Matrix Method of Structural Analysis
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Lecture – 14
Matrix Algebra Review (Contd.)

Welcome this is fourth lecture of module 3, we will continue with the solution of linear system of equation.

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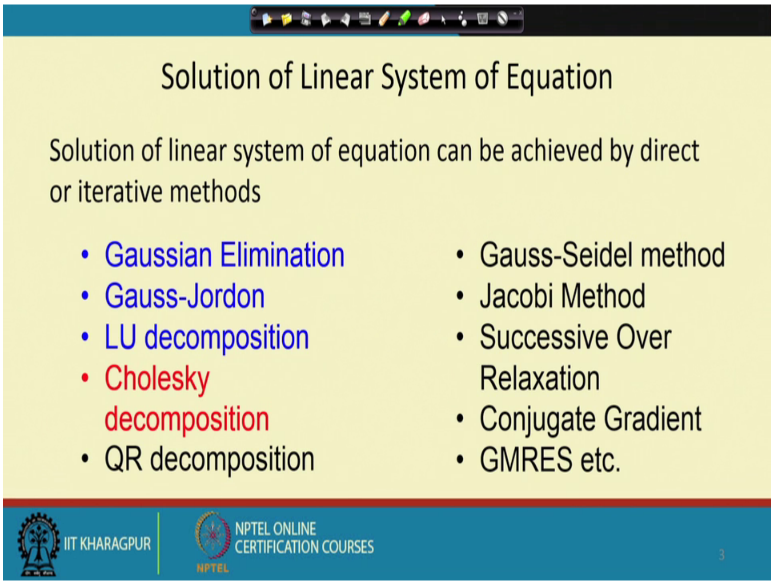

In the last lecture we have seen Gaussian elimination LU decomposition and previous to that, we have also seen Gauss Jordan elimination. So, in this class or in this lecture we will learn Cholesky decomposition. Cholesky decomposition is one of the important factorization matrix factorization scheme which we will be using. And then we will see some of the partitioning advantage of matrix partitioning.

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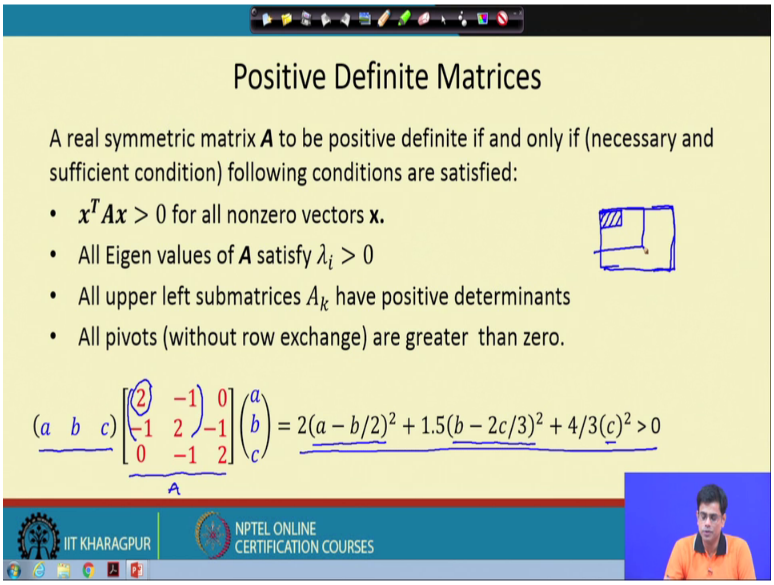

So, before doing the Cholesky decomposition, let us know one type of matrix which in the structural system we always observe so, that matrix is called positive definite matrices. Now a real symmetric matrix A to be a positive definite if and only if following conditions are satisfied. For instance, the matrix has to be for any nonzero vectors of x the x transpose A x is a scalar and which always should be greater than 0.

All eigenvalues of A must satisfy greater than 0 condition; that means, all lambda is if lambdas are eigen values then all lambda i should be greater than 0. Then all upper left sub matrices a k have positive determinants. So, this I will explain eigenvalues we are not discussing in this course, but I hope that all of you have an idea of what is eigenvalue. And then the last point which we have already learned in the last class is that all pivots without row exchange are greater than 0.

