Matrix Method of Structural Analysis

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Lecture – 15
Review of Matrix Algebra (Contd.)

Welcome. This is the last lecture of module 3, where we will briefly discuss about the rank of a matrix and linear independence.

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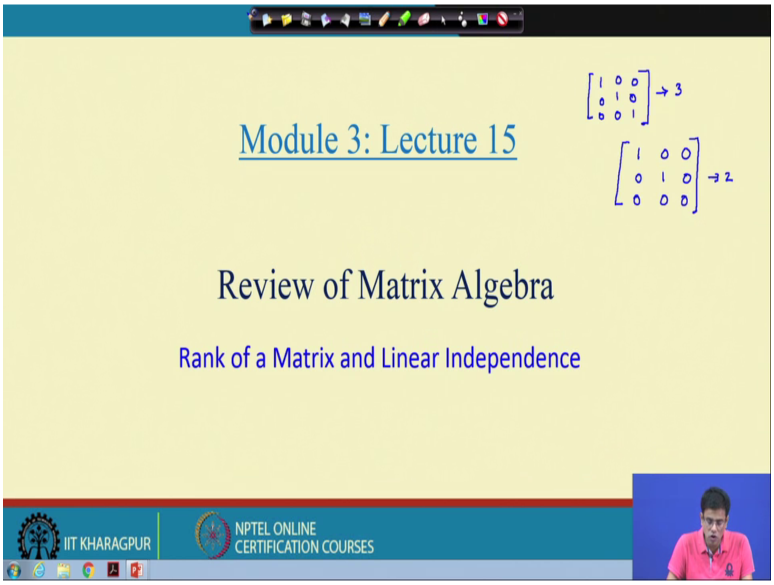

Actually both of these things maybe we will not directly use it, but it is better to know these things during these codes, because it is the foundation of when you learn the advance structural analysis like finite element method and other methods.

So, one important thing is rank of a matrix. Rank of a matrix you all of you know probably that a matrix is invertible, when it is determinant is nonzero. So, we are talking about a square matrix where the determinant is possible.

So, essentially what does this determinant shows is the singularity of the matrix. A matrix is essentially nonsingular if it is determinant is nonzero. So, here we will extend this concept that determinant of a matrix or more effectively the rank of a matrix. So, essentially what I mean this extension is if the matrix is rank deficient matrix, then it is determinant is not possible and it is determinant is obviously 0.

So, that we can easily find out for instance, if we see that there is an identity matrix, and then if we make suppose there is a identity matrix for which we know the determinant is 1. So now, if I if you encounter a matrix where one of the rows of the matrix is essentially 0, or b you can make it 0 by row elimination or the Echelon form that we have learnt earlier.

So, this matrix is essentially the determinant is essentially 0. So, that this matrix even though this rank of this matrix is 3, but here the rank of this matrix is 2. So, I will define the rank soon. So, if the primary thing is if there is a rank definition matrix, then we cannot invert it or it is a singular matrix.

So, this concept we will elaborate through first linear independence of vectors, and then we will define the rank of a matrix. So, let us see first, what do I mean by linear independence.

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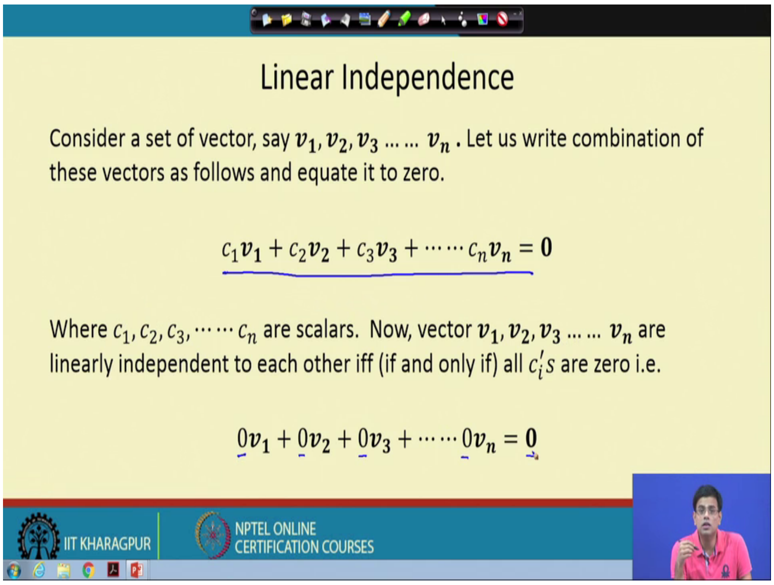

So, without any assumption or anything let us consider a set of vectors v 1, v 2, v 3 to v n. There are n number of vectors; which is of dimension say, 2 dimension, 3 dimension or n dimension or m dimension is possible.

