Matrix Method of Structural Analysis
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Lecture – 18
Matrix Method of Analysis of Trusses (Contd.)

Hello everyone let us continue what we have been doings in last 2 classes. We are trying to understand what is the Matrix Method for Analysis of Truss.

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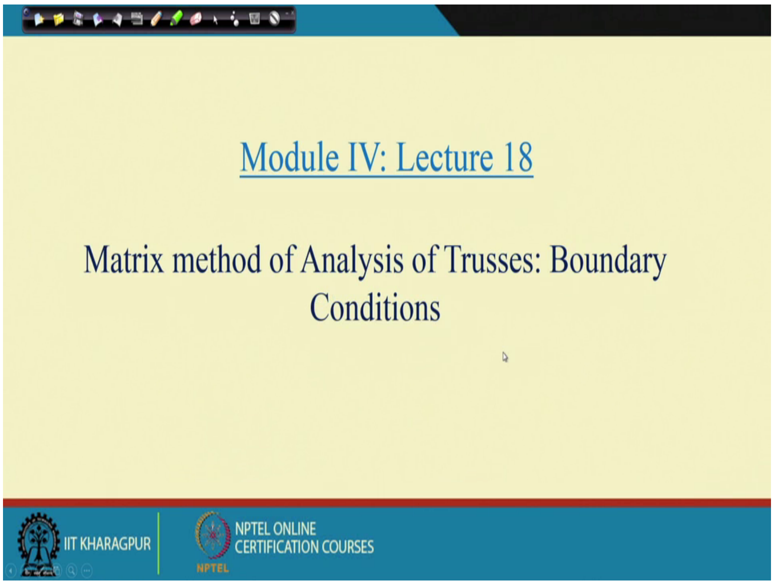

Today's topic is against an application of boundary conditions. If you recall the first 2 classes, the first class we discuss the general concept of analysis of truss and the second class we discussed how to obtain the member stiffness matrices and then assemble them to get the global stiffness matrix.

Now, once you have the global stiffness matrix next step is to applying the boundary conditions and now that is what we will be discussing in this class, ok.

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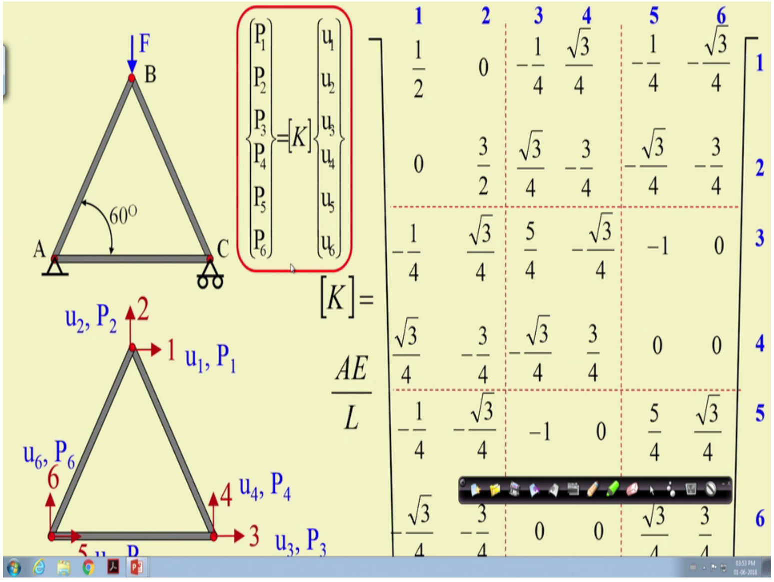

So, if you recall this was a problem that we have beenwe are trying to we are trying to solve this was a truss. And then we have we have member number we have different node number say node number one node number joint number 2 and joint number 3 and the corresponding member numbers. And then once you have these we determine all these stiffness matrices then assemble them and it is the global stiffness matrix.

And these 1 2 3 4 5 6 they are associated with degrees of freedom in the direction of 1 2 3 4, at each joints we have to degrees of freedom. So, total 6 degrees of freedom and these size of the global stiffness matrix is 6 by 6, ok. Now, then suppose these P 1, P 2, P 3, P 4 these are the corresponding forces P 1 P these are the corresponding forces and the corresponding displacements are represented by u.

