Matrix Method of Structural Analysis
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Lecture – 13
Matrix Algebra Review (Contd.)

Welcome this is third lecture of module 3; the module 3 is Review of Matrix Algebra. So, in this lecture we will discuss solution of linear system of equation.

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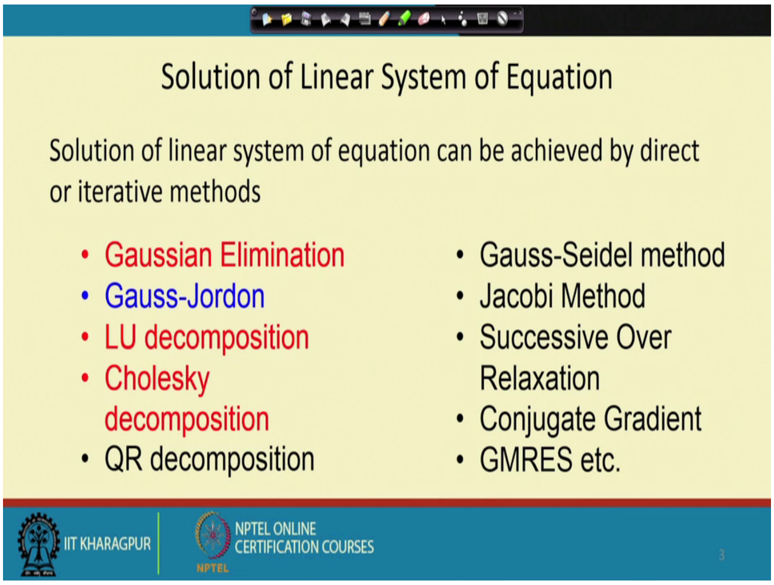

So, solution of linear system of equation can be achieved by direct or iterative methods. So, direct methods actually consist of Gaussian elimination which probably all of you know Gauss Jordon elimination which we have discussed in the last class, for in the context of matrix inverse and also we have discussed it for solution of linear system of equation. And there is a LU decomposition LU decomposition is very popular Cholesky decomposition, QR decomposition these are more or less the direct method of solutions.

An another kind of method where the full matrix is not factorized or method like Gaussian elimination or Gauss Jordon methods are not used is the iterative way. So, the iterative methods, where we solve the system of linear equations in an iterative manner ah

So, these kind of methods is very suitable for large scale of system the iterative methods, which does not required in this course, but it is very important to know what kind of iterative methods are there. So, for instance the Gauss Seidel method, Jacobi method, sword or successive over relaxation, conjugate gradient and then the several other types of method minimum residual method, GMA GMRES generalized minimum residual methods and all other things.

So, we will not discuss the iterative methods, we will discussed here mostly the direct techniques or direct methods of solution of linear system of equation of which we have already discussed the Gauss Jordon. And in this lecture we will discuss about the Gaussian elimination and the LU decomposition mostly.

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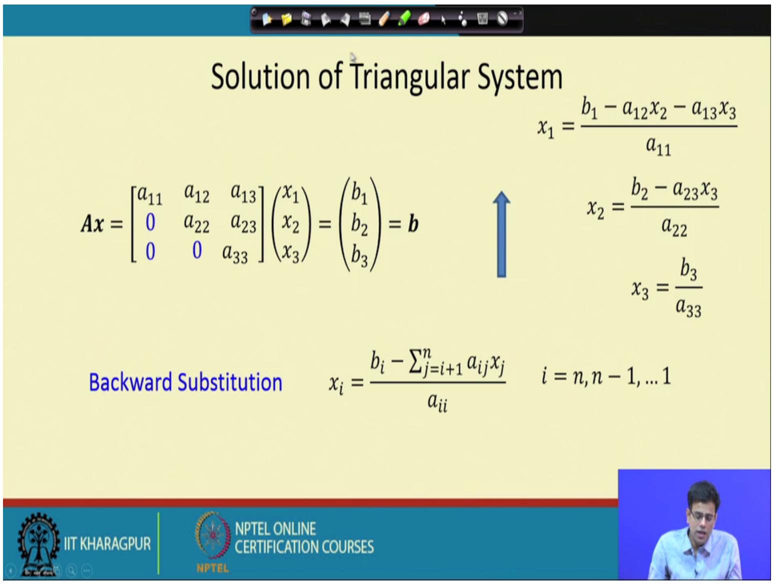

So, the Gaussian elimination all of you have known, but before Gaussian elimination let us a little bit investigate on the how we can easily solve triangular system of equations.