So, now; so, that first condition is essentially the nonzero, for all nonzero vectors x transpose Ax which is a scalar has to be 0. The second condition is the eigenvalue condition and the third condition is the upper left sub matrices Ak have positive determinants. So, the first condition there is an example let us say abc are vector so, which is nonzero vector.

So, if I transpose that vector and multiply with a matrix which is A is and then again multiply with that vector, then we obtain something like this. Now this scalar this is a scalar. So, all are you see that all quantities are positive because this is a square. So, this cannot be negative, this is also a square this cannot be negative and c is squared this cannot be negative.

So, this proves the first condition that all for any nonzero vector will obtain a scalar x transpose A x which has to be greater than 0 for positive definite matrices. Now second condition is that a eigenvalue condition all eigenvalues has to be greater than 0. So, if you find out a eigenvalue of this matrix A then you will see all eigenvalues are greater than 0. And the third condition for instance the all upper left sub matrices A k have positive a determinants what does this mean?

So, the first upper left sub matrix is actually the first element of that matrix. So, this is actually 2 which is greater than 0. So, the second upper left that matrix is actually this matrix. So, if you consider the determinant of this matrix, so which is 2 into 2 minus of minus 1; so, 3. So, this is the this is again greater than 0. Now the third upper left since the matrix order is a n so, this constitute the determinant of a.

So, if you compute the determinant of matrix a which will be greater than 0. So, in a what does this third condition means? Say if a if I draw a matrix. So, this is the first upper left matrix, then the second upper left matrix is this and the third upper left matrix is this. So, this is the first element of the matrix, then this is second element and then the whole element of the matrix for which the determinant has to be 0.

So, a it can be generalized for n cross n positive definite matrices. So, in that cases there are n such determinants we can find out and all such determinants has to be greater than 0. So, this is actually the positive definite matrix. And in a structural system we will actually obtain this positive definite matrices very often. So, because symmetric matrices we will also see these matrices are symmetric, because for instance in a reciprocal theorem you have learned in your that structural analysis course, which sometimes results in symmetric stiffness matrices and those matrices has to be the positive definite.

So, we will see when we discuss beams and trusses and frames. So now if the matrix is symmetric and positive definite, then what is the how we can compute its solution.

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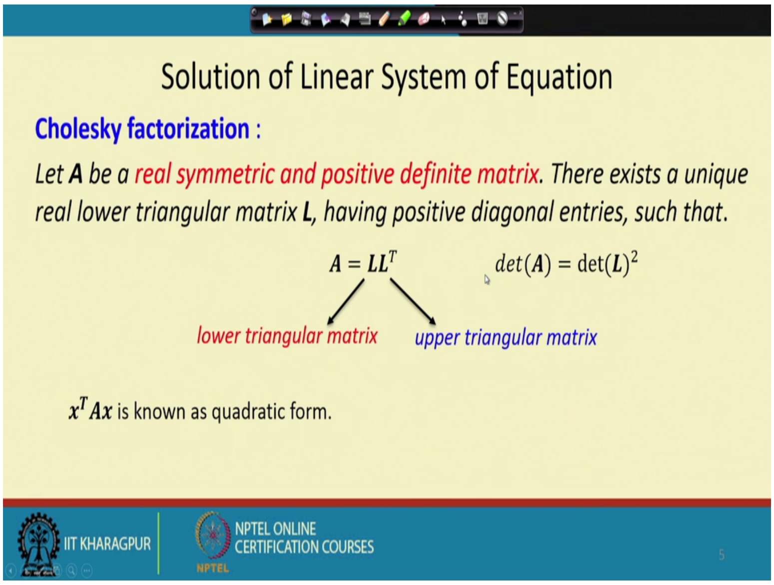

So, for instance there is one factorization known as Cholesky factorization. So, let us see what is this. So, let A be a real symmetric and positive definite matrix. So, there exists a unique real lower triangular matrix L having positive diagonal entries such that A can be written as LL transpose.

So, this is L lower triangular matrix L is the lower triangular matrix we know what is lower triangular matrix and so; obviously, L transpose will be the upper triangular matrix. Now if you if we can decompose a matrix in this form then; obviously, its determinant is one of the matrices square. So, um; so, this type of decomposition is known as symmetric Cholesky factorization which is valid for positive definite symmetric and positive definite matrices.