So, u 1 P 1, u 1 is the displacement in the direction of 1 and then P 1 is the associated force. Similarly, P 2 is the direction displacement in the direction of 2 and the associated force is P 2. Now, once these stiffness matrix what this exactly does is, this relates this P 1, P 2, P 3, P 4 with this right that was this that was the purpose of this stiffness matrix.

Now, P 1, P 2, P 3, P 4 and they displacement u 1, 2, u 3 displacement, ok. Now, if I substitute this stiffness matrix into, into here the form that we get is from we will get a form like this, ok.

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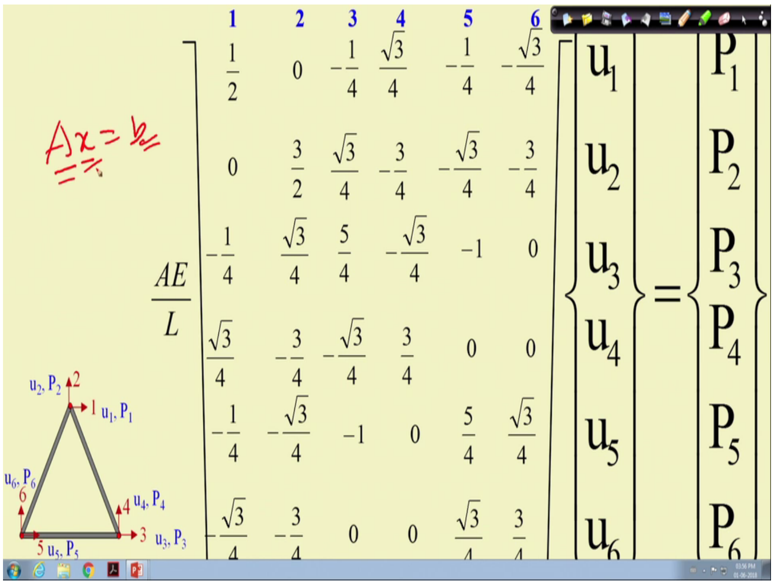

Now, you see this is the form we get, ok. Now, what this stiffness matrix essentially gives you? We have, we have been repeating this many times and probably I will be doing it again and again we if we if we have a system and if we apply some force to the system and analysis of structure is to find out what is the response of the response of the system.

So, in this case the response is the displacement, ok. So, if we apply a force then what would be the displacement, and this expression the force the stiffness into displacement is equal to force that is essentially the representation of our mathematical representation of that a system, ok. Now, there are some properties these stiffness must satisfy you can close the loop this that the stiffness matrix is symmetric. Stiffness matrix is symmetric which is a direct consequence of the reciprocal theorem if you remember Maxwell reciprocal theorem.

Another think the stiffness matrix is all these elements in the stiffness matrix all these diagonal elements in the stiffness matrix they are all positive, they are not zero they are not negative, ok. The stiffness matrix is all the diagonal elements are positive.

Now, but, you know if you try what essentially we have to do next is next is once we have this if we know the what are the forces you are applying to a system and then if we solve we need to solve it to get the displacement right.

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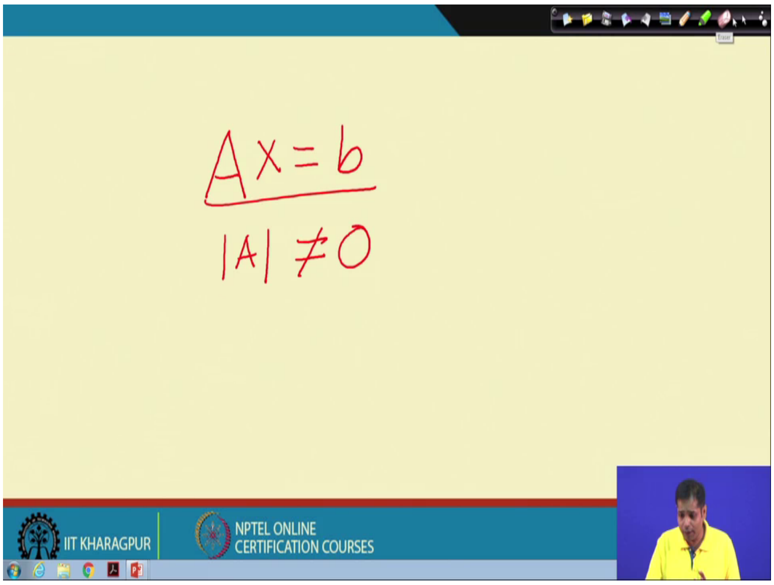

