Matrix Method of Structural Analysis
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Lecture – 21
Matrix Method of Analysis: Beams

Hello everyone, today, we are going to start the 5th week of this course. So, this week's topic is Matrix Method of Structural Analysis of Beam. Particularly in this lecture, we will talk about members stiffness matrix. You see by now, I believe you already have you already have a sense of what is the philosophy behind matrix method of structural analysis and what are the steps involved in this method just to recall all these steps.

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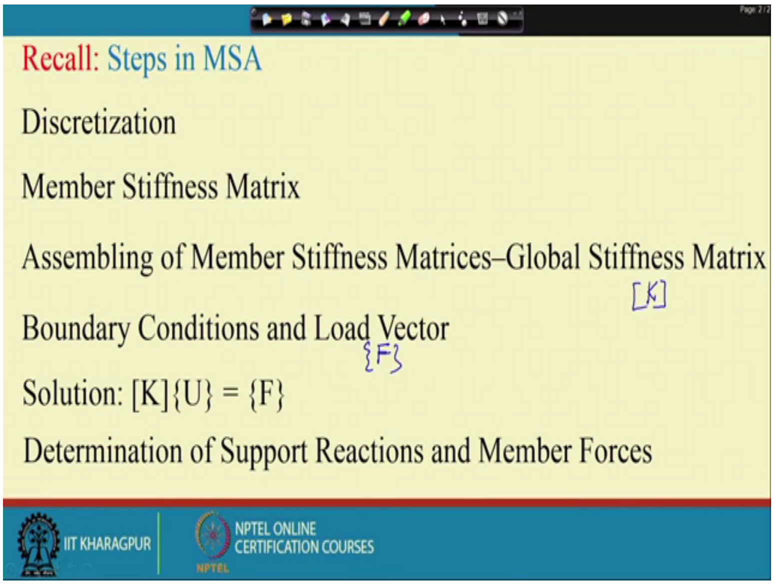

The summary of this is if you if a structure is given to us, then the first thing we do is we discretize the structure right. Discretize the structure means; we make the structure, break this structure into pieces. And then it also involves your numbering of the, we need to identify the members we need to identify the joints and then number them accordingly in discretization.

And then once we discretize this structure, we calculate the stiffness matrix construct the stiffness matrix for every elements every member of the structure. Once this stiffness matrix is for all the members are constructed are written, the next thing is to impose the condition of the connectivity of different members, how the different members are connected to each other and that we do by assembling this stiffness matrices. And that step, after the assembling the stiffness matrix that we get is called global stiffness matrix. This is this stiffness matrix of the entire structure.

Now, once the global stiffness matrix is obtained. The next step is we have to then apply the boundary conditions and write the load vectors recall the global stiffness matrix. If we try to invert them, it is not invertible. It is a singular matrix, which is the physical manifestation of this is if there is no constraint in the structure, the structure is unstable. Now, we you have to make it stable, in this case, we have to provide constraint. And then once the constraints are provided, then we also need to identify write the load vectors.

And then what we get is we get the next step will be solution of this solution of we write the all these the system equations linear equations, and finally we solve them. Now, this is the global stiffness matrix K. And this is the after the boundary conditions, load vector is F. After the boundary condition, your stiffness matrix is reduced. And this is the load vector. And then, we solve these equation solve for the unknown displacements, once we have the unknown displacement. Next, next step is to find out other support reactions in the member forces. So, this is the basic steps involved in matrix method of structure analysis.