So, that means, the components of vector could be any dimension. So, now let us write the combination of these vectors as follows and equate it to 0. For instance, I write this that c 1, v 1, c 2, v 2, c 3, v 3. Here c n v n equals to 0. So now, here c 1, c 2, c 3, c 4 and c n these are the scalars scalar quantity. So, what it means for instance, if there is a vector 1 0 0 I multiply with the scalar say, 1 or 2 or anything minus 1 minus 2 something like that.

Now, the definition of linear independence is, if these vectors are linearly independent to each other, if I have already expressed what if means; that is, if and only if all c i's are 0 and then if I multiply these all constants if I take all constants 0 and then; obviously, the right hand side has to be 0. So, that means, there is no other possibilities of nonzero c i's such that the combination this combination will be make it 0. For instance, we will see through an example what this means.

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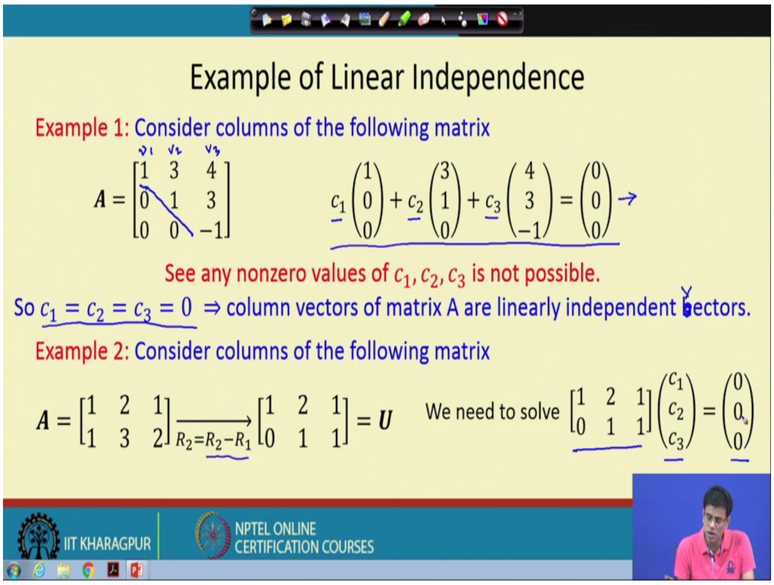

Let us see through an example consider the column of this matrix A. Now, this matrix has 3 columns. The columns are 1 0 0 3, 1 0 4 3 minus 1. Now as per our definition I multiply c 1 with this vector, c 2 with the second vector and c 3 with the third vector, and equate it right hand side equals to 0. So now, I have to find out c 1, c 2, c 3 whether any possible values of c 1, c 2, c 3 is either there or not that is our goal.

Now, see if you look carefully if you, if you investigate carefully by looking at these equation. This equation you see that c 1, c 2, c 3 if nonzero values are not possible. Because suppose that if I consider these 2 case suppose. Now c 2, if c 2 is nonzero and c 3 is nonzero, then certainly this quantity you cannot make it 0 equate 0 with the right hand side. I mean c 2 is if I take say 2, and c 3 is suppose I take 3 or 4 anything any number. And then c 2 into 0 and 4 into minus 1 that is minus 4 which cannot be 0.

So, similarly for c 1, c 2 and c 3 you cannot make this quantity 0 or this quantity second equation 0 without having any nonzero ; means, without having c 3 is c 2 and c 1 is 0. So, obviously, c 1, c 2 and c 3 has to be 0 for this case. Otherwise this equation is not satisfied. So, that means, there is no other possibility of nonzero c i's arises here. So, that is why we say these vectors these vectors are v 1, v 2 and v 3, these vectors v 1, v 2, v 3 are linearly independent vectors. So, these are linearly independent vectors.