For instance if the coefficient matrix coefficient matrix is a triangular matrix of this form this is an upper triangular matrix, and this matrix with a vector x 1 x 2 and x 3 with the right hand side b 1 b 2 b 3 we want to find out the solution of x 1 x 2 and x 3. So, if you look carefully that if you that x 3 is directly obtainable by just dividing by a 3. So, once we know the x 3, then from the second equation actually if we substitute the value of x 3 and then we can directly find out the x 2.

Similarly, with the help of x 3 and x 2 we can find out x 1 from the first equation. So, you see this kind of substitution is first we compute the x 3 and then x 2 and x 1. So, it is the backward way of solving the or finding out the solution. So, this type of substitution is known as backward substitution; now if there is a n cross n matrix which is an upper triangular and then you have n unknown variables or n equations, which can be written in a upper triangular form.

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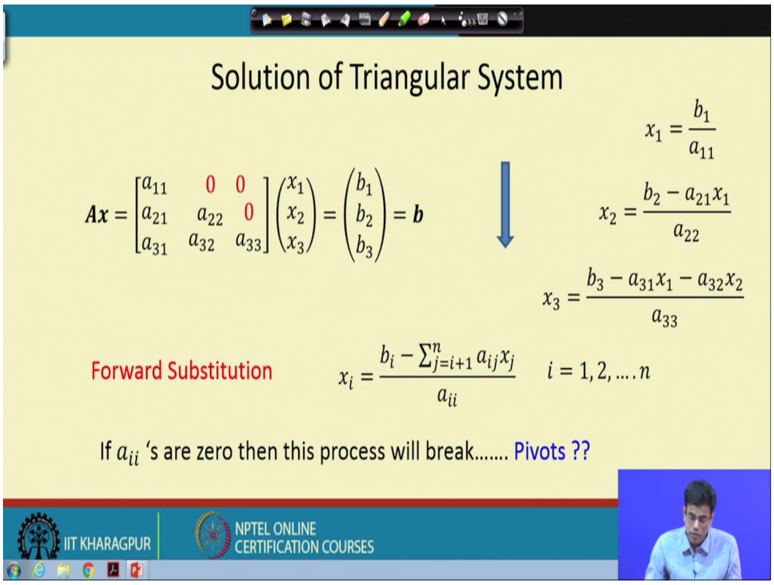

Then the generalized formula for the backward substitution is this, which we can easily write for the backward substitution formula.

So, now similarly a matrix can be lower triangular also. So, if it is lower triangular, then the process is just reverse instead of proceeding from the last variable, we proceed from the first variable. So, this is a kind of lower triangular matrix where the upper portion of the matrix is essentially 0 and then this lower part is nonzero.

So, it represents again 3 variables 3 unknowns 3 by 3 matrix, and then here what we do we first estimate the first variable x 1 which is essentially b 1 by a 1 1 and then again compute with the help of x 1 we compute x 2, and then x 3 simultaneously. So, these kind of substitution if you look carefully, this is the forward progression or forward substitution. So, if again the general formula for this is this.

Now, if you look carefully the even the forward a backward substitution and forward substitution now you see you are actually dividing by this a 1 1, a 2 2 and a 3 3. Now if one of the these values are 0 then the procedure will; obviously, break now. So, we have to ensure one thing that these diagonal entries which by which we are dividing the right hand side of the solution is actually should not be 0 so, if it is 0 then our procedure will break. So, this actually leads to an important concept called devote which will be used in the Gaussian elimination.