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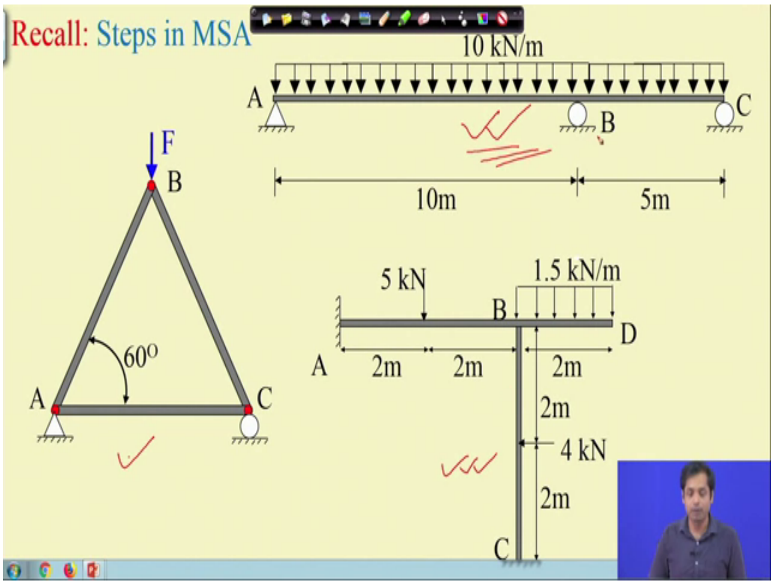

Now, we already have demonstrated these steps for truss problem. This is the problem that we discussed in details in the previous in the last week, this week. This problem, we have discussed already in the last week. Now, this week we discuss how to apply all these steps that we have we just now, we revisited how to implement all these steps in the context of these kind of problem the beam problems. And the next week, we will do the similar exercise for frame problem this one so, let is start with this problem ok.

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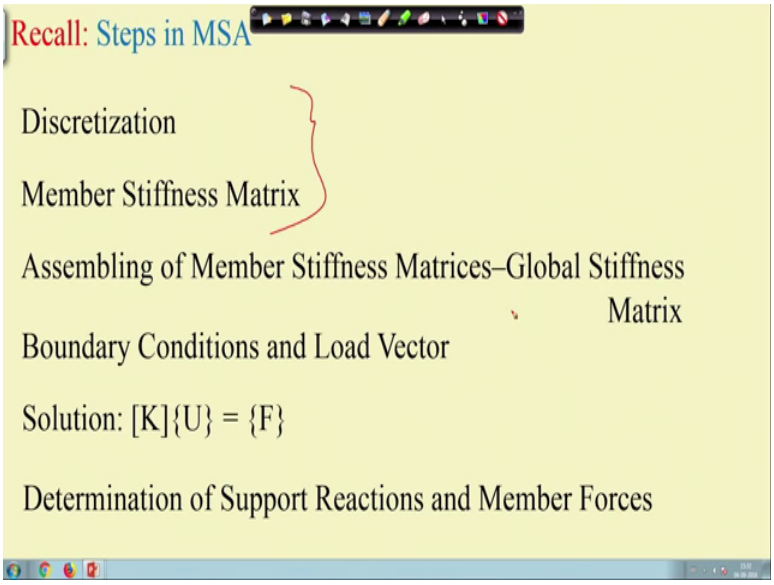

Now, in this week, in this lecture today particularly, we will talk about these two steps the discretization, these two step for this week, discretization and member stiffness matrix. Next week, we will see next class we will see how to assemble this stiffness matrices ok.

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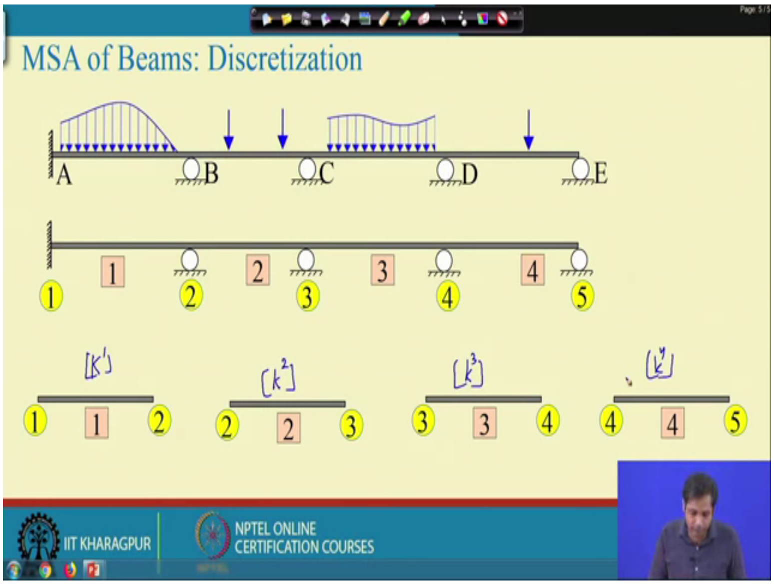

So, let us demonstrate the entire process through one example. Suppose, this is a problem, it is just an representative problem, you can have a any problem in beams ok. See, one good thing about beam problem is unlike trusses and also unlike frame. You see in the truss, your members are oriented in a different direction, it could be vertical member, inclined member even in frames, you have members oriented in different configuration.