Now, consider another example, where it is a rectangular matrix. So now, this matrix if I consider the all columns of this matrix. Now this is we can easily see that this row operation, if I do that is R 2 dash or R 2 equals to R 2 minus R 1, then I can obtain the Echelon form of this matrix or the upper triangular form of the matrix.

So, you see in the previous example. Previous example we have seen one important property, that this matrix the upper this matrix is essentially upper triangular matrix. So, from where we can conclude one thing that if there is a upper triangular matrix and if it is diagonal elements are nonzero, then we can easily say that the vectors these column vectors of those matrix are essentially linearly independent vectors.

Now, if we now look at the rectangular matrix which is 1 0, 2 1 and 1 1. So, what essentially we are doing? We need to solve essentially this into c 1, c 2, c 3 to 0. So, this is that means, we want to find out c 1, c 2 and c 3. So, we have to solve these equations. So, how to solve these equations? Let us see. So, it is a rectangular equation naturally these rectangular equations will have multiple solutions or a indefinite amount of solution we will see that.

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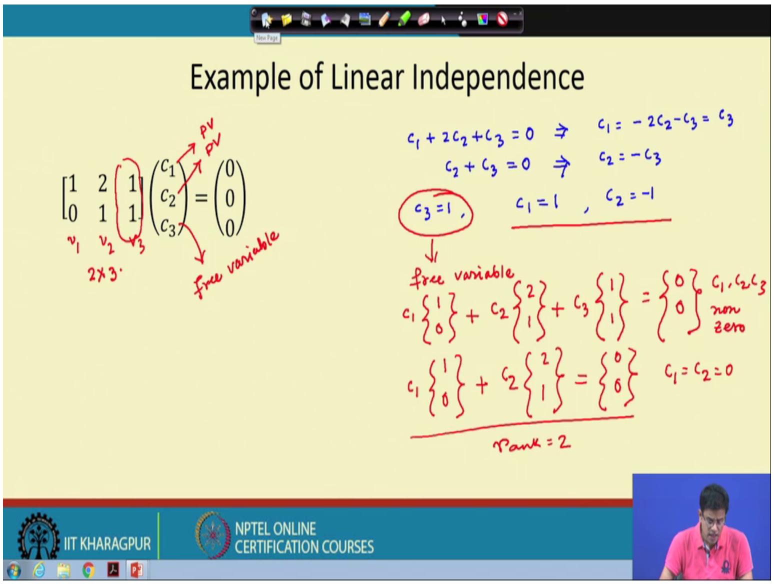

So, if I write it carefully now the equations of this matrix say c 1. So, if I write it c 1 plus c 2 2 c 2 plus c 3. This has to be 0 and c 2 plus c 3 has to be 0. So, we have 2 equations 3 unknowns. So, one of the variable I have to assume, then we can find out the other variables.

So, here if you look carefully these variables if you if I solve it now. If I write it so, c 2 or c 3 is essentially c 2 is essentially minus c 3. So now, here c 1 is essentially minus 2 c 2 minus c 3. So, essentially it is if I substitute c 2. So, it is minus c 3. So, essentially it is c 3 right.

So now here you see c 1 and c 2, I have represented in terms of c 3. Now if I assume c 3 there are different values for different c 3's. So, if I assume c 3 is 1 suppose and then I can now find out c 1 equals to 1 and c 1 equals to 1 and c 2 equals to minus 1.

So now here you have to understand very carefully, that here I have assumes c 3 then only these values are possible. So, these variables essentially is known as free variable; that means, that any possibilities are infinite number of values I can assume an and I can get infinite number of solutions. So, you see go be going back to our linear definition of linear independence ; that c 2 c 1, c 2, c 3 and nonzero possible.

So, if I write now that c 1 into this vector, that is 1 0 plus c 2 into this vector that is 2 1, plus c 3 into 1 1; I can produce this right hand side as 0 0 right. With these values if I now write c 3 equals to 1 and c 1 equals to 1 and c 2 equals to minus 1, I can produce a right hand side 0 0.

So, as per our definitions. So, I can now say this vectors that is v 1, v 2 and v 3 they are linearly dependent vectors. So, that means, also we learn from one thing here that these dependency is essentially due to these free variable c 3, where all solutions c 2, c 1 are dependent on c 3. So, if I assume c 3, I will get different values I will get c 1 and c 2. If I assume different values of c 3 I will get c 1 and c 2 different. So, that means, infinite number of solutions are possible. And so, these variables are essentially known as the free variables. And these variables are essentially known as pivot variables. Probably, you have heard this name earlier so; these variables are called pivot variables, c 1 and c 2.

