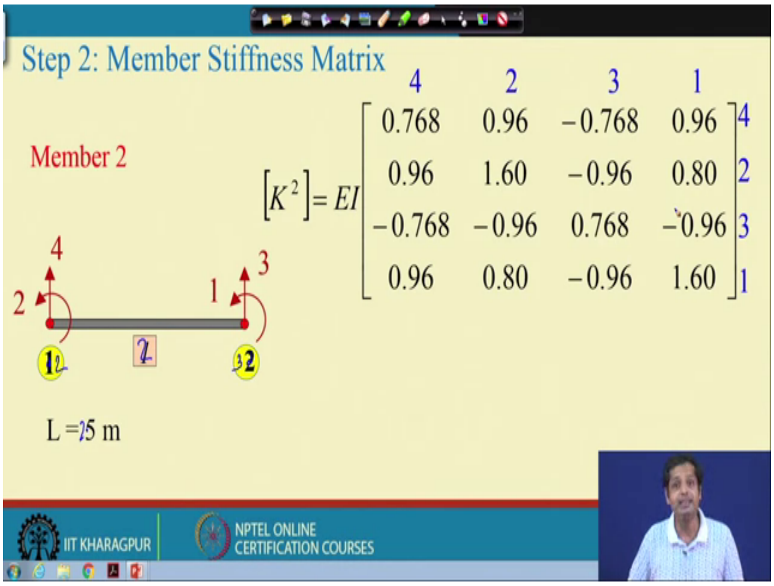

And this if we do that exercise, the final stiffness matrix for this element will be this. So, please this is your, this will be 2, and this is again 3, and this is element number 2, and this is 2.5 mm. So, this is the stiffness matrix K 2.

Now, once we have this stiffness matrices for all the members. Then next the third step is to assemble those stiffness matrices and get the global stiffness matrix. Let us do that.

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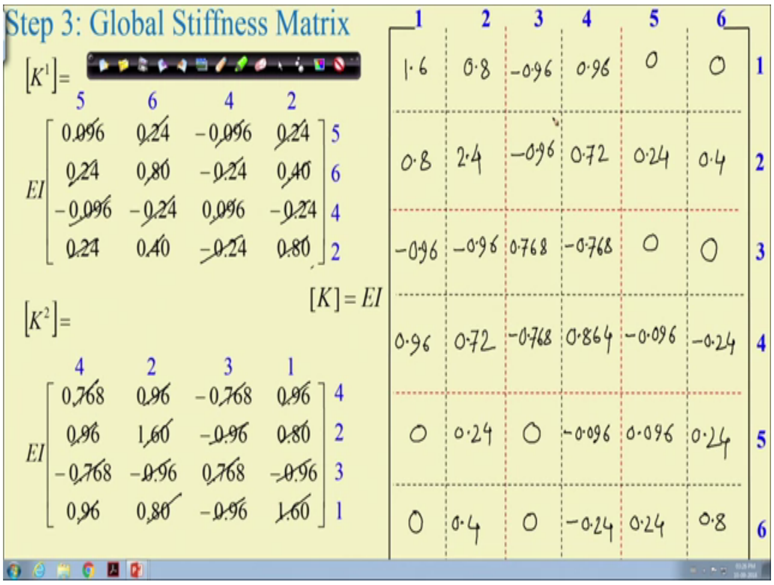

Now, so the third step is the assembling of stiffness matrix to get the global stiffness matrix. Now, this is the stiffness matrix K 1 and this is K 2. And remember this spot is very important, and the way we numbered this, it is you have to assemble this stiffness matrix based on that numbering.

Now, suppose these are the so since we have 3 nodes, and every node we have 2 degrees of freedom so, total 6 degrees of freedom stiffness matrix will be 6 by 6. Let us, all the stiffness all the elements in the stiffness matrix let us assume 0, then we have to populate those, those corresponding rows and columns depending on their value in the depending on their value in the member stiffness matrix and the connectivity of different members.

Now, let us first find out what is 1 1. So, yes now 1 1 if we take, then this will be 1 1, we have 1 1 here it is 1.6; so this is 1.6, we have E I outside. Then 1 2 we have here it is 1 and then 2 this is 0.8; so this is gone and this is 0.8, 0.8. So, this is we have taken.