Now, suppose if we have essentially then what we have to do is. Suppose you have a, so this is a common thing that you have learned in your matrix theory Ax is equal to b, if we have an equation like this and then we have to at the set of linear equations if we have. Then, then if we have if you want to solve it the one the important condition that that the matrix A must satisfies it should be non singular. Means determinant of A, determinant of A should not be 0 right if the determinant of A is 0 then we cannot have unique solution of this system right that we all know.

Now, let us try to understand. So, essentially we also if we take a system like this if we take a system like this. Here also we are doing essentially same thing we are we are solving Ax is equal to b, where A is equal to this stiffness matrix, b is equal to the load vector and the x is equal to the displacement the unknown displacement field, right.

Now, but if you calculate the, if you see what is the determinant of this stiffness matrix then you will see that these determinate is 0. But the condition if condition for a system of system of linear equations to be solvable to have a unique solution the determinant of the coefficient matrix should not be 0 the matrix should be non singular should, not be singular. But in this case the stiffness matrix is singular.

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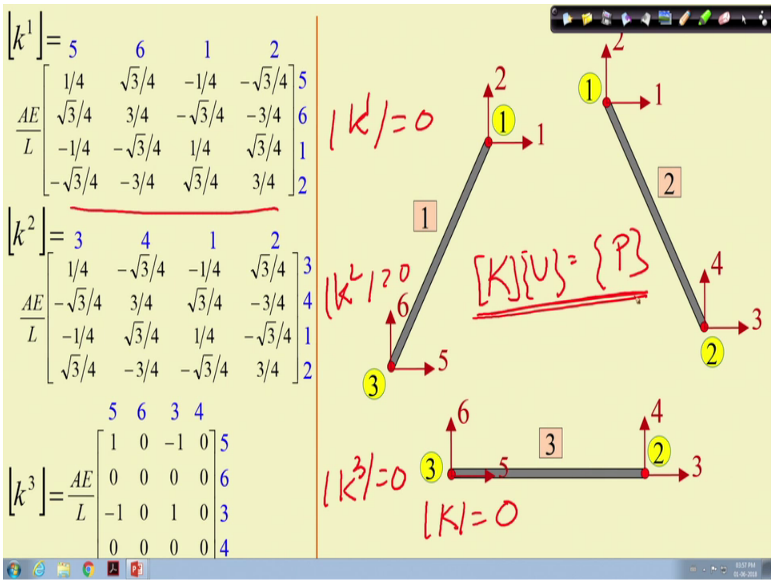

Not only that if you take if you take individual element stiffness matrix, member stiffness matrix. All these all these matrices if you look they satisfy, they are symmetric, they are they have the all the diagonal terms are positive but in addition to that one important feature of these stiffness matrix you can see these all these determinant is 0. So, if you take determinants of K we will get 0 determinant of K 2 we will get 0 similarly determinant of K 3 K 3 we will get 0, ok.

If the determinants are 0 then what does it mean and the not only that if you when you assemble it assemble the stiffness matrix is also 0. If the determinant of the stiffness matrix is 0 it means that we cannot have this solution we cannot determine the solution uniquely the expression that we have here that stiffness into stiffness into displacement is equal to force in this representation with represented this entire truss we cannot we cannot have an unique solution to this system.

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Now, what does this physically mean? You see once Ramanujan says, Srinivasan Ramanujan say then an equation means nothing to me unless it expresses a thought of god. Well, I would like to interpret this expression in this way that in every behind every equation every equation has a story, every equation tells you something, every equation tells you something about some physical process, blindly knowing the equation without understanding what story it tells what is the underlying physical process it is meaningless, ok.

So, when I say that when I say that representation of the mathematical representation of the system is equal to force stiffness into displacement is equal to force and then the stiffness is singular then what does this physically mean.