Now, we will see how we can do such decomposition. So, as we have learned in the previous slide that a positive definite matrix should also satisfies s x transpose A x should be greater than 0 and for your information, this quantity is often known as quadratic form. So, even though we will not use this in this course, but it is better to know this. So, now, we will how to solve such positive definite matrices or how to solve such positive definite system of equation, where the coefficient matrix is positive definite let us see.

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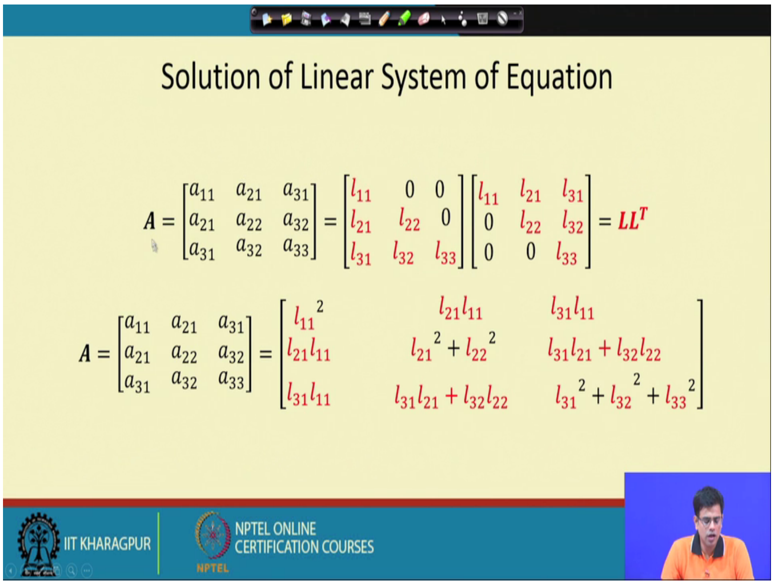

So, as per our definition A matrix can be decomposed into 2 such thing like LU decomposition we have learned in the last lecture. So, here instead of the matrix u, which is the transpose of L the lower triangular matrix. So, if we do such things then by the matrix multiplication, we can easily get the easily get the final form of the matrix.

So, now from here by comparing the each element of the matrix, we can find out what is the l 1 1, l 2 1, l 2 2 and so on. So, let us see how it takes this l 2 1 and l the lijs how it takes.

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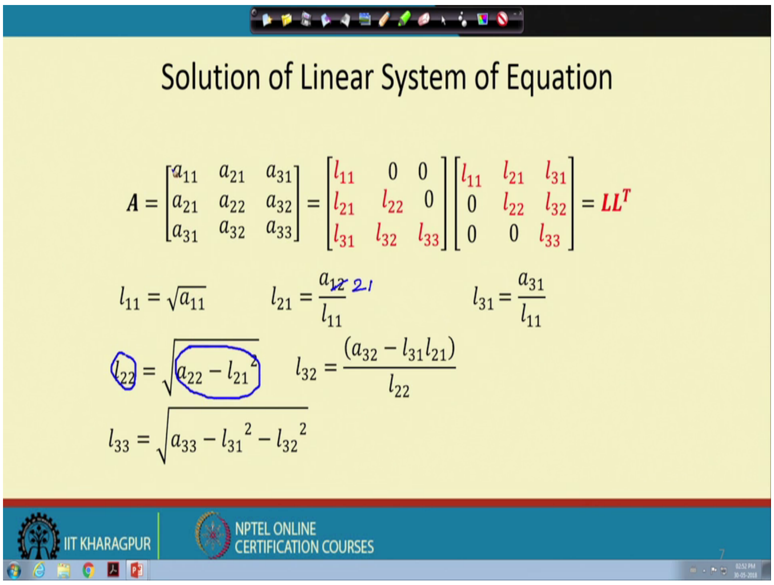

So, if we do this then it turns out that l 1 1 is root over of a 11. So, the first important thing is that that a 1 1 has to be the to find out the liis or l the component of the lower triangular matrix, you have to take the square roots.

So, this is important difference from what we have learned in the LU decomposition. So, similarly if I want to compute l 2 1, which is sequential once you find out l 1 1 then a 1 2 by l this has to be the l 2 1 or a 1 2 because this is a symmetric matrix. So, this has to be the l 2 1. So, now this l 2 1 can be find found out by l dividing by a a 2 1 divided by l 11.