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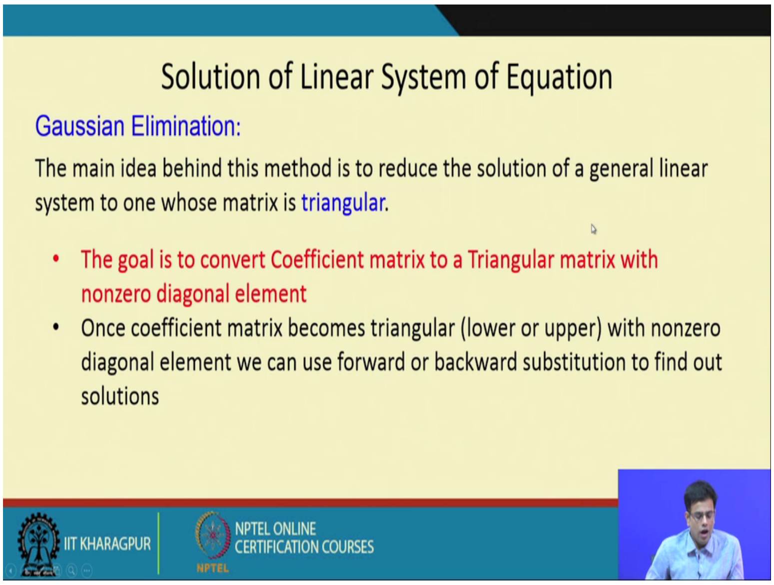

So, let us start with the Gaussian elimination, the main idea behind this method is to be reduce the solution of general system of equation, whose coefficient matrix becomes triangular matrix. Because triangular once we obtain the triangular matrix, the solution is very simple either backward substitution or forward substitution.

So, the goal is to convert the coefficient matrix to a triangular matrix, but we have to keep in mind that the diagonal nonzero diagonal elements we have to ensure that the diagonal elements are not 0. If it is 0 then our backward substitution or forward substitution will fail so, as we have seen in the last slide. So, once the coefficient matrix becomes a triangular lower or upper triangular with nonzero diagonal element, we can use the forward or backward substitution to find out the solution.

So, basically the goal of Gaussian elimination this method Gaussian method of solution is to make the coefficient matrix triangular so, with non zero diagonal element.

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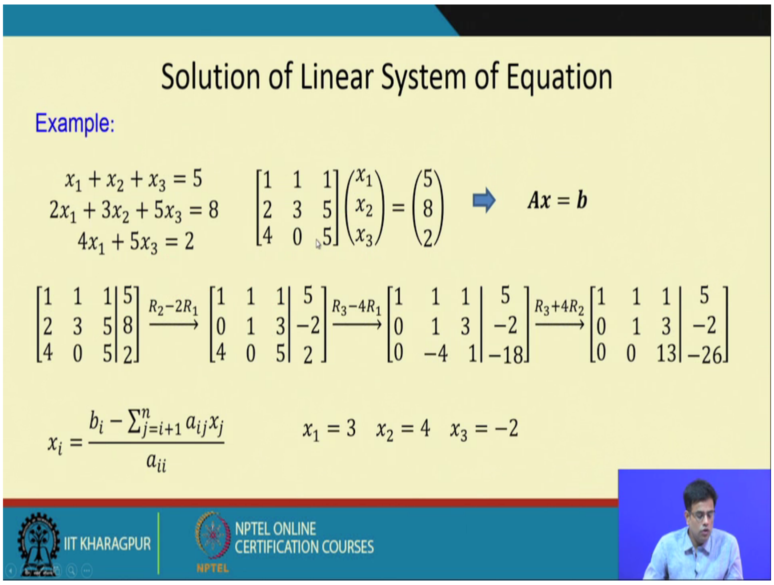

So, let us see with an example which we have solved for the Gauss Jordon method elimination. What we did is actually we made these coefficient matrix is the identity matrix, and through the row operation. But here we will not go till the coefficient matrix becomes identity matrix before that we will stop.

Let us see. So, here the 3 variable and 3 equation the matrix vector form of these equations are of this. So, these this is the coefficient matrix a and with the Gauss Jordon elimination we have learnt last class is that we write the augmented form of the matrix with the right hand side of the vector as the augmentation augmented part with the coefficient matrix.

So, here what we do, we do the row operations. So, these row operations the first row operations are R 2 minus 2 R 1 that is R 2 is substituted R 2 times R 2 is minus 2 times of R 1. So, if you do this row operation then matrix changes and as well as the right hand side also changes, and similarly like this if we perform the row operations, we obtain this kind of triangular system.