And therefore, once you construct the matrix in local coordinated system, we need to transfer them with respect to the global coordinate system right. But, here that step that exercise is not required, because the beam always your orientation of the orientation, where the member is all in unidirectional ok in a particular direction all this members.

Now, so this is take any representative problem, this is the problem. Since, we are talking about how to calculate this stiff matrices first, let us remove the load loads we will come to that point later, so remove the load. And the next is identify the different members and the joints and number them accordingly. The members are this is say member number 1, member number 2, then member number 3 and member number 4. And then, similarly we have join number 1, join number 2, join number 3, join number 4, and join number 5.

Now, depending on the structures we can have different number of members, different number of joints ok. So, once we have this, then in this problem we have 4 members. So, let us isolate all these 4 members, let and then if we do that, so these are all four members. The member number 1, the connective between connectivity is very important member number 1, the connectivity is it is between node number 1 and 2. Member number 3, it is between node number 3 and 3 3 and 4 and member number 4 is between 4 and 5 ok.

Now, once we have that what we have to do is, we have to write stiffness matrix for each such member. We have to write this stiffness matrix for this one say this is k 1 and we have to write stiffness matrix for k 2 and k 3, and so on, k 4 and so on ok. Now, let us take any arbitrary member any arbitrary member m, which is connected between two points say ith point and jth point.

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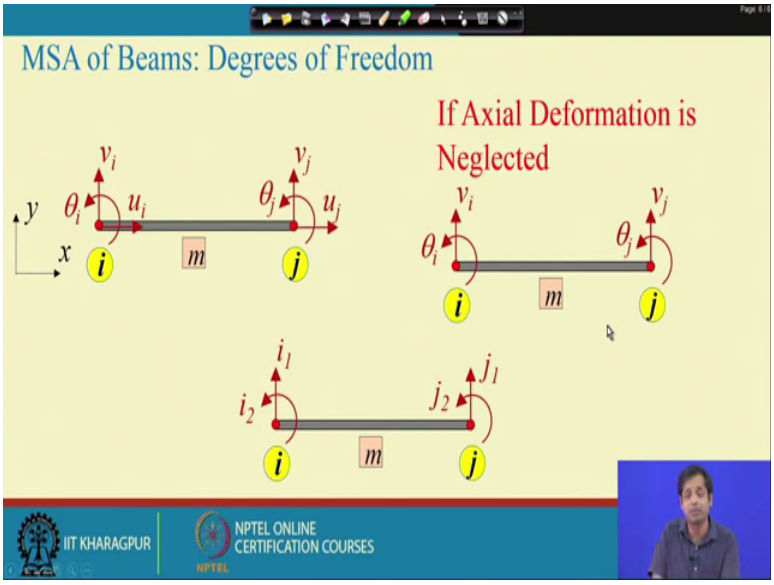

So, if we do that, take any arbitrary member say m, this is the connecting the connected between the connected between i and j points. Now, before we see how to what are the elements of this stiffness matrix, it is important for us to understand or to define what are the degrees of freedom it has?

You see in if you know in two dimension and any object any point if we take, it have 3 degrees of freedom right. It can displace in these direction, it can displace in these direction and it can rotate like this. So, two translation in two directions and rotation about the third axis. These are the three degrees of freedom we can have.

Now, let us take these degrees of freedom at ith node we have three degrees of freedom, and jth node also we have three degrees of freedom. So, suppose these three degrees of freedoms are u i, v i, theta i. u and v i are the translation in x and y direction at ith point. And theta i is the rotation with respect to the z axis.