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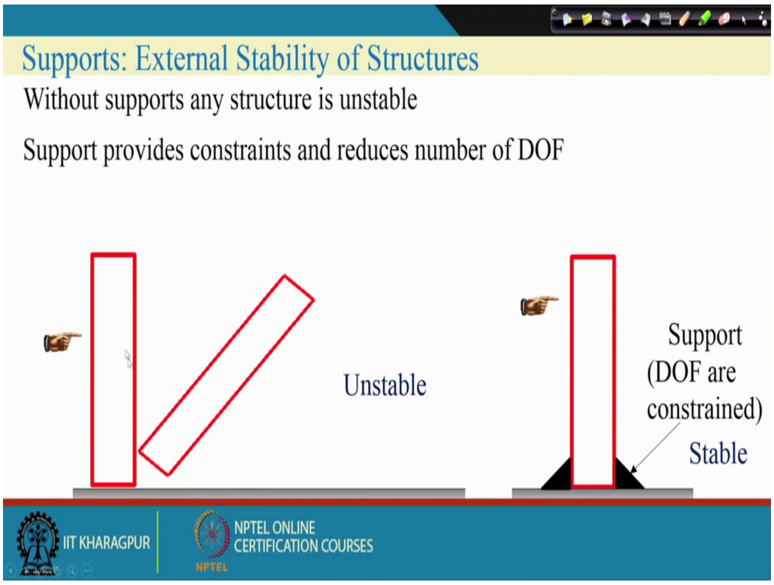

If you recall if you recall in the in the first week when we were when we are reviewing a some of the concept of structural analysis one I showed you this slide. If we take a system say for instance this one and if we apply a force then if there is no constraint in the system then it is unstable, right And then, but if you if you if you provide some constraint then what happens and then, but if you if you if you provide some constraint then what happens?

If you do not provide a constraint this these a 2 dimensional object this 2 dimensional object has 3 degrees of freedom 2 translation and 1 rotation and in 3 dimension it has 6 degrees of freedom 3 translation and 3 rotations. But then, if you constrain then your you are reducing the degrees of freedom and once you do that meaning the structure is not free to move structure has lots is freedom to move. Some of the degrees of freedom the structure has lost and this makes this structure stable, ok.

Now, if we if we see this expression in this expression this is the stiffness matrix which is the property of this of this structure the property material and the geometric property of this structure, right. In this case in this stiffness matrix we have not given any constraint, if you do not give any constraint then if you do not give any constraint then this structure is unstable, right. In order to make this structure stable we have to reduce some of the degrees of freedom of this structure we have to provide some constraint and that is exactly we have to do here as well, ok.

So, what constraint we have that information we have to put, we have to substitute that information in the stiffness matrix. If we do not substitute that if you do not give this information then these stiffness matrix is for the entire structure assuming that every nodes and all the degrees of freedom but if every node has all the degrees of freedom it then means the structure is structure is not stable. And this is manifested by the fact that this stiffness matrix is singular stiffness matrix means we cannot have an unique solution.

So, we cannot have the unique solution the physical, interpretation of that, the physical they the physics or the physical process behind this is the structure is unstable in this case, ok. So, in the physical problem we have in order to make this structure stable you have to substitute we have to put the constraint, we have to reduce some of the degrees of freedom exactly here also you have to do the same thing. So, if your model is correct then whatever you observe through your model through the mathematical representation you should be able to observe physically as well. If you do not do that then probably there is some problem there is some something wrong in our model, ok.

Now, so, we have seen that this is singular and the physics the physical processes it is it is unstable which is very consistent. Now, let us see how to enforce this boundary condition how to provide this information about the, concede and this is exactly do exactly what we are going to do in this class ok.