So, in the Gauss Jordon process, we further did the row operation so, that the matrix becomes this coefficient matrix becomes triangular. But we stopped here because if we look carefully our them our converted or the change matrix is a triangular matrix. So, now, we can use the backward substitution, this is the general formula for backward substitution.

Now if we use the general backward substitution formula to find out the solution of x 1 x 2 and x 3. So, Gaussian elimination is essentially connected with the Gauss Jordon elimination or Gauss Jordon process, in which we stopped early once we reach the triangular form of the matrix.

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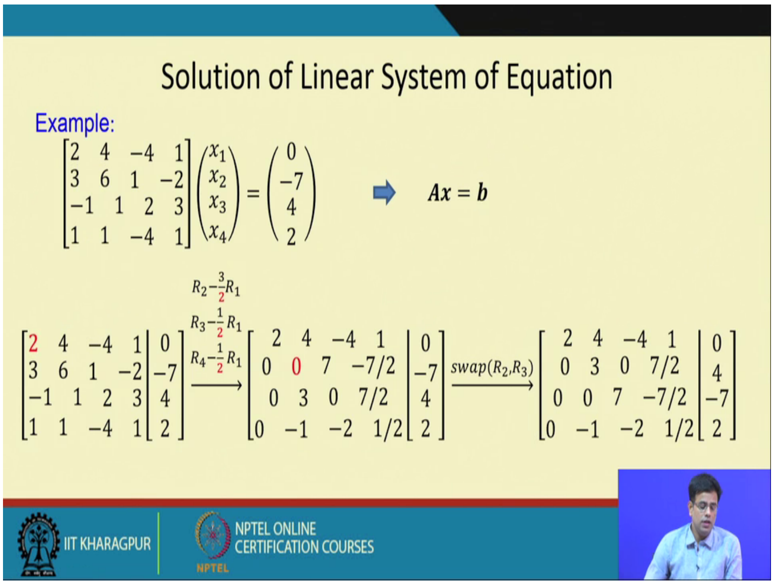

Now,, but let us see a complicated example where we have a 4 equation and 4 unknown. So, here what is important we will learn actually what is the pivot? Now if you see carefully this matrix is my coefficient matrix. So, again as in the previous case we write the augmented form of the matrix and this augmented form is actually written here. So, now, here again we do the row operations.

So, what is the objective here, the objective is here from the second equation if we can remove the x 1 and then from the third equation if we can remove x 1, and fourth equation if we can remove x 1 similarly from the second third equation if we can remove x 1 and x 2, from the fourth equation if we can remove the x 1 x 2 and x 3 then we can just find x 4 directly.

So, keeping this in mind so our objective is to remove the x 1 variable from the second equation, third equation and fourth equation. To do that now if you see if these 2 if I subtract 3 3 by 2 times of row 1 from the row 2, we obtain this row 2 with 0 in the first element of the row 2.

Now, similarly if I do one by half times of R 1 from if I minus 1 by half R 1 from row 3 we obtain 0. Similarly we have 1 by half of R 1 for R 4 if we minus, then we get 0. Now if you look carefully these operations are why this term 2 is coming. So, if you look carefully this first entry of the coefficient matrix a 1 1 is actually the second 2. So, this term is known as the pivot. So, this is very important term or the very important element of this pivot. So, these 2 is playing a great role in reducing this matrix in a triangular form.

Now, in the second case now once we obtain x 1 is removed from second third and fourth equation, now our next objective is to remove x 2 row and x 3 from the third and fourth equation. So, if I want to remove x 2 from the third equation, then we have to again look for the pivot. Now you see the pivot here is 0. So, again this once we reach a 0 pivot, this procedure will break.

Now,. So, here we do as we did earlier for the Gauss Jordon case we swap the row. So, we swap the row 2 and row 3 so, that we do not encounter with the 0 pivot. So, here we swap the row we interchange the row or row exchange we did. So, so that the second or a 2 2 element of that matrix becomes nonzero. So, here we obtain this 3. So, our new pivot will be 3.