Now, then 1 3 1 3 we have here, there is no 1 3 here; 1 3 is minus 0.96, so this is also taken. Then 1 4, 1 4 there is no 1 4 here; 1 4 it is 0.96. 1 5 there is no 1 5, so straight away it becomes 0; 1 6 straight away becomes 0.

Now, let us take the second column, second row 2 1. 2 1 there is no 2 1 here, so 2 1 this is 0.8. And then which is obvious, because 1 2 should be equal to 2 1 this is symmetric matrix, but when you assemble it to start with when you were writing the program first time to start, we do not take this symmetric, this property do not assume that property. Let us assemble them, and then check whether the matrix, you are getting whether that is symmetric or not that will be a check, the accuracy of your correctness of your assembling.

Now, let us say in the 2 2 there is 2 2 here which is 0.8 here. And then 2 2 we have 2 2, 1.6 so it is 0.8. And then 2 2, 1.6 becomes 2.4 so these becomes 2.4. Similarly, 2 3 2 3 there is no 2 3 in K 1, 2 and then 3 2 3 becomes minus 0.96 minus 0.96.

Then 2 4; 2 4 is 2 4. And then there is there is there is 2 4 here, this is 2 4 so 0.96 minus 0.24. So, these becomes 0.72, 0.72. Then 2 5, we have there is no 2 5 here. So, 2 5 becomes 0.24. And then finally, 2 6 there is no 2 6 here 2 6 is 0.4,. So, this is for second row.

Let us do the same thing for third row. Third is 3 1, 3 1 is 0 point minus 0.96 minus 0.96. And then 3 2 will be 3 and then 2 again minus 0.96, 0.96. Then 3 3; 3 3 will be this is 3 3, 3 3; there is no 3 3 here, 3 3 will be 0.768. And then 3 4, there is no 3 4 here, 3 4 is minus 0.768 minus 0.768. 3 5 there is no 3 5 here there is no 3 5 here. So, 3 5 will be 0; and then again 3 6 will be 0.

Do it for row, fourth row send 4 1, 4 1 0.96. 4 2 4 we have 2 0.96 here. And then again, 4 2 4 4 2 minus 0.24; so 0.96 minus 0.24 become 0.72. And then we have 4 3, 4 3 there is no 4 3 here, 4 3 minus 0.768.

Then 4 4, we have 4 4 here is this is 4 4, 0.768; and then we have another 4 4, 0.096, if we add them we get 0.864. 4 5; 4 there is no 4 5 here in K 2, then we have 4 and then 5 0.096 minus 0.096 and then 4 6, 4 and then 6 minus 0.24.

Then fifth row, 5 1 there is no 5 1 here, 5 1 there is no 5 1 here straight away it become 0, 5 2 these there is 5 to here which is 0.24. And then 5 3, there is no 5 3 here there is no 5 3 here, this becomes 0. Then 5 4 there is no 5 4 in K 2, here 5 4 is minus 0.096. And then 5 5; 5 and then 5 is 0.096, 0.096 and then finally 5 6; 5 6 0.24.

And the last row sixth row, 6 1 there is no 6 1 here, there is no 6 1 here straight away it becomes 0. 6 2 6 2 there is no 6 2 here 6 and then 2 is 0.4, 6 2 is 0.4. And then 6 3, there is no 6 3 here, no 6 3 here this is becomes 0. 6 4, 6 4 minus 0.24 minus 0.24 and then 6 5; 6 5 is 6 and then 6 5 is 0.24. And finally, 6 6; 6 6 is 6 6 0.8.

You look at all these elements we have considered in these members stiffness matrices both the member stiffness matrices and this is the final global stiffness matrix. Look at this global stiffness matrix, you see this matrix is symmetric. And another important part is all the diagonal entries in these stiffness matrix is positive.

Now, I leave it to you to know, we will discuss that sometime in the in the subsequent ways why this property, what is the interpretation of this property of stiffness matrix. But so you try to find out why the all the elements in this stiffness matrix, diagonal element of this stiffness matrix always has to be positive. Now, so this is the global stiffness matrix, so step 3. Now, once we have that global stiffness matrix, so finally your global stiffness matrix look like this.