And similarly, at j point we have these 3 degrees of freedom, please note here, the sign convection that we are taking. Sign convection is your in vertical the translation in upward direction is taken as positive. And then rotation about rotation in anti-clock wise direction is taken as positive, rotational movement an anti-clockwise direction is taken as positive.

You can have your own sin convection, but whatever sin convection you take, you have to you have to construct the stiffness matrix accordingly ok. We have to be consistent with that sign convention. So, this is the sign connection that we will be using for formulating the method.

Now, here we make an assumption, assumption is there is no actual deformation. If you see is equal in when we talked about truss, truss is all members are or all members are two force members. Essentially, every member is subjected to either compression or tension along the longitudinal axis so, it has just actual deformation.

Now, in this case, no other deformation in truss member, but in this case we assume there is no actual deformation takes place. So, if there is no actual deformation takes place though, then these degrees of freedom we can neglect ok. So, if we neglect the degrees of freedom associated with the actual deformation along the length of the beam, then we are left with only two degrees of freedom per node.

When we talk about frame problem, then we will not neglect the actual deformation. We will consider the actual deformation, this is for beam we neglect the actual deformation. We will also see that if we have to consider actual deformation, what would be the changes required in stiffness matrix. But, for the time being there is no actual deformation, so that degrees of freedom we can neglect. So, if we neglect that, then we are left with only two degrees of freedom per node that is translation in vertical direction and then rotation about z axis.

Now, let us just for the ease of writing these expression, let us not use v i, theta i, and v j, theta j and so on. Let us use i 1, i 2 and j 1, j 2 so, i 1 is actually v 1 and i 2 is actually theta i, and j 1 is v j, and j 2 is theta j. Even more even simplifying that we will not let us not even use i 1, i 2 and j 1, j 2, because writing expression every time i 1, i 2, j 1, j 2 would be tedious.

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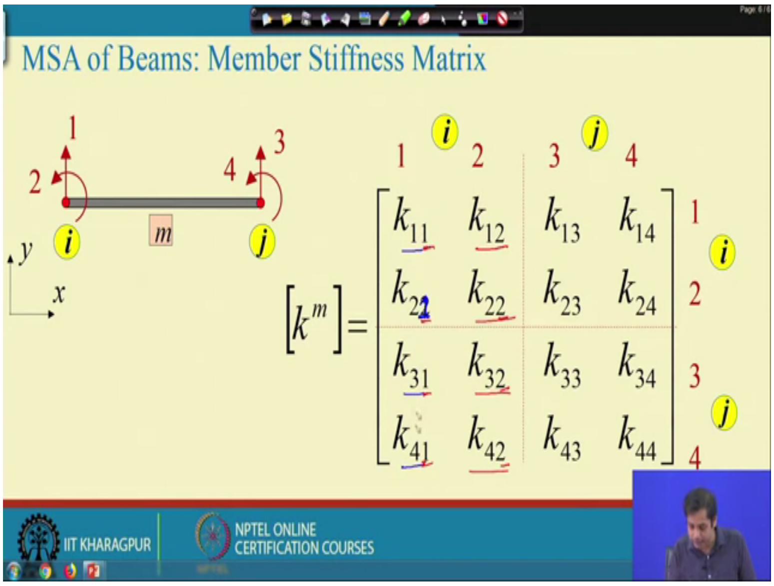

So, let us take 1, 2 and 3, 4, so, 1, 2 is essentially the degrees of freedom at node i, and then 3, 4 are the degrees of freedom at node j. So, 1 and 3 are the translation and 3, and 2 and 4 are the rotations ok. So, it will be easier for us to write the elements of those stiffness matrices representing them.

Now, what we are interested is to determine, what is the form of the elements stiffness, member stiffness matrix of this given member? Now, for different members these nodes will be different, so with the same expression, we will find out a general expression. And then for different members differing on their depending on their flexural rigidity, depending on their length, we can have different values of this stiffness matrices.

But we are now in a process of deriving a general form of stiffness matrix. So, by just looking at this structure, we can say the stiffness matrix size will be 4 by 4, because we have total 4 degrees of freedom. And then the form of this stiffness matrix will be something like this. So, these members stiffness matrix these are the k 1, k 2, k 1 1, k 1 2 and these are the elements of this stiffness matrix.