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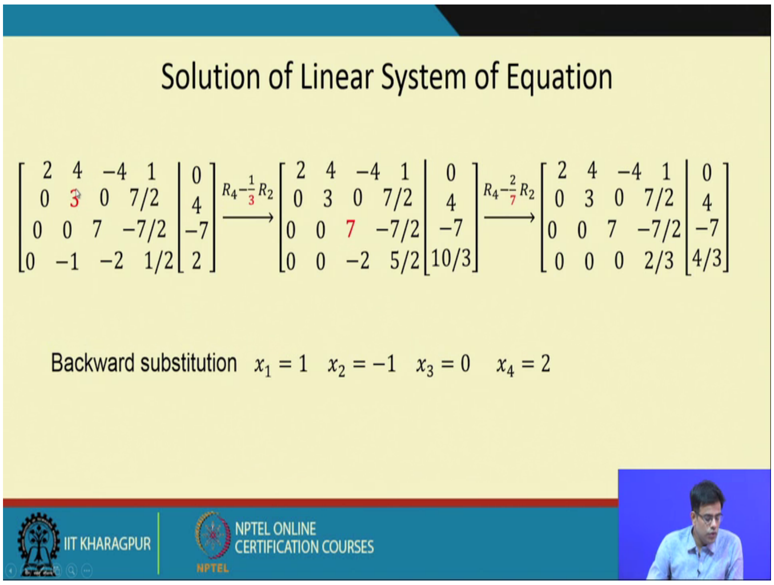

Now, with this 3 again we do the row operation. If you look carefully that by row exchange it is already this x 2 is removed from the equation 3. So, we have to remove x 2 from the equation 4 that is fourth row. So, again we do one third times we subtract one third times of R 2 from R 4, then we obtain this matrix.

So, if you look carefully that from the third equation and fourth equation x 2 and x 1 is removed, and then again we look for the pivot which is 7 here which is nonzero. So, again we do the row operation to remove third variable from the fourth equations. So, again if we use two third if we subtract two third 2 by 7 times of R 2 from row 3 a row 4, and then we obtain the final triangular form of this matrix.

So, if you look carefully this form is a triangular form. Now once we reach the triangular form of equation then our solution procedure is very simple, and this is a upper triangular form. So, you will use back backward substitution to find out the solution of variables x 1 x 2 x 3 x 4.

Now, here you have to look carefully two things. So, one is our objective is to make the coefficient matrix triangular and second, we do not encounter with 0 pivot because if we encounter 0 pivot the procedure will break. So, what is the solution for that? We do row exchange or row interchange so, that we do not encounter the 0 pivot as we have seen in the this example.

Now, this is overall Gaussian elimination it will be very helpful for us to use this method for the matrix structural analysis course.

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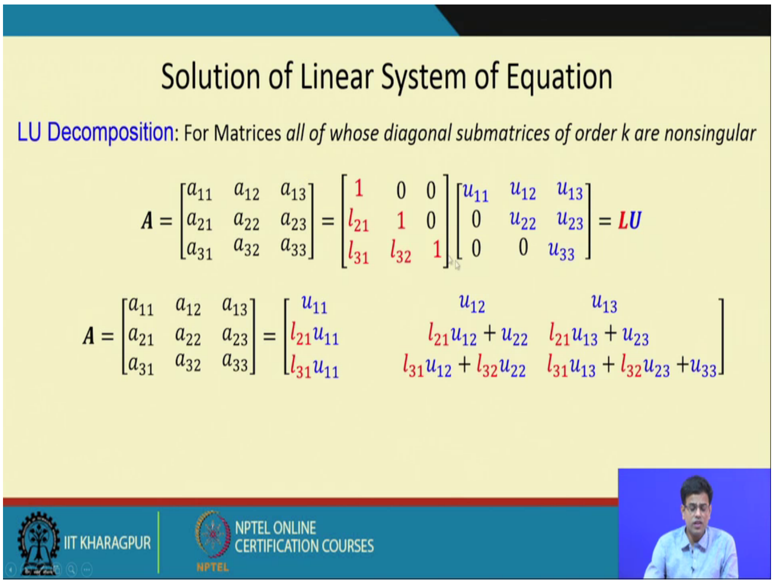

Now, another way of solving this linear system of equation is LU decomposition. So, LU decomposition is essentially a matrix is decomposed in 2 sub matrix or 2 matrices which is of L the lower triangular matrix with diagonal element 1 and U the upper triangular matrix with the lower part of it 0.