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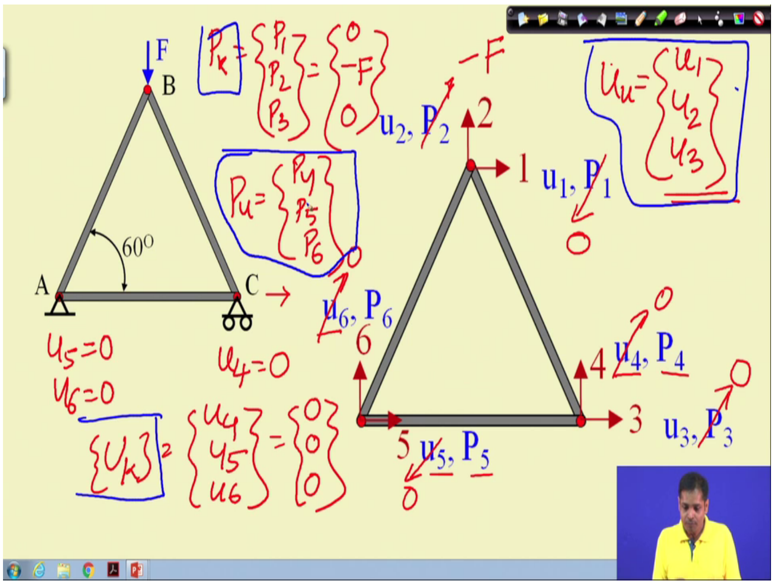

Now, you see let us see what are the information that we have about this problem about this a problem. Now, if I draw the,you see these end is fixed end, so in this case all the, so in this case all degrees of freedom are there is no degrees of freedom adjoint A, ok. So, in this joint u 5, u 5 is equal to has to be equal to 0, and u 6 is equal to 0, u 5, u 5 and u 6 at the degrees of freedom adjoint at this joint, ok.

Now, what the joint C, joint C has degrees of freedom in this direction, but joints C is the roller support so if do not have it does not have degrees of freedom in the vertical direction and therefore, the vertical direction degrees of freedom is u 4. So, it is therefore, u 4 has to be equal to 0. So, this is the information about the displacement we have, ok. So, u 5 is 0, u 6 is 0 and then u 4 is 0, ok.

So, suppose if we define u which is u known, K for known then u known are u 5, u 4, u 5 and u 6 suppose and what are these values these are 0, 0, 0, ok. So, this is u known. Now, what is u unknown? Then we define another u which is u unknown u small u which is equal to then u 1 is unknown u 2 is unknown and similarly u 3 that is also unknown. So, we need to determine what is this unknown displacement.

Now, let us see what information we have about the forces. Now, the external force F we have a vertical force Fat this adjoint B. So, f P 2 will be minus F, minus F because the force is in the opposite direction of 2, ok, so P 2 is minus F. We do not have any external force in this direction, so then P 1 has to be equal to 0.

Similarly, we do not have external force in the in these direction you do not have any external force in these direction. So, therefore, P 3 has to be equal to 0, ok. But then we do not know about P 5, P 6 and P 4 there are no external forces, but since they are constrained there could be some support reactions as well which will be equal to P 5, P 6 and P 4. So, what are those support reactions? We do not know. So, P 5, P 4 and P 6 they are unknown and P 1, P 2, P 3 they are known.

So, similarly similar to u if we define P which is known that is equal to P 1, P 2, P 3; P 1, P 2, P 3 which is equal to which is equal to P 1 is 0, P 2 is minus F, and P 3 is 0. Again and similarly if we define P unknown which is equal to P 4, P 4, P 5, P 6, ok. Now, what information we have? Now what information we have is, we have this information we have this information, this information, this is and we have this information. With this information we have to determine what is this, and then what is this, ok.

Now, with the information above off from this and this, ok. Now, what we do now is we will see how to how to get these information once we get in this information then how to get these information and the subsequent other information that we need that we will discuss in the next class, ok.

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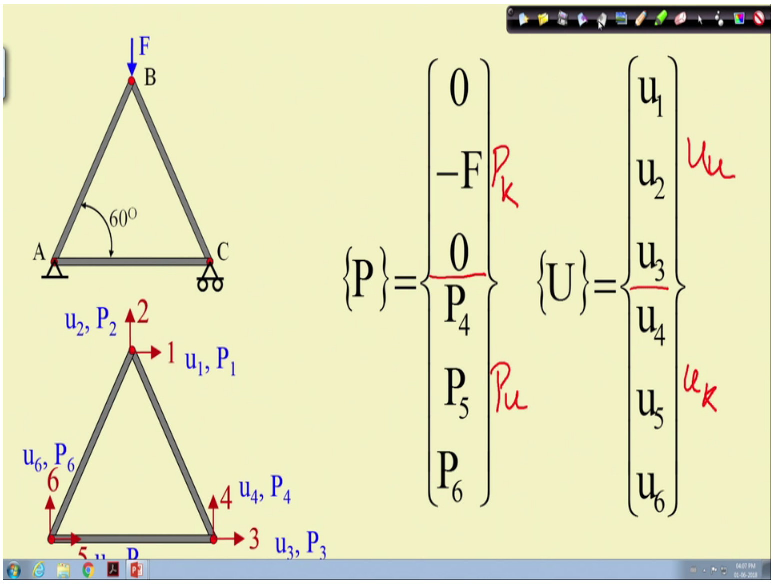