So now if I look carefully or we will learn it that this matrix is having rank 2 why 2? Because there are 2 pivot variables. So, if I write this matrix without the third column, without the third column of this matrix, then you will see that we cannot have c 1 and c 2 different here means a nonzero here to produce right hand side 0. What I mean by this? That that if I write these only 2 columns of this vector that mean 2 columns of this matrix, that is 2 1, I cannot have a 0 0 ; unless c 1 and c 2 is essentially 0.

But here in this case c 1 c 2 and c 3 may be nonzero, non 0 is possible. So, here you see if I remove the third column of this matrix, then it is the rank is 2, because 2 all of them are pivot variable rank I can say it is 2. Now I will define rank formally later. So, what we have learned? Among these c 1 and c 2 and c 3, these are pivot variable, these are also pivot variable, these are also pivot variable, but this is known as free variable. So, if there is a free number of free variable is; so, this is a finally, 3 cross 2 cross 3 matrix. So, there are 3 independent 3 unknowns. So, among that one of them is full free variable. So, another 2 is pivot variable. So, that is why the rank becomes 2.

Now, let us now formally define what is essentially rank.

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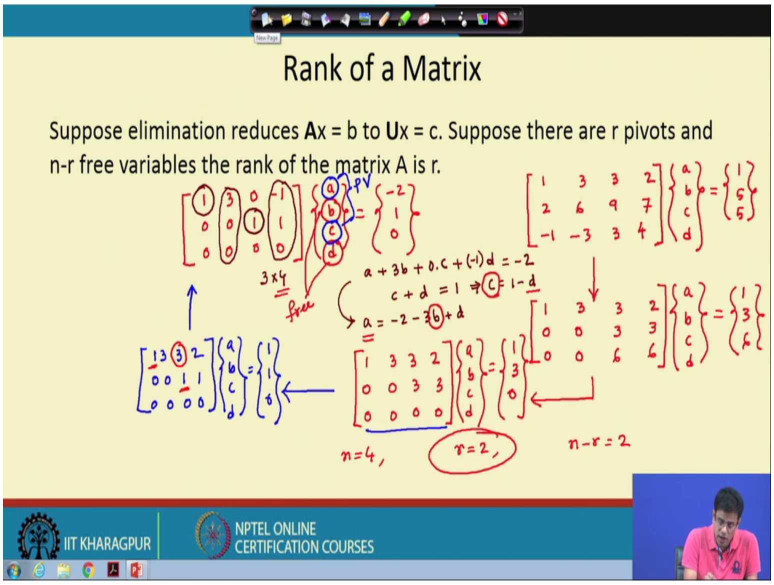

So, suppose the elimination reduces x equals to b 2 u x equals to c. And suppose there are r pivots. And n minus r variables r pivots are n minus r free variables, then the rank of the matrix A is r. So, essentially what I mean by this we will see through an example let us consider a simple example with a matrix.

So, suppose I want to find out the, this equation say 1 3 3 2, and then 2 6 9 7 minus 1 minus 3 3 and 4. And I have 4 unknowns, suppose let us say a b c d, right 4 unknowns. And I have right hand side also which is 1 5 5 5.

So, this I am doing for it rectangular matrix a vector equation, but all the rules are applicable for the square matrix. So, this, if I have the square matrix then we can connect it with the determinant of the matrix. But in case of a rectangular matrix we will not we cannot connect it for the determinant and inverse is essentially right inverse or left inverse all those things are there. But we are not required those concepts here.

So now if what we will do? We will do it is Echelon form, row reduced Echelon form or upper triangular form. So, I can just do this row operation that we have learned earlier. So, 1 3 3 2 let us make this thing same. And then I can just multiply 2 times with the first row, row with the and subtract it from the second row. So, it will be 0, then it will be 0, then it will be 9 minus 6 that is 3, then 7 minus 2 into 2 4 that is 3. And then this one in case of a third case, third row I will just add the first row with the third row then this will make 0, 0. And then this will make 6, and this will make again 6.