Similarly, a l 3 1 can be found out by l a 3 1 by l 1 1 and again once we calculate l 2 l 2 2 this is again this we have to take the square root. So, this can be observed from the multiplication and comparing the matrices. So, the important information here is that once we perform the Cholesky decomposition, we have to take the square root of the entries.

So, now, certainly if this quantity this a 1 1 and these quantities, if you look carefully are not greater than 0 you cannot take the square root because it will be negative and the square root will be imaginary so, which is not our requirement. So, the Cholesky decomposition; obviously, will fail if the these quantities these quantities comes as negative for instance the first element which is negative a 11.

So, this a 1 1 we cannot take square root, if we take the negative if the element is negative.

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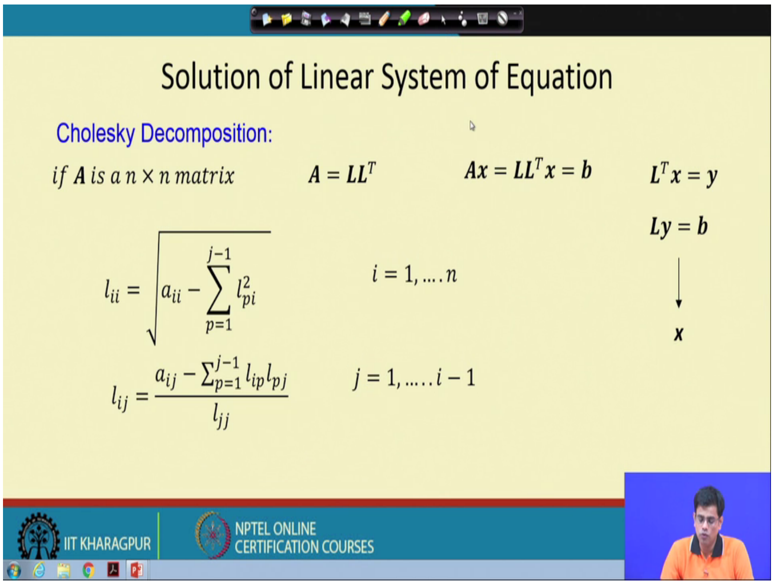

Now, if we do such things, then for n cross n matrix we can generalize it. So, which is a as similar to the LU decomposition we have seen the generalization of l LU decomposition. So, the formula turns out to be like this.

So, once we compute the ls, the solution is pretty similar with the LU decomposition. So, here Ax can be written as LL transpose x and as we have assumed that in LU decomposition L transpose x equals to y and l Ly equals to b, and then we first do the forward substitution to compute y and then back substitution to compute the x. So, this is the overall procedure for the a Cholesky decomposition.

But now we will see a modification of this Cholesky decomposition or the more general form of the Cholesky decomposition, where it is known as an alternative form also known as an alternative form this is known as LDL decomposition.

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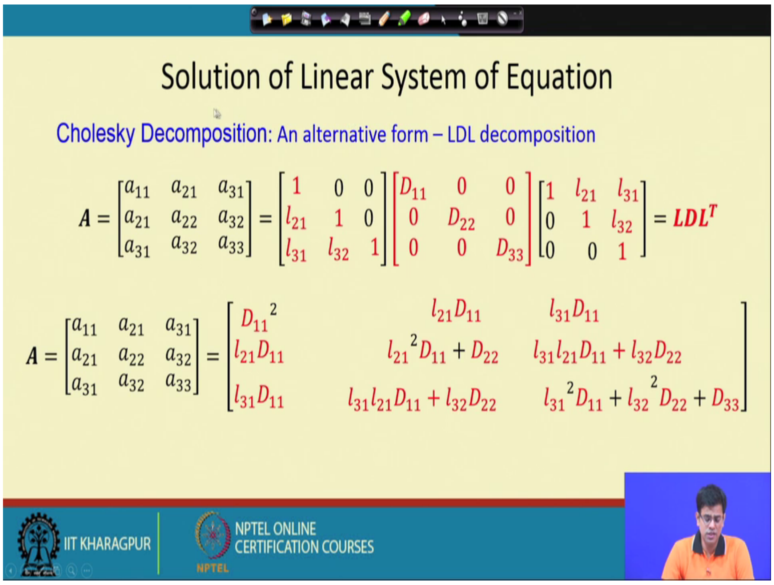

This is more general decomposition and actually you need not to find out the square root of any element. This has a specific advantage also that matrix need not be strictly positive definite.