Now, these elements understanding these elements is very important, we already discussed in the previous week. Just we again for better comprehension of the all the discussion to follow, let us again revisit that you see k 1 1 is essentially what is definition of stiffness is a load per unit displacement right.

Now, k 1 1 is essentially this one is if I apply a unit displacement along the degrees of freedom 1 in this case the translation, then k 1 1 is essentially is the forces developed in the 1st degrees of freedom. k it would be k 2 1 please correct it, then k 2 1 will be the force developed along the degrees of freedom 2.

k 3 1 will be the force developed along the degrees of freedom 3. And k 4 1 will be the degrees of force developed along the degrees of freedom 4, but all these will be due to the unit value of degrees of freedom 1 ok.

And similarly, if I give an unit rotation, supposing the degrees of freedom 2, if I give unit value along the degrees of freedom 2 degrees of freedom 2, then the force developed along the degrees of freedom 1 will be k 1 2. Force developed along the degree of freedom 2 will be k 2 2, force developed along the degrees of freedom 3 will be k 3 2 and force developed along the degrees of freedom 4 will be k 4 2.

And similarly, all other elements are defined like this. So, what we are now interested to find out these elements. So, what we do is now remember one thing, when I say that k 1 1 is the or for instance say k 2 1. It is the force developed along degrees of freedom 2, due to the degrees of unit value of degrees of freedom 1, when we say that it means all other of freedoms the value all other degrees of freedoms are constrained ok. So, what we do is we now find out all this expression, all the elements of this expression ok let us do that ok.

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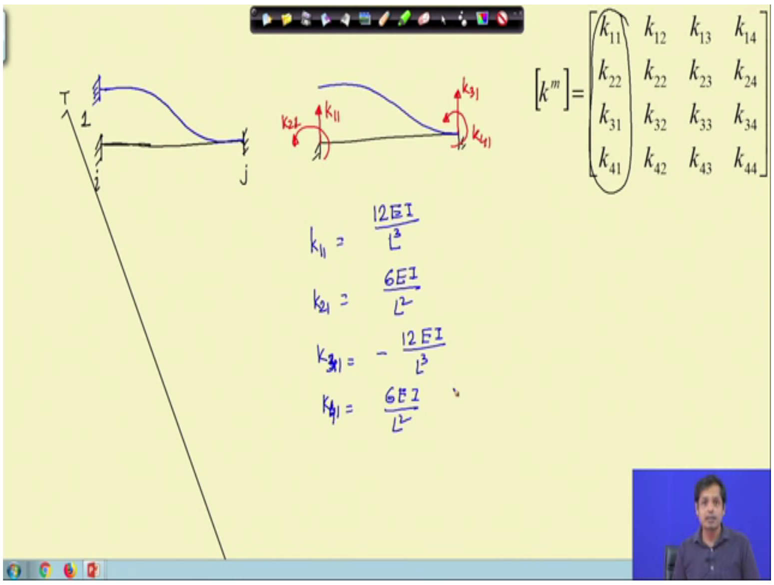

So, this is the stiffness matrix, we have just now discussed what are the what are these what is what are these values means, let us now let us now find out. Suppose, first we will give an unit a unit value along the degrees of freedom 1 and then find out what are the forces developed in other degrees of freedom means, we will try to find out all these values ok.

Now, so suppose the beam is this and, if you do that, all other degrees of freedom will be constrained ok. Suppose, this is the all degrees of freedom first we constraint and now this is ith point and this is jth point ok. Now, next what we do is next we give an unit displacement along degrees of freedom 1, so this is degrees of freedom 1. So, along if we give unit displacement, suppose the deformed shape will be like this ok. And these value is 1 and this is 1, so these value is 1 these value is 1 ok.

Now, next is what we have to do is, once we have this degree once we give that unit displacement along this one. The next what we have to do is next we have to find out what are the forces developed in along a different degrees of freedom. Let us see, what are the forces developed along degrees of freedom?

Now, again draw the same thing suppose, this is the un deformed configuration once again and corresponding deform configuration will be like this ok, these value is 1. Now, let us see what are the forces developed. The forces developed will be these force and then a rotation will be developed here similarly, these force and a rotation will be developed here ok.