So, similarly I have to do it for the right hand side. So now, here this will becomes 1 and then 5 minus 2 that is 3. And then 5 plus 5 plus 1 that is 6. So now, again I can subtract it from the here. So, I will just write 1 3 3 2 0 0, then I can write it 3 3 and then 0 0, then 2 times of second row subtracted from the third row, it will be 0 0 and then this will be my a b c d. And then it will be my 1 3 and 0.

So now you see carefully these among these matrix this matrix is now one row of this matrix is essentially 0. Now if you look carefully this row that means, it is automatically satisfied that that means, what this row 0 row means; that means, 0 into a 0 into b 0 into a plus 0 into b plus 0 into c plus 0 into d will be 0. So, which is actually true so, that means, a this equation has no meaning essentially. So, it is so that means, here there is some free variable involved here.

So, we will see that, what is that free variable. We can now do this more means this operation, I can do it more, I can actually divide by 3 by this second row and find out the find out the reduced from 1 1 3 3 2, then 0 0 1 1 0 0 0 0. And then I can write it a b c d, and then I can write it 1 1 0. So now, you see we can also go for pivots that means, here what will be the pivots? So, as we know we make the entries just top of the pivot 0.

So, we can just simply do again subtract the first one. So, we want to make this quantity that is this one will be 0. Because this will again a pivot element and this will also pivot element. So, we can make a this is 0 by just simply doing another row operation. So, one 3, then 3 times of second row minus first row that is 0. And then because these quantities are 0 these quantities are 0. So, it has no effect so, then again multiply 3 times with one and subtract 2 minus 3. So, it will be minus 1, then again 0 0 1 1, then 0 0 0 0.

So now here again I have to do the right hand side. So, a b c d and then right hand side we will be 2 minus. So, the simply minus 2, because 3 times of 1 1 minus 3 is minus 2 and then 1 then equals to 0.

So now you see. So, final matrix the m row reduced Echelon form is this. So, here we can see carefully that only your this one is the pivot element and this one is pivot element. So, top of that is also 0. So, that means, as per our rank definition so, there are r number of pivots. So, here it is 2 number of pivots and then what are those and then naturally this matrix is 3 cross 4 matrix. So, there are 4 variable 4 variable, among 4 variable some variables are pivot variables, some variables are free variable. So, which variables are free variable? If you look carefully that these 2 columns actually has the corresponding variable of these 2 columns are actually free variable. So, this we can also verify if we write in the equation form.

So, if I write it in a equation form; that is, a plus 3 b plus 0 into c plus minus 1 into d equals to minus 2. And the second equation will be c plus d equals to 1. So, from here I can write c is actually 1 minus d right. Now here a from here I can write a equals to minus 2 minus 3 b and then plus d. You see so, if I assume now d and if I assume c if I assume d then I can find out c, what is a value of c. And here once I have assume d, I have to also assume b then I can get the value of a. So, that means, b and c are possibility are in find it possibility of b and c are infinite.

So, that means, b and c are essentially b and d are essentially free variable. While a and a and b c are essentially dependent variables; that means, the pivot variable. If you look carefully that in the in this matrix, the column one is essentially the pivot column. So, it contains a pivot and column 3 is also the contents the pivot element.

So now here column one corresponds to column corresponds to variable a, and column 3 corresponds to variable c. So, that means, if we know the pivot elements belongs to with column. So, we can also identify the, which variables are pivot element, which variables are pivot variable. So, a and c are essentially known as the pivot variable. And b and d are essentially b and d are essentially known as the free variable.

So, you see in this matrix the number of variables is 4. So, 4 minus. So, let us summarize so, n is actually 4 here. So, there are pivot variables r is actually 2, and then free variables n minus r is actually 2. So, that means, rank of the matrix is 2. So, we know how to calculate the rank of the matrix? So now, see for a square matrix if it was a 4 cross 4 matrix, then rank has to be. So, is suppose there is a square matrix. So, let us see a square matrix.