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Step 3: Global Stiffness Matrix

$$[K] = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1.60 & 0.8 & -0.96 & 0.96 & 0 & 0 \\ 0.80 & 2.4 & -0.96 & 0.72 & 0.24 & 0.4 \\ -0.96 & -0.96 & 0.768 & -0.768 & 0 & 0 \\ 0.96 & 0.72 & -0.768 & 0.864 & -0.096 & -0.24 \\ 0 & 0.24 & 0 & -0.096 & 0.096 & 0.24 \\ 0 & 0.40 & 0 & -0.24 & 0.24 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

This is the global stiffness matrix. Now, this is corresponding you see you may give a numbering in any form, but when you write the global stiffness matrix, you write this global stiffness matrix as per the degrees of in chronological order; 1, 2, 3 (Refer Time: 17:11) 1, 2, 3, 4 and then corresponding row 1, 2, 3, 4. Now, this is the global stiffness matrix.

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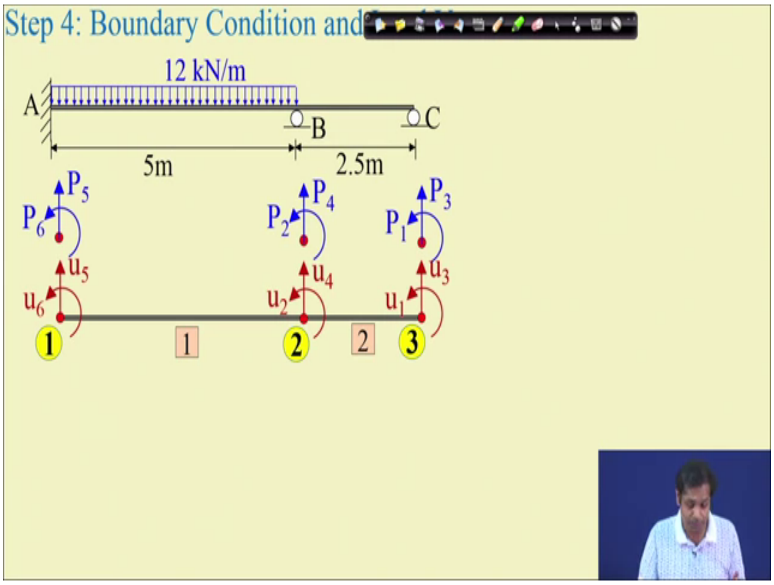

Now, once we have that the next step is to solve it means, now you see so essentially what you have to do is suppose now, these are the degrees of freedom, this is how we define the degrees of freedom. And these are the nodal vectors are equivalent joint load that we discuss in the last class. And then we have to solve this equation, these two are related to each other; related with each other as this.

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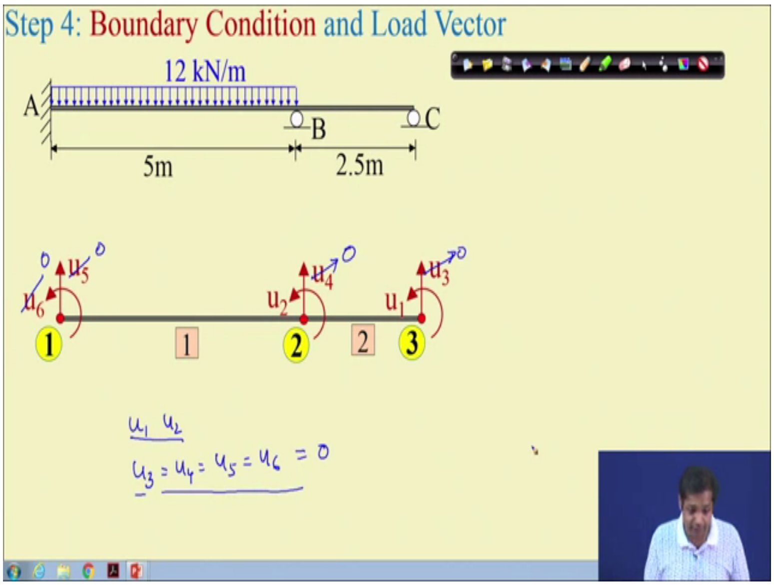

Before we solve it one important step, the step 4 is not the solution, step 4 is the boundary conditions and load vector. Let us, see what are the boundary conditions we have here. Once we apply the boundary condition, then the solution comes let us see what are the boundary conditions we have here. Now, you see this is the fixed end, so naturally u 5 will be 0; and u 6 will be 0.

And then this is constraint the so u 4 will be 0 and then u 3 which is also be 0. So, only unknown is u 1, so u 1 and u 2 are only unknown, and all other degrees of freedom u 3 is equal to u 4 is equal to u 5 and u 6, they all are 0. Now, you look at the advantage of numbering like this is the unknowns are u 1 u 2 and rest is u 3 u 4 u 5 6 are all are 0. So, it will help us to partition the stiffness matrix. Now, so once we have this the boundary condition let us now calculate the load vector.