Now, these value is unit displacement and the because of these displacement what is the corresponding what is the corresponding forces developed here will be k 1 1 that is the unit displacement, because of the unit displacement. Similarly, what is the force developed along these degrees of freedom, it will be k 2 2 and this will be k 2 1, this will be k 3 1, and this will be k 4 1 ok.

Now, what we have to find out, we have to find out what are these values of k 1 1, k 2 1, k 3 1, k 4 1. We will not determine, we will not analyze the structure and find the value of find the values of these stiffness, these elements of this different elements. We already have structural analysis one, where you have studied already, how to determine indeterminate structure, determinate structure you studied different methods like slope deflection method, consistent deformation method, you can apply any method to solve this structure.

This problem is you take a fixed beam, apply some unit translation at 1 joint. And then find out the forces developed, what are the reactions developed at other joints. And if we do that, then the values that we will get is like this. See k 1 1 will be k 1 1 this will be the reactions the vertical reaction, these value will be 12 E I by 12 E I by L cube.

And then k 2 1 k 2 1 is moment developed at the fixed energy at the at the ith end, this will be 6 E I by L square. And then, similarly k 4 1 will be minus 12 E I by L cube and k 3 1 will be this will be k 3 1 the k 3 1 and k 4 1 will be 6 E I by L square. So, you can please do this exercise and then check whether these values are you are getting on these values are not. What you have to do is, you have to take a fixed beam. And apply you already have done in a slope deflection method or mogue three moment theorem. If there is a support settlement then how do, what is the effect of support settlement, it is exactly that.

You have a fixed beam; you give a support displacement or support settlement at one particular support. And find out due to that due to the unit displacement, unit settlement of support what would be the corresponding reactions at different tension. If we find out those reactions and those reactions will be these components of the stiffness matrices ok.

So, this is for first column so, this column correspond to the forces developed along different degrees of freedom due to the unit value of degrees of freedom 1. Let us, now do it let us now give unit value of degrees of freedom 2 and find out what are the forces in different what are the forces developed in other along other degrees of freedom, do the same exercise.

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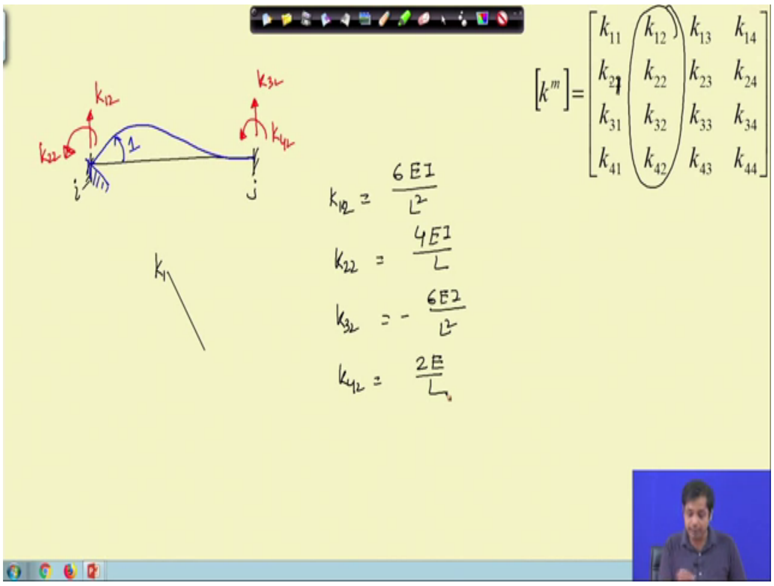

So, now we are interested in to find out this column ok this is 2 1 ok. What you have to do is again first draw the un deformed configuration, un deformed configuration is this. And this is fixed end and then we have to give a unit displacement, recall our unit displacement second degrees of freedom, second degrees of freedom is rotation. Recall our sign convention is rotation along anti-clockwise direction is positive.

So, let us rotate it and when we rotate it, when we apply unit value along degrees of freedom 2, then all other degrees of freedom will be 0 ok. So, if we do that, it will be something like this. So, essentially the support this is something like this ok this value is unit rotation, this value is 1 ok.