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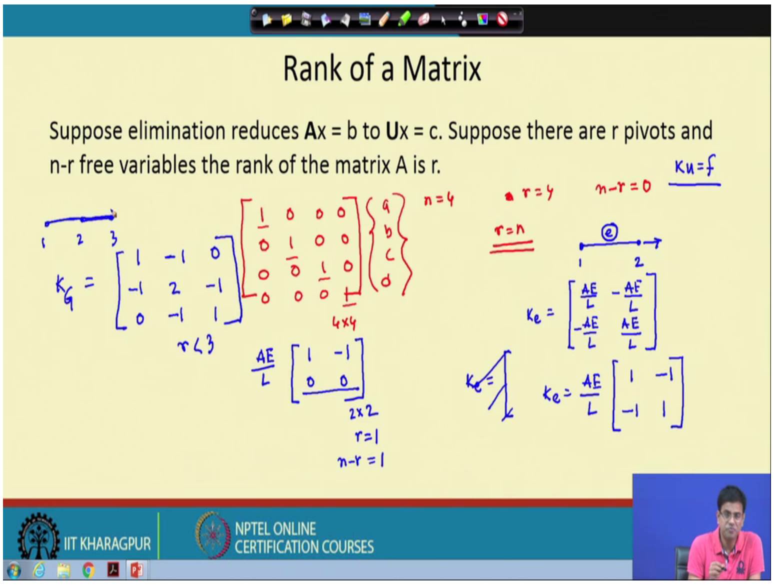

So, the simple square matrix is identity matrix. So, as I have discussed earlier 1 0 0 0, 0 1 0 0 0 0 1 0 and then 0 0 0 1.

So now here this is a 4 cross 4 matrix. So, that means, here n is actually 4. Now this matrix is having how many variable if I write the matrix vector equation. So, say a b c d. So, 4 variables so, among these 4 variables if n if; that means, here if I if you can see that if I write here a b c d all variables are actually pivot variables. Because these are the pivot elements, these are the pivot element. So, that means, r here is actually 4. So, n minus r that means, number of free variable is actually 0. So, when the rank is the for a square matrix; that means, where the rank is essentially the number of the dimension of the matrix equals to the dimension of the matrix we can invert that matrix. Or we can solve the matrix the matrix vector equation uniquely.

So, that means, since there is no pivot variable; that means, there is no assumption required for the other variables to be found out. So, the solution of the matrix vectors equation that is Ax equals to b will be unique. So, that means, if r equals 2 n then we can invert the matrix, or we can have the non singular matrix or determinant is nonzero.

So, this is actually having a very important application in our case. So, probably by this time you have learned what is actually the stiffness matrix of a truss, or stiffness matrix of a beam, stiffness matrix of a frame. So, probably also we have reduced will reduced it that is if I have a cross element. So, we will have the Axial degrees of freedom only, and then the elemental this is element e. So, in the element we have first degrees of freedom, and the second degrees of freedom, corresponding to node 1 and node 2. So, element stiffness matrix k e e we have found out from the unit load method or any method. So, which is we know that AE by AE by L minus AE by L minus AE by L and AE by L.

So, what is A? A is essentially the area of the member and, E is the young's modulus of the member. So, if I write it now you see this k e is essentially k e is essentially your AE if I common if I take outside, then this becomes 1 minus 1 minus 1 1.

Now, you see if I do the this matrix whether it is full rank or not if it is full rank; that means, it is determinant will be nonzero and if it is rank deficient; that means, it is determinant will be 0. So, this matrix will not being invertible. So, you see by just doing the row operations, you can just add the first row and second row, and replace it with the second row. Then this row will be 0 that means, 1 minus 1 will be 0 here minus 1 1 will be 0 here; that means, one row of the matrix will be 0. So, this is essentially a rank definition if the rank deficient matrix and rank will be 1.

So; that means, this has a physical significance 2. So, that means, our stiffness matrix for the truss if I want to find out reduced form. So, 1 minus 1 0 0, I can write it like this. So, here you see that one row is essentially 0. So, that means, this is a 2 cross 2 matrix. So, one variable is free variable; that means, there are pivot variable is r is 1, and 1 is free variable. So, this matrix is not invertible.

So now if I find out an elemental stiffness matrix of a structure, and then it is not invertible. So, how can I find out the deformation? If you know the stiffness form k u equals to f that you are going to solve. So, k is your global stiffness matrix u is your displacement that you want to solve and f is the force vector.

So now if k matrix is not invertible, even this is true if you do connect with a multiple element this rank will be deficient. That also we can say for instance if you write 2 truss element, 1 another 1 2 truss element 1, 2, 3. So, the final global stiffness matrix will look like this. So, 1 minus 1 0 then minus 1 2 minus 1, then 0 minus 1 1 so, this will be our global stiffness. If you try and find out the rank of this matrix, you will see there is rank is less than 3. So, this matrix will also not be invertible.