So, in case of a positive semi definite matrix, this procedure can be applied and which is pretty general we will show with an example also. For instance this what we do is that instead of putting a diagonal element some value, we put we restrict the diagonal element of the lower triangular matrix l is 1. So, the upper triangular l transpose will also be diagonal element will 1, but instead of multiplying ll transpose, we multiply L D which is again a diagonal matrix and L transpose.

So, if a matrix can be written in this form LDL transpose, then again we can find out the lijs and d iis; so, by comparing the each element of the matrix.

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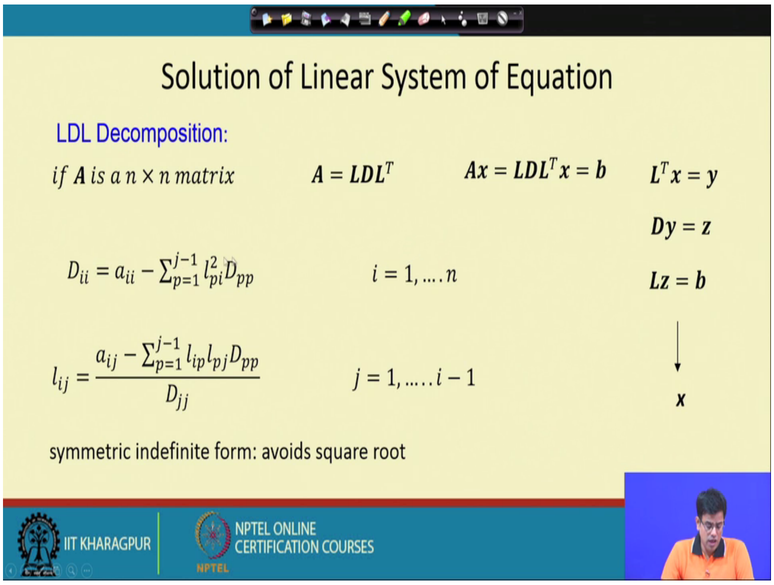

So, let us see for instance this if A is an n cross n matrix then this comes out to be the general formula for l and Dii lij and the D iis. So, as we have seen in Cholesky decomposition that we need to for the diagonal element of the lower triangular matrix we need to perform the square root here we can avoid that square root.

So, this is relaxation of the strict condition of positive definiteness of the coefficient matrix, which can be symmetric in definite form. So, we can see with an example what does this means. Now once we have the decomposition the solution or the finding out the x is very similar, which is Ax equals to LDL transpose y. So, if you assume that L transpose x is y, then again we have to assume because we have a 3 matrix here.So, d is also assumed Dy equals to z and then Lz equals to b which finally, gives the solution of x. Now we will see how with an example between these two factorization.

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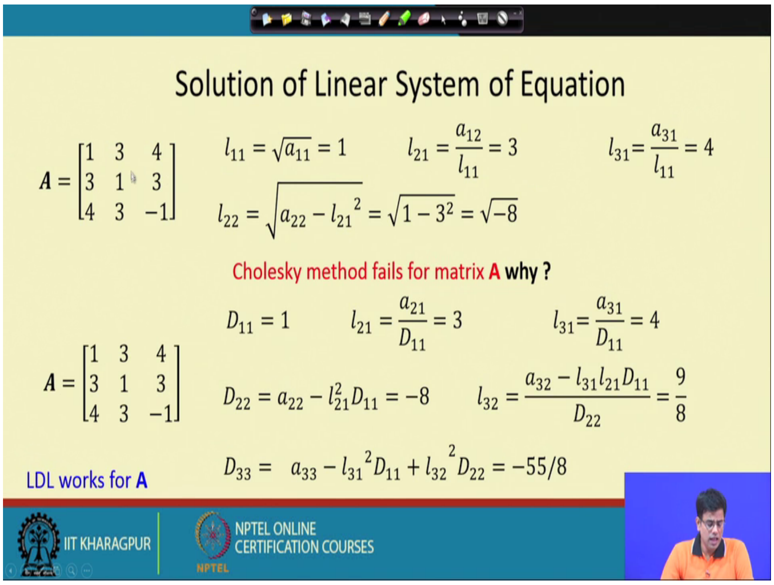

Now, let us see let us take a matrix, which is a 3 cross 3 matrix and this matrix is the we need to we will do it with the Cholesky decomposition, we will factoring it with the Cholesky decomposition and LDL.