Now, this is ith point this is ith point and this is jth point. Now, if we have this, then what would be the corresponding forces developed, the corresponding forces developed will be, you have a vertical reaction in this direction. And then you have corresponding moment in this direction, we have vertical reaction in this direction and corresponding moment in this direction. Now, as per definition, because of this unit displacement at 2 as per definition then these values will be k these will be k 1 2and then these will be k 2 2, and these value will be k 3 2, and these finally will be k 4 2 these entire column.

Now, again if we apply any method that we have learnt in structural analysis 1, apply unit rotation and find out the reactions at different points and if you do that, the values that we will get is I am just writing the values k 1 2 will be say k 1 2 will be will be 6 E I by L square L square, k 2 2, which is the moment this will be 4 E I by L. And then k 3 2, this vertical reactions these vertical reaction this will be 6 E I by minus 6 E I by L square and final k 4 2 will be 2 E I by L ok. So, this I leave it to you to check the values, so this is for the 2nd column.

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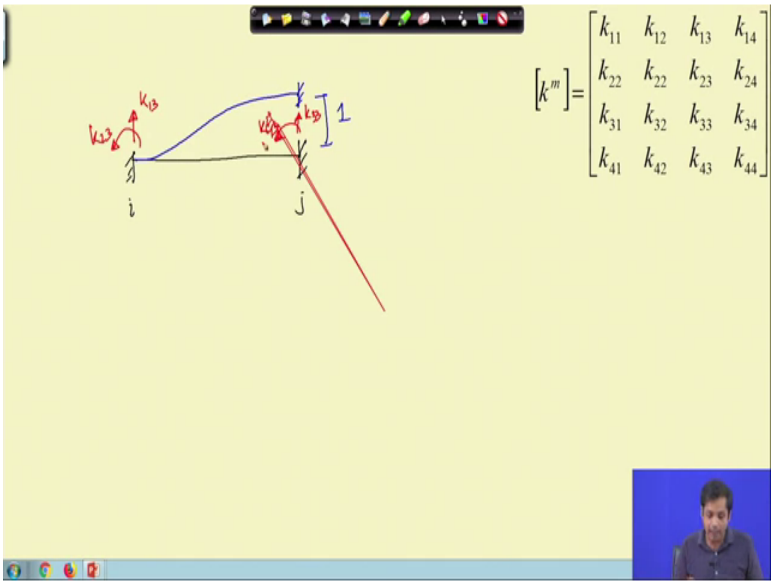

Now, let us do the do it for 3rd column, 3rd column is what third column is give unit value along degrees or degrees of freedom 3 and find out what are the reactions along other degrees along all the degrees of freedom. So, 3rd will be again the same thing if we draw the un deformed configuration, this is the un deformed configuration, we have to give the degrees of freedom unit value along degrees of freedom 3.

Degrees of freedom 3 was it is ith point, it is this jth point, degrees of freedom 3 was the translation vertical direction at jth point. Then draw the corresponding deformed shape, the deformed shape will be so these value is 1 right. If these value is 1, then we have a force and then reactions.

And similarly, we have force and reactions we have the force in these direction and reaction in these directions right. And these values will be this will be k 1 3 k 1 3 the force developed along degrees of freedom 1 due to unit value of degrees of freedom 3. This will be k 2 3 force developed along degrees of freedom 2 due to unit values of degrees of freedom 3, k this will be k 3 3, this is force developed along degrees of freedom 3 due to the unit values of degrees of freedom 3. And finally, this will be k 4 3ok so, these are the all the values and similarly, you can solve it and find out the expression of different these stiffness coefficients.