So, this has a very physical significance because you see here if I rotate, if I rotate, if I have a truss element here now, if I do not have put some constraint to the truss element it can go anywhere.

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Suppose, if I do not put any constraint to this truss element, it can go for here from here to here. So, there is no deformation in the truss element, it is just translation.

Similarly, it goes from here to here and rotate it. So, there is no deformation. So, it is called rigid body translation and rigid body rotations. So, unless you restrict the one end of the truss or beam or the any structure, it will not have the deformation. So, this phenomena is actually reflected in the singularity of the stiffness matrix.

So now here when we derive the stiffness matrix, we have not essentially put this constraint here. So, what is this constraint? This constraint is the boundary condition. So, boundary condition means; that means, probably that n displacement is 0. So, this boundary condition when we incorporate into the stiffness matrix, then only it is non invertible. So, that means, then only it is determinant will be nonzero, then only we can invert this matrix.

So, then only we can find out the deformation of the other nodes or the position. But if we do not put the constraint; that means, our matrix will be singular, and this singularity means that u v w or any deformations rotations in case of a beam can have any value, and which is physically suppose this is this. So, it goes here and then rotate. So, anything any translation any rotation is possible.

So, this phenomena is actually related with the rank of the stiffness matrix. So, when we encounter a rank deficient stiffness matrix; that means one of the reason maybe we have not put the boundary condition correctly into the stiffness matrix.

So, similarly in case of a beam if we see you know the beam stiffness matrix. So, this is a beam stiffness matrix that we know very clearly now.

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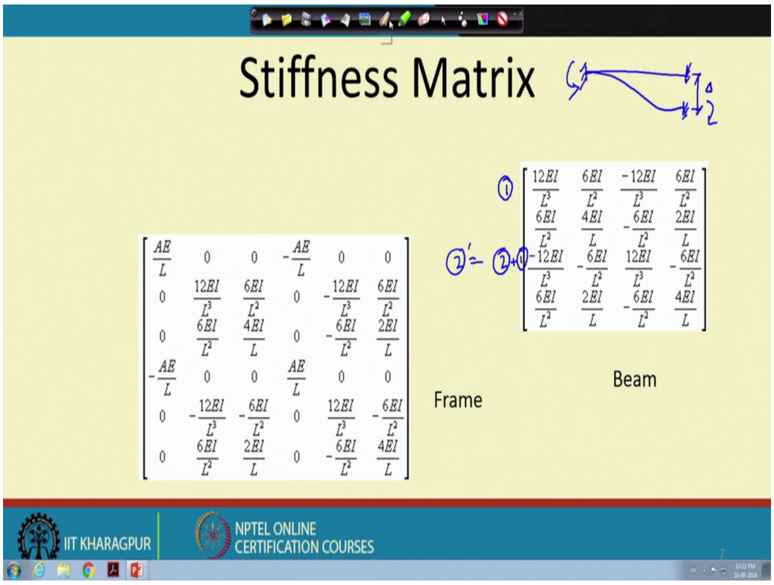

So, this stiffness matrix is also a singular stiffness matrix. Because probably will do it through the conjugate beam theory; where we will essentially put one beam, and then we will make this thing delta amount slide. So, this will give me delta and we will take you need delta, and then calculate what are the moments here and here what are the moments. So, all those things through this we found out the stiffness matrix.

Now, here you see in this stiffness matrix if I add these 2 rows first row and second row, you will see and replace it again in the second row. So, that is 2 plus 1. If I do, then you will see this row will be essentially 0. So, that means, it is again a rank deficient matrix. So, these matrix is cannot be invertible.

Similar case is obtained for a frame. So, in the frame if you in case of a frame we add Axial stiffness also, because this AE by L A minus AE by L and minus AE by L these quantities will be there. So, same argument if we do and same row reduced Echelon form or reduced row equivalent form if we do, we can see that this matrix is also sorry, this a matrix is also rank deficient matrix.

So, as a whole if the idea is why we need to know the rank of a matrix here for the structural analysis case; is that if there is a rank deficiency, sorry. So, if there is a rank deficiency we cannot the invert the matrix. So, here today we stop here. So, we will continue in the next module from the next week.

Thank you.