Now, so go back to our stiffness, so this is P and this is total P and total u, and in this total and total P we have just to this is this part is P known and this part is P unknown. If we see this part is P unknown and this part is P known, ok. And then this part is u unknown and this part is u known, ok, here we can say this part is u know and this part is u unknown, ok.

Now, if I take the stiffness matrix, this is the expression of stiffness matrix, yes this is 1, this is fine, ok. I will just keep it here.

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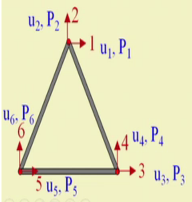
$$
\frac{AE}{L}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{ccc|ccc}
\frac{1}{2} & 0 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\
0 & \frac{3}{2} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\
-\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{5}{4} & -\frac{\sqrt{3}}{4} & -1 & 0 \\
\hline
\frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 \\
-\frac{1}{4} & -\frac{\sqrt{3}}{4} & -1 & 0 & \frac{5}{4} & \frac{\sqrt{3}}{4} \\
-\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 & \frac{\sqrt{3}}{4} & \frac{3}{4}
\end{array}\right]
\end{array}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \hline u_4 \\ u_5 \\ u_6 \end{Bmatrix}
\begin{array}{c} u_u \\ \\ u_k \end{array}
=
\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \\ P_6 \end{Bmatrix}
\begin{array}{c} P_u \\ \\ P_k \end{array}
$$

K_{KK} K_{KU} K_{UK} K_{UU}



Now, similarly what we can do is, so this is a new partition the load vector as well as the displacement vector. And if you partition it is u, it is u unknown, it is u known and then P unknown this is P known and this is P unknown, right.

Now, similarly once you have partition this now in this case it is a consequence the it is a coincident that that first 3 displacement first 3 degrees of freedom unknown and these last 3 degrees of freedom, first 3 are unknown and the last 3 are known. But when but for some structure it may so happen that you are known degrees of freedom and the known loads are not serially numbered, so in that case we have to rearrange the corresponding rows and columns of all these load vector displacement and the stiffness matrix to get this number serially. We will do some example towards this example like that towards the end of this, end of this week, ok.

Now, similarly let us partition the stiffness matrix as well now we partition the stiffness matrix as well, ok. Now, suppose name this, this is K unknown, suppose this is K unknown and then we partition this as suppose this you name it K known K known. And then it is K, K u, why I am writing K u and K KK it will be clear shortly, and then you are take it say K u K and then we have K u u, ok.

Suppose, this is partition like this and if it is partition like this then what we can write that there is something, ok, what we can write is as this you see.

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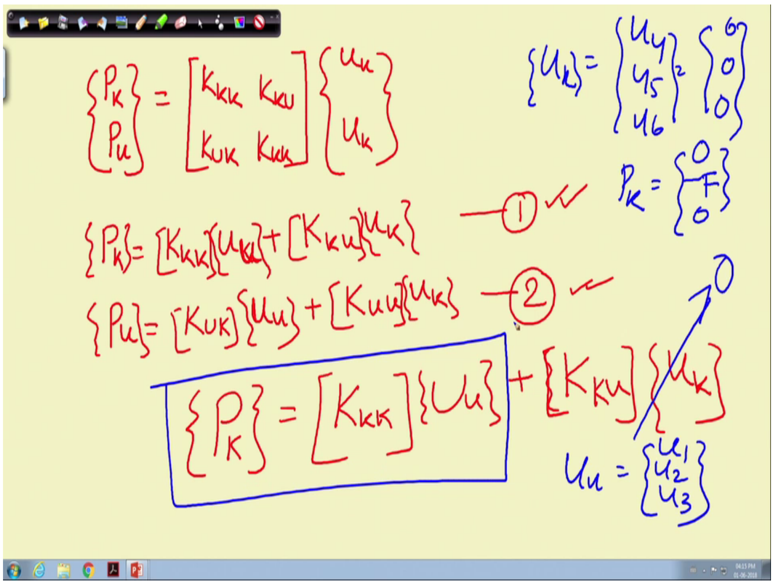