Now, the first term as you have seen in the last slides that l 1 1 is root over a 1 1 which is because a 1 1 is positive. So, we can take the square root. Similarly we can find out the l 2 1 and l 3 1. So, that is fine which is because these divisions if as long as l 1 1 is nonzero so, this division works fine. Now l 2 2 when we calculate l 2 2 which we have seen from the Cholesky method is that a 2 2 minus l 2 1 square.

So, now l 2 and a 2 1 we have found out 3 now a 2 2 is actually 1. So, 1 minus 3 square is root over minus 8. So, here you can see that Cholesky methods fails. So, this matrix even though it is symmetric, but it is not positive definite why? The reason is that we can quickly see what is the reason. For instance if you remember the third condition of positive definite test, that all left upper left sub matrices the determinant of all left upper sub matrices has to be greater than 0.

So, if you now test that; so, if that first upper matrices is the first component of the matrix, which is one and the one determinant of one or the scalar is one itself so, it is greater than 0. Now if you see the second upper left sub matrices, which is this which has to be greater than 0. Now if you take this portion if you see the determinant of 1 3 3 1, you will see that the determinant is negative and which is actually the minus 8.

. So, that is why these matrix cannot be said as positive definite, but very interestingly the determinant of this matrix is positive. So, if you compute the determinant of the whole matrix which is a 3 cross size matrix it is positive, but the second upper left sub matrices or this matrix 1 3 3 1 which is not the determinant of which is not positive. So, Cholesky method will fail. But the on the contrary if we use LDL transform or LDL decomposition. So, the diagonal element first we calculate which is D 1 1 is 1.

Then again we calculate l 2 1 which is a 2 1 by D 1 1 which is 3 and then l 3 1 is a 3 1 by D 1 1 which is again 4. Similarly we calculate D 2 2 which is a 2 2 and l 2 1 square D1 1, which is minus 8. You see since there is no square root it does not fail here. So, similarly we can find out l 3 2 and similarly we can find out D 3. So, the decomposition works here LDL decomposition works even though matrix is not strictly positive definite or we can say the matrix is positive semi indefinite.

So, now the advantage of LU decomposition is that the even though LU decomposition is valid for general matrices will mostly use the symmetric matrices and mostly use the symmetric positive definite matrices. But what may happen is that in a all numerical compute computation for instance the determinant when you calculate, when you have a large matrix a where all entries are different order. For instance one of the entries is 10 to the power 4 one of the entries is 10 to the power minus a.

So, rounding of error and all those things occur in the numerical computation. So, the determinant which we compute in a computer code may be nearly singular or may be a very not greater than 0 sometimes. So, in that case we may have to use the LDL transform LDL decomposition method to solve the problem. Now this we will we will see when we will learn for the structural systems.

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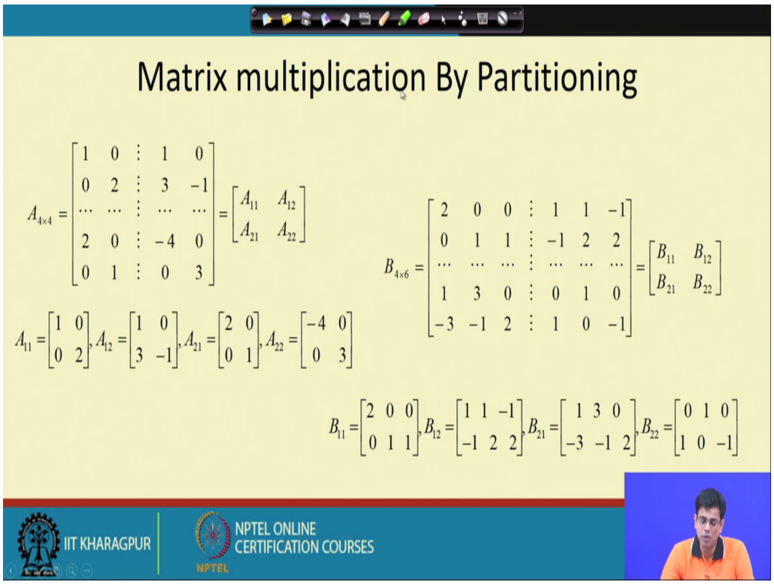

So, now there is another important thing here we need to know is that the matrix multiplication by partitioning or the matrix partitioning. Suppose you want to multiply a 1000 cross 1000 matrix with another 1000 cross thousand matrix. So, first in a computer code what you have to what you generally do is that you store that thousand cross 1000 matrix and then multiply it.