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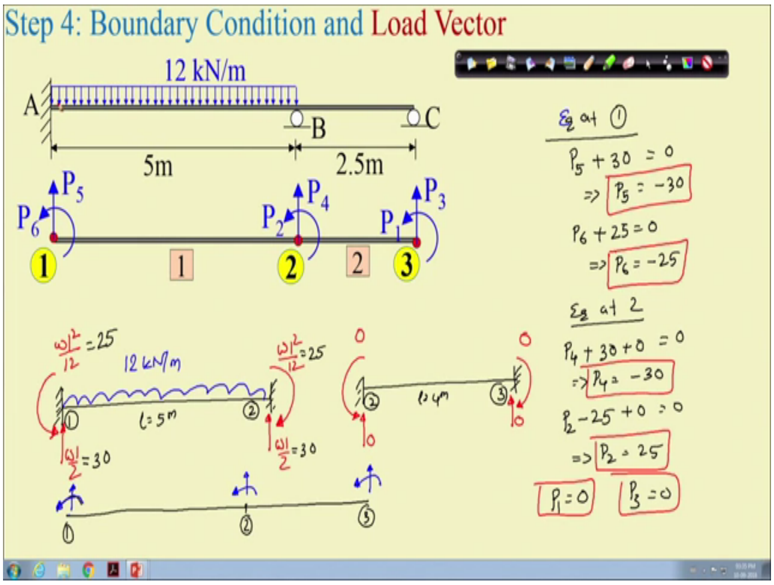

Now, load vector is if you recall, how do I calculate load vector, first we have to do first we have to find out the fixed end moments. Fixed end moments means, we have to we have to consider all these segments separately, and assuming their aims are fixed and calculate the stiffness, calculate their fixed end moments.

Now, what are the, so first segment will be like this, this is the first segment. And this end we assume these ends are fixed end. And then second segment is this, and we assume these ends are fixed end. Now, this is subjected to uniformly distributed load like this which is 12 kilo Newton per meter, and there is no load in this segment.

Now, this is your joint number 1, and this is joint number 2; and this is joint number 2 and this is joint number 3. And what would be the fixed end moments for this? Fixed end moments will be now if recall for a beam fixed beam with uniformly distributed load, the fixed end moments are w l square by 12. So, fixed end moment these will be w l square by 12. Similarly, fixed end moment these will be w l square by 12.

Now, recall our, as per our convention anticlockwise rotation and anticlockwise moment is considered as positive. So, essentially in that sense; here fixed end moment will be positive, here fixed end moment will be negative. Now, but then here it is this fixed end moment it will be 0, here also fixed end moment is 0 right.

Now, the reactions fixed end reactions, here the reactions will be w l by 2, here the reactions will be w l by 2; these reactions will be 0, and these reactions will be 0 right. Now, so, l is equal to in this case say l is for this problem l is equal to 5 meter; and this is l is equal to 4 meter, w is equal to 12 kilo Newton per meter.

So, if we substitute that value, that value comes around this comes 30, this is let us this comes 30 and this is 25. So, this is 30 and this is 25 and this is anyway 0 0. So, here fixed end moment is 25 in this case the fixed end moment will be minus 25, because it is clockwise direction, here the fixed end reactions are 30; in this case fixed end reactions are 30.

Now, we have to find out equivalent joint load now; if you recall the equivalent joint load will be at this point, if we draw this entire beam together, this is node number 1, this is say node number 2, and this is node number 3. And, if the equivalent joint loads are like this, we take anticlockwise direction as positive I use different color.

If we take this is our equivalent moment and corresponding this is equivalent force and this forces this so this will be the equivalent force. So, this is equal to P 1 this is P 2, P 3, this is P 2 and this is P 4 and this is P 5, and this is P 6; how do we get these values by satisfying the equilibrium at every joints right.

So, what is their equilibrium? Equilibrium will be now it is the total vertical reactions will be so total vertical reaction is this plus because of the fixed end action that should be equal to 0. So, if we satisfy the equilibrium at 1, equilibrium at node number 1 if we satisfy, then we have then we have; if we satisfy the equilibrium at node number 1, then what we have is this.