Say P K, P known, P unknown that is equal to stiffness matrix and u unknown and u known and this is K KK, K Ku, K uK and K KK, ok. Now, where K KK is equal to, if we K KK is equal to this part K KK is equal to your K KK is equal to this part this, the entire part and K is another part, this is another part and this is another part, ok.

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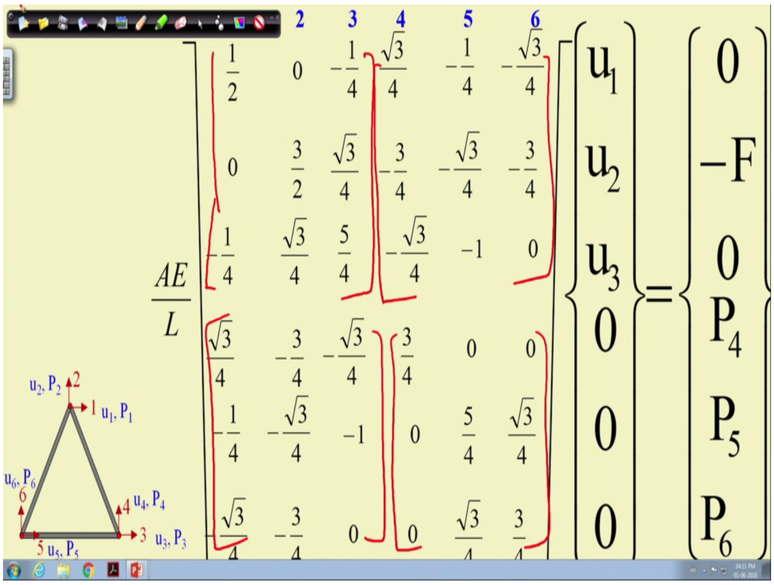

Now, now so we have this right, now if we have this then can we write P K is equal to KK into uK plus uu, uu plus K Ku into uK. Similarly pu is equal to unknown is equal to K uK K uK into u uu plus K uu into uK uK, ok. Now, from this we will come to this. So, there are equation number 1 and equation number 2, ok. It is all these expressions are these a matrix expression this is a matrix, these are all matrices stiffness matrix partition stiffness matrix is, this is known displacement vector, unknown displacement vector, these unknown displacement vector, unknown displacement vector, ok.

Now, let us consider the first equation today and we will see how to how to use second equation what is the use what is the use of second equation in the next class, ok. Now, the first equation is, so let us write it in write it again. So, P K is equal to K uu and then please plus K Ku this is matrix and uK. Now, in this we have already seen these is equal to 0 these is equal to 0 is not it, because uK is equal to if you remember uK is equal to that is u 4, u 5 and u 6 which are is equal to 0, 0, 0.

So, if it is this then essentially we are left with these expression, and we know that this expression and in this expression what we know we know that P K is equal to, P K is equal to 0 minus F 0, and u u is equal to uu is equal to your u 1, u 2 and u 3, now let us solve this equation.

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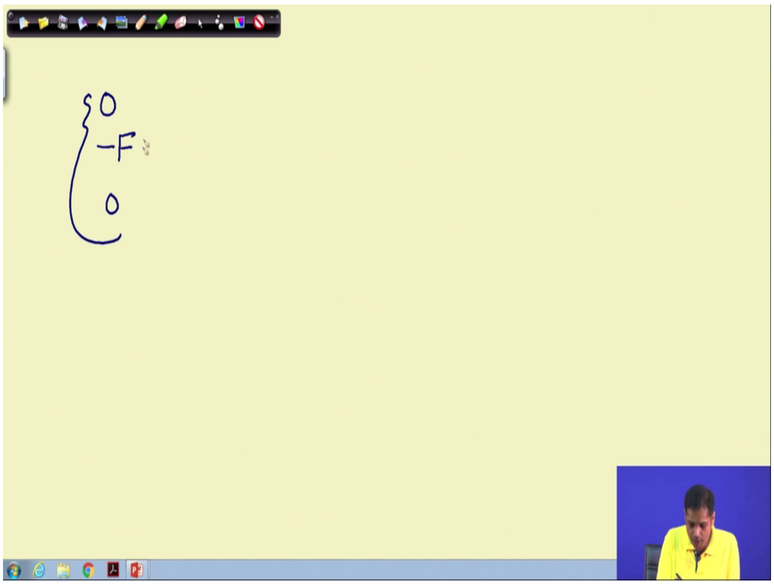