So, for matrices all of whose diagonal sub matrices of order k are non singular, these kind of matrices can be decomposed through LU decomposition what does this mean? Actually if you look this the meaning of this sentence is this it has to be a 1 1 has to be greater than 0 and the a 1 1, a 1 2, a 2 1 a 2 2 this sub matrix has to be greater than 0, the determinant of this sub matrix has to be greater than 0.

Similarly, the total matrix the determinant is has to be greater than 0; because if the matrix is the final determinant the whole matrix determinant is not greater than 0, then we will be reaching to the non singular system a similar system of equation for which the matrix inverse it does not exist watch you what we have learned in the last class.

Now, if you use this kind of decomposition. So, finally, if you multiply these 2 matrices and write it in this form, then we will see the how to calculate these l ijs and uijs.

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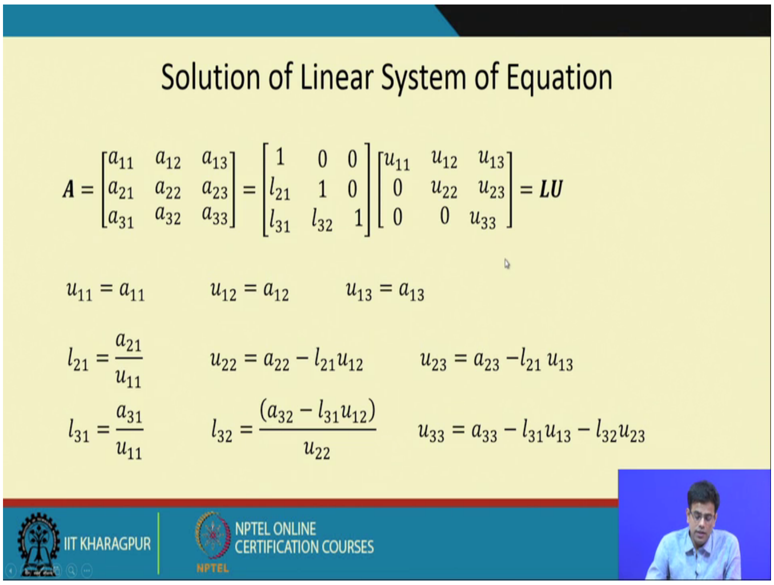

So, if you now compared these 2 matrices, from this previous slide this slide this u 1 1 and a 1 1 should be same so, u 1 2 and a 1 2 should be same u 1 3 and a 1 3 should be same.

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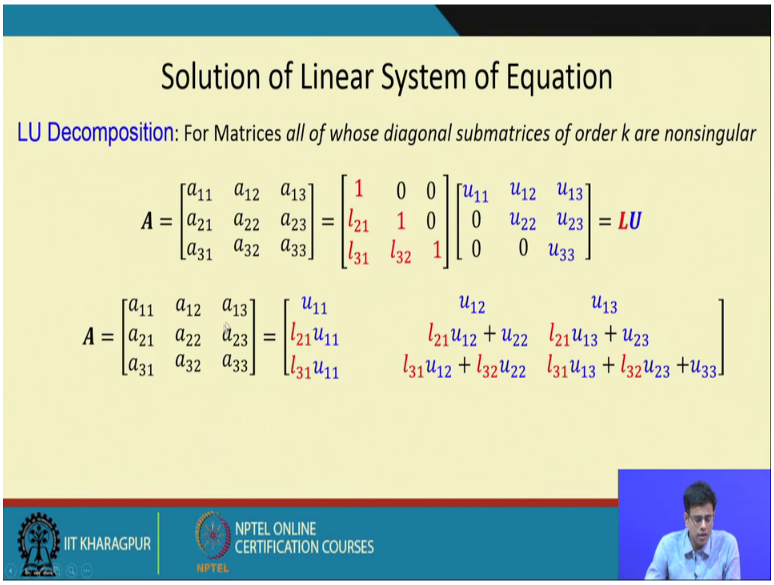

Similarly a 2 1 we can write is l 2 1 u 1 1. So, once we know u 1 1, l 2 1 we can directly find out by dividing with the a 2 1 and u 1 1.