But this is not only a very huge memory will be allocated in the computer and your process could be slow due to that, but there is an way of doing that if you partition this matrix in such a form that, say this is a 4 cross 4 matrix and we partition with a 2 cross 2 form. So, this this is the partition matrix we partition this matrix and this block matrices A1 1, A 1 2, A 2 2, A 2 1 and 2 2 we write in this block format.

And then this is an another 4 cross 6 matrix and we want to multiply 4 cross 4 with the 4 cross 6 matrix. So, if we partition it again, it is 2 cross 3 at each block matrix. So, B 11, B 1 2 and B 2 1 B 2 2 we can write. Now this is A 1 1 and B 1 1 we can see; now if we want to multiply these 2 matrix a and b which is a 4 cross 4 and 4 cross 6 matrix.

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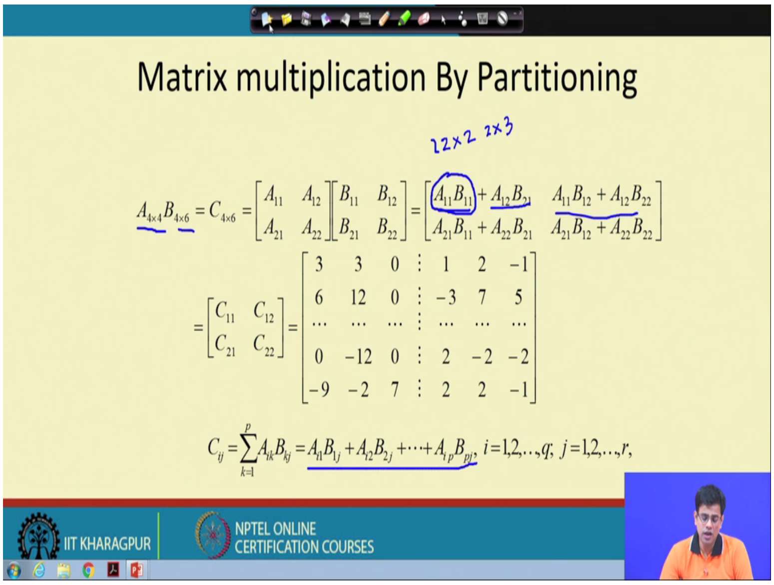

So, we could use the partitioning algorithm or partition a matrix partitioning method to do the multiplication efficiently. For instance here we multiply; so, since a is A 4 cross 4 matrix and B is a 4 cross 4 matrix its multiply multiplication will results in a 4 cross 6 matrix. So, if we multiply A and B together with the partitioned block matrix form.

Then if we write it, it is A1 1, B 1, A 1 2, B 2 1 and so on. So, this form if you see importantly this is actually the multiplication of A 1 1 is 2 cross 2 A 1 1 is 2 cross 2 and B 1 1 is again 2 cross three. So, similarly A 1 2 is also here 2 cross 2 and B 2 1 is 2 cross 3. So, in this way we can reduce the storage of the matrix.

Um. So, this is a general formula if you want to write it in a general format. So, if there is a number of partitions if you decide and then its multiplication will look like this. Now this has a clear advantage in our this course which we will see a through an example let us see what is that.