If you solve it so these equation is, so we have we have 0, then minus F and 0, that is the, ok.

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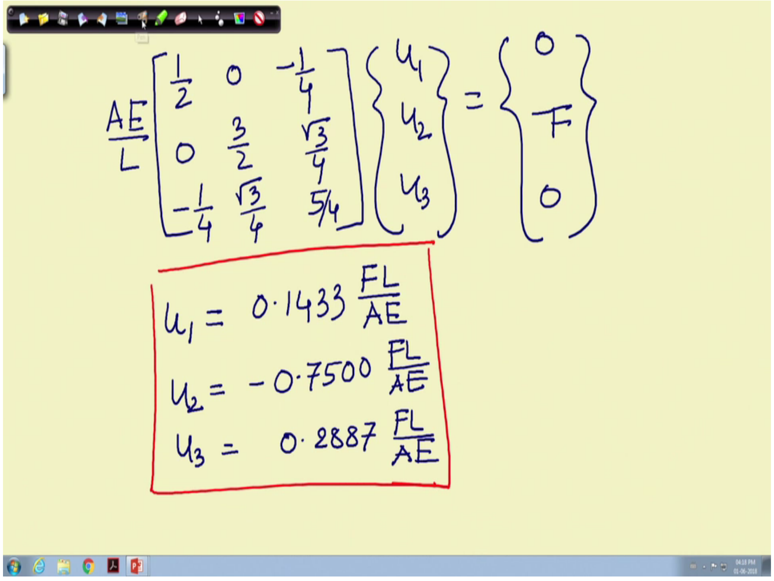

Let us, the matrix will be AE by if we take that matrix this is half you can check it minus 1 by 4 0 3 by 2 and then root 3 by 4 this is minus 1 by 4 root 3 by 4 and then 5 by 4. This is the stiffness matrix the first part of the stiffness matrix and then this into u 1 u 2 u 3 that is equal to 0 minus F and then 0, ok.

Now, in this expression you check your this stiffness matrix is not singular. Now, why it is not singular? Because you have already given the constraint and so this stiffness matrix the reduced stiffness matrix or the after the partition the stiffness matrix you have this stiffness matrix is with the information from the global stiffness matrix, you applied constraint and reduced the stiffness matrix the global stiffness matrix to this.

So, when you write this is the stiffness matrix this is the stiffness it means already you have already you have enforced the boundary conditions. And that is why these the structure is now stable and which is the and in these expression that is getting manifested as it is not a singular stiffness matrix.

Now, if you solve them, if we solve them what we get is u 1, u 2, u 3 is equal to u 1 is equal to you can check this solution 0.1433 FL by AE, u 2 is equal to minus 0.7500 FL by AE, and u 3 is equal to 0.2887 FL by AE. So, this is the solution of the displacement, solution of the displacement, ok. Now, once we have solved for the displacement next is we have to determine what are the support reactions and the member forces, ok. So, support reactions and the member forces how to determine then we will discuss in the next class.

So, please the partition here partition of the stiffness matrix is a very important step, and please we can demonstrate only the process through one or two example at the most if you look at books there are many examples given with different numbering different boundary conditions different number of members. And you see those examples you yourself try to find out the member stiffness matrix, you try to assemble them and then construct the load vector and the displacement vector and solve them. And if you can do that then probably some of the concepts will be clear.

So, we will stop here today next class we will start with the same global stiffness matrix and partitioned global stiffness matrix, and then see with information about the displacement unknown displacement that we have already determined today with that information how to determine the support reactions and the member forces, ok. I will stop here today. So, see you next class.

Thank you.