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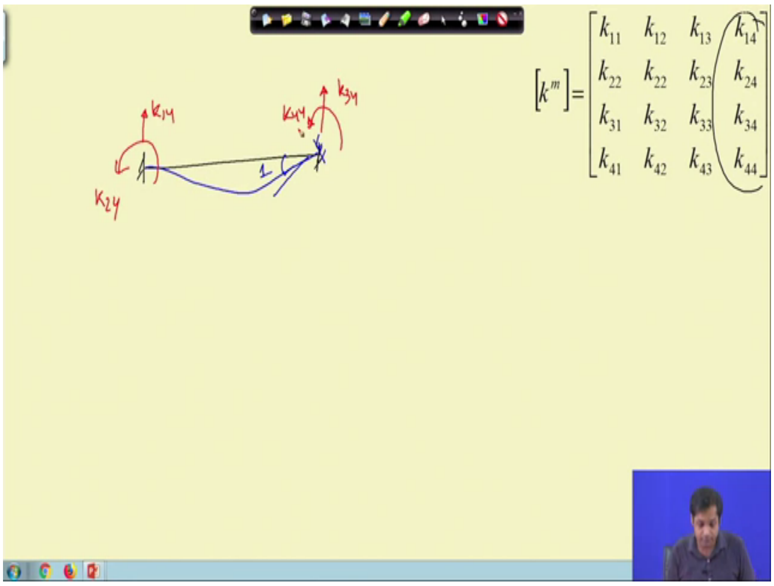

And the finally, if we take the last one last column, this is the column, where you have to give unit degrees of freedom at along the degrees of freedom 4. So, this is the un deformed configuration and what will be the corresponding deformed configuration, we have to give unit values.

Again recall your rotation along anti-clockwise direction is positive so, this deflection will be something like this so, this is your 1. And then what are the forces we have, these force will be k 1 4 then reaction will be k 2 4 and then these force will be k 3 4 and final reactions will be k 4 4. And again we can solve it and find out what are the components of these values.

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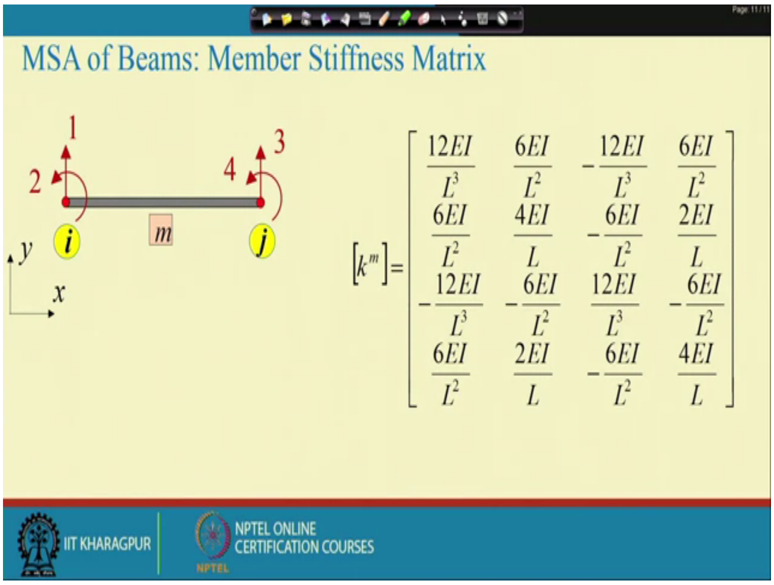

Now, finally if you do that, then what we finally have is this, the final stiffness matrix, we will be having like this. You have to it is now I leave it to you, to check the components of different stiffness matrices. But, remember when we have written this stiffness matrix our sign convention is the translation in upward direction is positive and the rotation anti-clockwise direction both moments and the rotation anti-clockwise direction is positive. And then, we have this stiffness matrix stiffness matrix.

So, this is a stiffness matrix for any arbitrary member, whose length is L and young's modulus is E and second moment of area is I. Now, we can have different such members in a structure having different lengths, different flexural rigidity and therefore we can have different stiffness matrices for different members.

And the next class what we do is we see take an example, where we have more than one member. And find out and write the stiffness matrices for both the members and then see how to assemble them to get the global stiffness matrix. This is the locals this is the element stiffness matrix or the member stiffness matrix these gives you the force displacement relation for a given member, but what we want is we want the force displacement relation for the entire structure for that we need global stiffness matrix.

Next class, we will see what would be the how to assemble them, we have done that exercise, in the context of truss; we will see how that is to be done for beam problem. So, next class we will see what is we will assemble them and find out the global stiffness matrix. I stop here today; see you in the next class.

Thank you.