Then we have P 5, P 5 plus 30 is equal to 0, this gives us P 5 is equal to minus 30. Similarly, if we take P P 6, P 6 which is anticlockwise direction positive plus, this w l this 25 is equal to 0; so this gives us P 6 is equal to minus 25. So, this is the equivalent joint load at node number node number 5; at node number node number 1 and corresponding degrees of freedom 5 and 6.

Similarly, if we satisfy the equilibrium, at this say equilibrium at joint 2 if we satisfied; equilibrium at 2 if we satisfied, then what we have is this. Then we have P 4 plus we have 30, this is 30 plus from these join it is 0 is equal to 0; so P 4 is equal to minus 30.

And, similarly we have P 2 and this is this is anticlockwise direction positive; this is clockwise direction negative, minus 25 and from here it is 0 is equal to 0, so this gives us P 2 is equal to 25. So, this is the equivalent joint load at node number 2.

And similarly if we apply, if we apply equilibrium at node number 1, we get P 1, P 1 is equal to 0 and P 3 is equal to 0, these are all equivalent joint load. Please note that these equivalent joint loads are not reactions for instance at node A we have support reactions; at node B we have support reactions vertical direction force; at node C we have support reaction is the force in vertical direction. These equivalent loads equivalent joint loads are not the reactions.

The total reactions will be equivalent joint load, which is obtained by assuming there is no degrees of freedom at the end plus the contribution from the deformation. Actual beam will deform, and whatever is the contribution from the deformation that plus the fixed end moments will be the equivalent, will be your reactions. We will come to that come to that point. So, this is the equivalent joint load.

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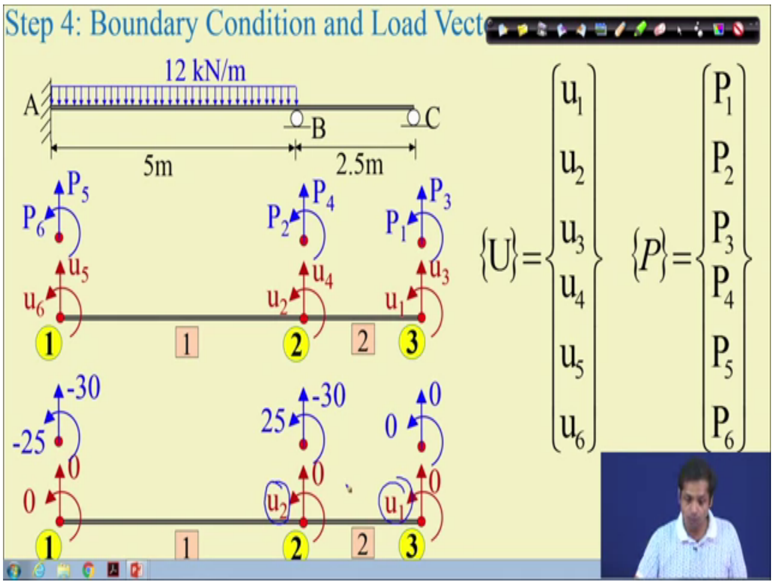

Now, next once we have the equivalent joint load, these equivalent joint loads are like this. So, these are the so U is a vector, and P is a P is a equivalent joint loads. So, what we have is, this is the equivalent joint loads we have. Now, only unknown here is this, these are the only unknown; u 2 is unknown and u 1 is unknown, we have to solve for these 2 unknown. Let us, let us do that so step 5, once we apply the boundary conditions and the load vector is constructed, step 5 is to solution for the unknown displacement.

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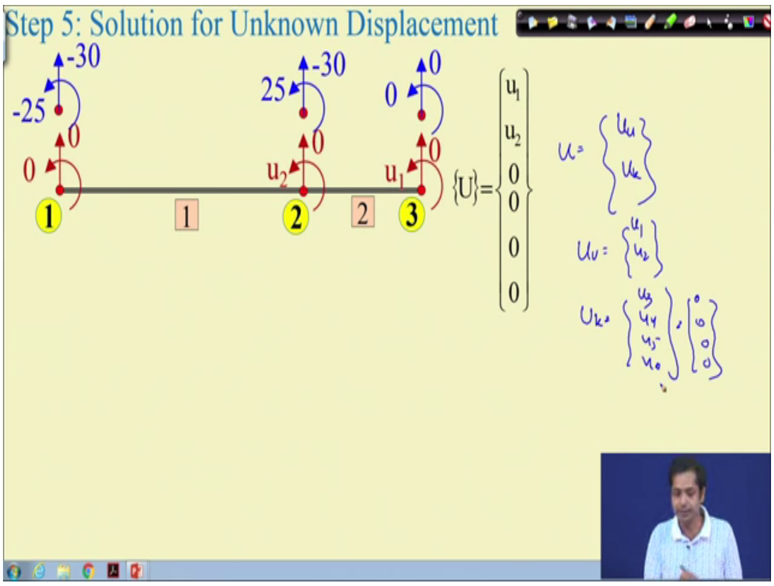