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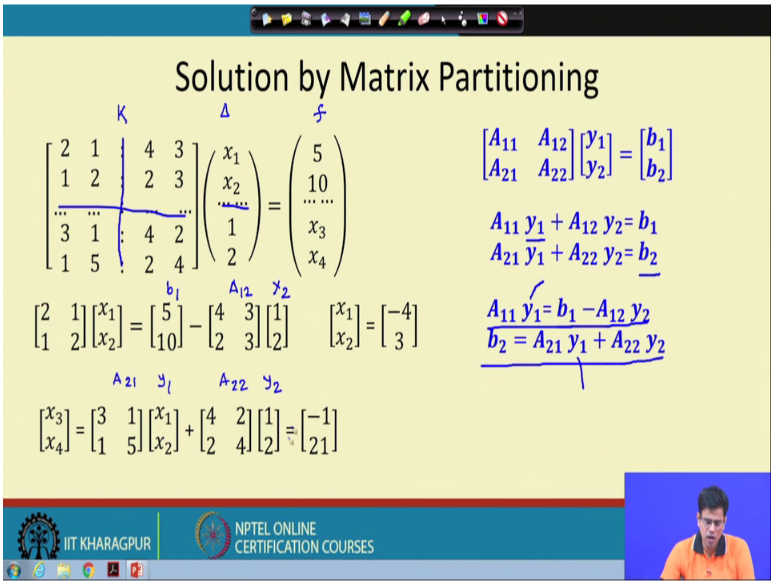

So, suppose we want to solve this matrix system of equation now here what happens is that my x 1 and x 2 are unknown and x 3 and x 4 are unknown, this kind of situation occurs when we see what is a stiffness method where the x 1 and x 2 are the unknown displacements and x 3 and x 4 are unknown forces. So, corresponding to these unknown displacement we have 5 is the force and x 2 is the 10 corresponding to that force is 10.

So, now this x 3 and x 4 is the unknown force, where we know the displacement for instance the boundary condition. The cantilever beam you have the boundary condition is known at the fixed end it is displacement and rotations are 0, but you do not know what is the moment due to the applied load and shear force; so, this kind of situation as we may observe in matrix method. For instance this matrix could be a stiffness matrix k and this matrix is the delta, and this matrix is the force matrix.

So, this such situation in such a situation we have partial delta known and partial f known. So, we what we need to do is that we need to first find out the unknown deltas with the known fs, and then use that solution to find out the unknown names in such cases the partitioning matrix partitioning is very helpful. For instance here we as per the unknown deltas we partition this matrix in again 2 cross 2 form and then write it in this form.

So, a 1 1 and a 1 2 a 2 1 and a 2 2 is y 1 and y 2. So, y 1 is essentially x 1 x 2 and y 2 is essentially 1 2 vector. So, now, similarly the force matrix or force vector we can also partition. So, which is b 1 and b 2 and b 1 is a essentially is your 5 and 10 which is a known vector, but a b 2 which is unknown for us.

So, now if we write this system of equation in the block format; so, it is A 1 1 y 1 and A 1 2 y 2 and b 1 and A 2 1 y 1 and A 2 2 y 2 equals to b 2. Now here you see the y 1 is unknown for me and b 2 is unknown for me we know y 2 and b 1. So, from the first equation we can calculate y 1 which is A 1 1 y 1 b 1 and A 1 2 y 2 because since we know y 2 we can multiply with the coefficient block matrix A 1 2 and then subtract it from b 1 which will give me the right hand side of this system of equation.

And then we can find out y 1 and similarly once we know the y 1; that means, we know the unknown displacements and then we want to compute the unknown forces which is b 2. So, again A 2 1 y 1 since we have found out y 1 A 2 2 y 2 and y y 2 is known to me because y 2 comes from the boundary condition of the problem.

So, if you do this then we can find out the b 2 the unknown displacement. So, similarly this in this example this a first block matrices is A 1 1 is multiplied with the x 1 x 2 and which will give me the modified right hand side which is 5 10 minus. This second block matrix which is A 1 2 and then the known displacement vector, which comes from the boundary condition and this is the b 1.

So, once we do that we get the modified right hand side and we can use any of the decomposition methods, we have learned or any of the solution system a solution of linear equation um any of the method for solving linear system of equation, we can use to find out x 1 and x 2. So, once we know the x 1 and x 2 then the force vector which is unknown at the known boundary condition n, we can find out x 3 by x x 3 and x 4 which is again the block matrices A 2 1 and A using A 2 1 and A 2 2 which is actually the y 1 here and this is the y 2.

So, we substitute known x 1 and x 2 from the previous equation in this equation and then we just do the matrix multiplication to find out the unknown force vector. So, this technique is worth remembering, because we will be using this technique or in this course and this is how we actually implement the boundary condition in the system matrix or the stiffness matrix. So, when we will learn specific problems or specific structural type we will use this kind of partitioning to solve our system or finding out the unknown displacements; so, that is all. So, in the next class we will learn what is matrix rank and its linear independence.

Thank you.