So, step 5 is solution for unknown displacement. Now, this is the displacement vector, all these displacements are known. Suppose, if you recall if we can write this, we can express u as; we can express u as; say u is equal to u unknown, and u known. And u unknown is equal to in this case, u 1 u 2; and u known in this case is u 3 u 4 u 5 u 6 which is 0 0 0 0 right. So, this is how we can write.

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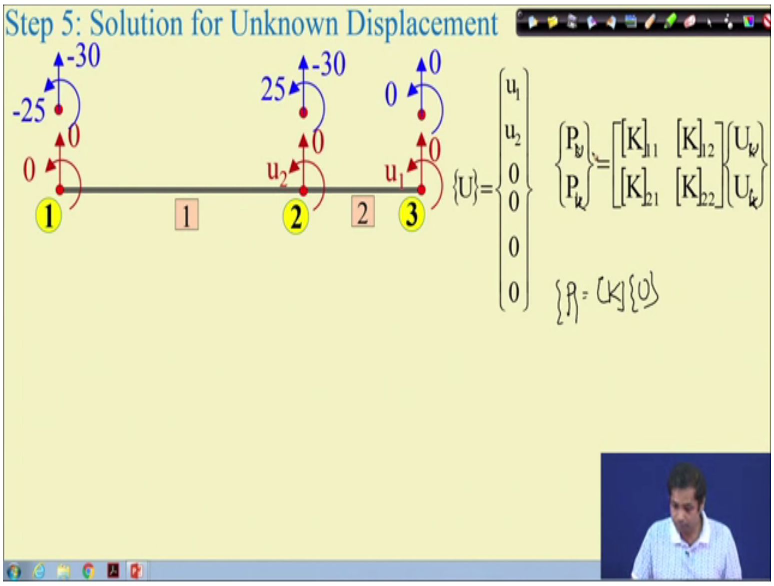

Now, so once we have that and then you recall the total stiffness matrix, eventually what we have, eventually we have these expression, we have these expression that P is equal to K into U right; where K is the stiffness matrix, lower stiffness matrix, P is the load vector and U is the displacement.

Now, this entire equation can be expressed in terms of this can be rearranged is such that the all the known displacement you can; all the known displace all the unknown displacements are unknown displacement these will be unknown displacement, this will be unknown displacement, these will be known displacement. This will be unknown displacement, this is known this is. So, this can be rearranged like this.

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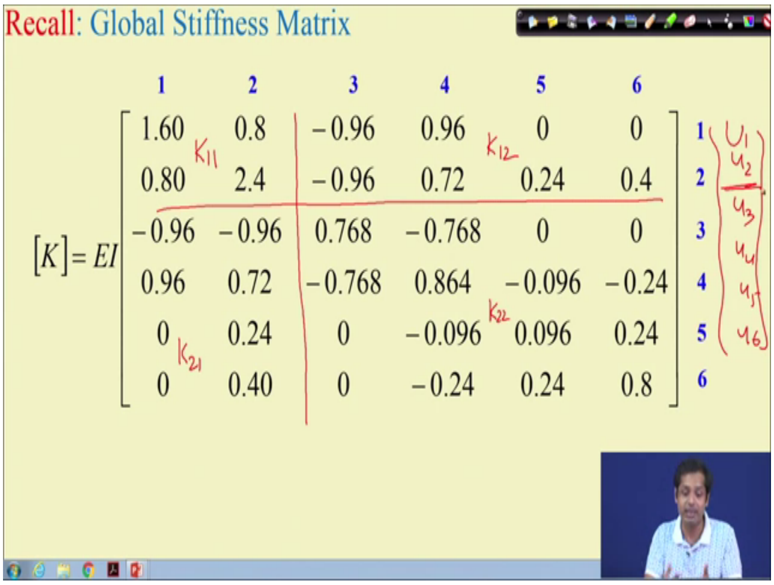

Now, and accordingly the stiffness matrix can be partitioned. So, if we have this you see here, this is the reason why we chose the numbering of degrees of freedom accordingly. Now, what we have to do is, we have to partition this matrix like this, because only u 1 and u 2 are unknown; so these will give us K 1 1, this will be K 1 2, this will be K 2 1 and this will be K 2 2. And this will be u 1 u 2 and then u 3 u 4 u 5 and u 6 all this displacement vector, partition of that displacement vector, similarly partition of that node vector.

So, in this case since you have chosen the numbering of the degrees of freedom in such a way that first two degrees of freedom are unknown, and the rest are known; in that case you have to only scooped out these first 2 rows and column from the from the stiffness matrix. And then the partition of the stiffness matrix will be easier you do not have to rearrange, the rows and columns. But if you choose the numbering of the degrees of freedom in other way, there is absolutely nothing wrong in that you can do that, but then when it comes to partitioning of this stiffness matrix we have to rearrange them; in terms of in such that or known displacement together and all the unknown displacement together.

So, it needs the rearranging of corresponding rows and columns, but in this case rearrangement is not that exercise is not required, because first two our degrees of freedom are unknown that is how we named the degrees of freedom, number the degrees of freedom.

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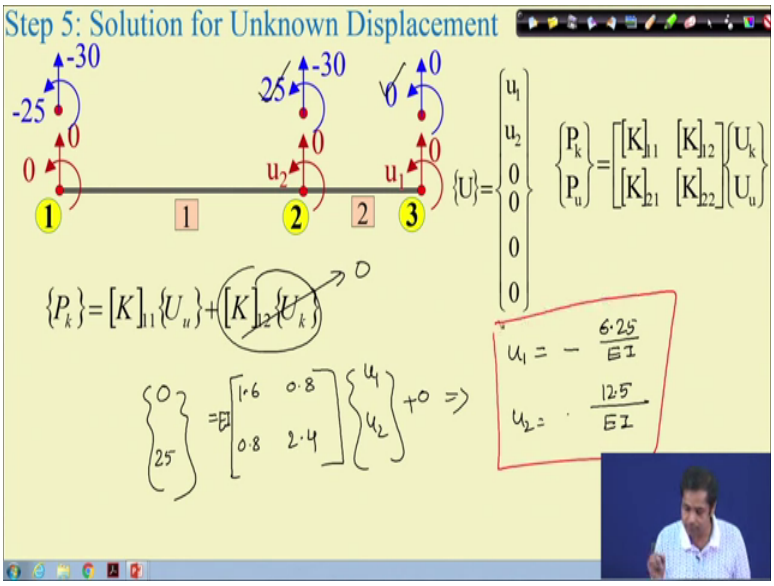

Now, once we have this then what is next, next is this equation, if you recall, so this becomes these becomes this right. Now, so if we substitute what are the values of this, what are the values of this, so this becomes if we recall these equation becomes, your stiffness matrix is the first is 1.6 and then 0.8; and then 0.8 and 2.4. This is and then u unknown, this is u 1 and u 2 plus these U known is 0.

So, these part the entire part will be 0 this is equal to so this is corresponding to that degrees of freedom u 1, so this is 0 and then this is this is 0, these value and then corresponding u 2 this is 25 this is this.

Now, E I so, this is the equation we have to solve this is. So, if we solve them, then the final results we get u 1 is equal to minus 6.25 by E I. And then u 2 is equal to 12.5 by E I. So, this is the solution for unknown degrees of freedom.

Now, we have all the degrees of freedom are known. 0 degrees of freedom u 3 u 4 u 4 5 6 and u 1 u 2 which were unknown by solving them we get this degrees of freedom. Now, once we have the degrees of freedom unknown displacement, next part we have to find out is the support reactions, but remember these equivalent joint loads they are not support reactions. They are the support, they are the fixed end reactions right, where we assume your degrees of freedom are constrained, but in actual structure degrees of freedom are not constrained.

So, because of the deformation there will be some reaction, some contribution to the reactive forces. So, the total reactions will be this plus the effect of deformation so, how to find out that we will discuss in the next class. So, we stop here today. Next class, we will continue with this example. And then, see how to find out the support reactions, and then we also demonstrate this method through another example; where the flexural rigidity, are not constant for or not same for different segments. I stop here today; see you in the next class.

Thank you.