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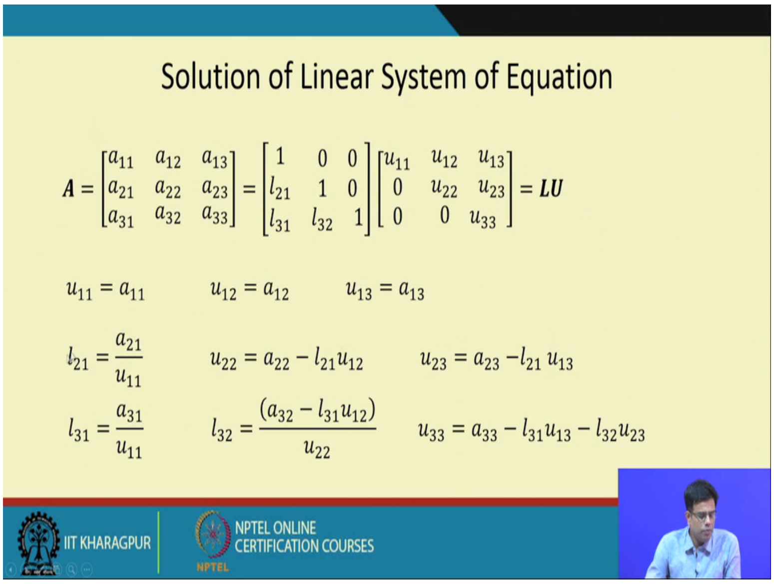

So, similarly u 2 2 we can find out which is essentially a 2 2 minus l 2 1 and u 1 2. So, you see this l 2 1 is used in finding the u 2 2. So, there is a sequence you just directly cannot find out from the last entry u 3. So, you have to come up from the forward sequence; so, the first element then 1 2 element then 1 3 element and so on.

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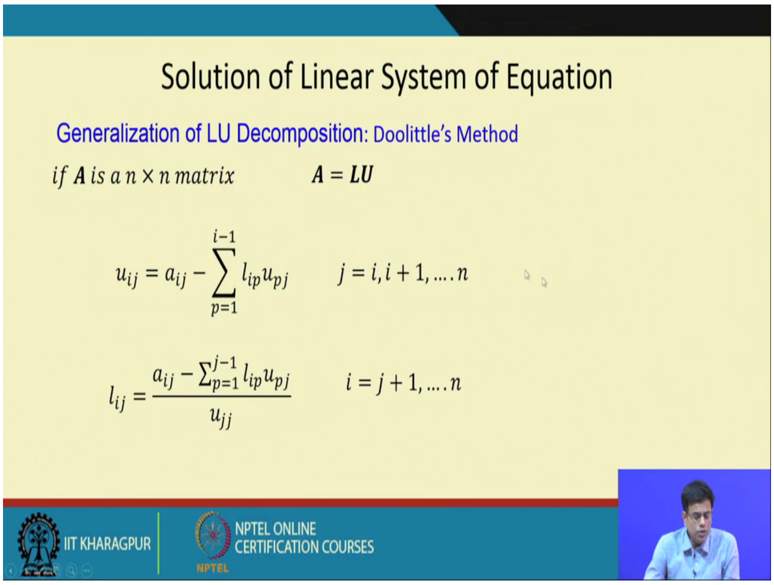

So,. So, once we do this we find out all lijs and uijs. So, once we do that so, this is the generalization we have seen for the 3 cross 3 matrix, which is a which can be generalized for any dimension means n cross n matrix and this generalization is how to do it. So, this is the formula for the generalized n cross n matrix. So, this is very helpful in coding because if you want to code it in a computer, you need to use this kind of formula.

So, here if you see carefully these l i p and u p j these terms are dependent. So, finding out u i j is dependent on l i p where i is where j or the p running from 1 to i minus 1. So, this is important. So, once you code it you will see that it is sequential as I was talking about in this that is just directly cannot find out u 3 three from in a first step, because to find out the u 3 3 you need to know you l 3 1 and l 3 2.

So, to find out l 3 1 and l 3 2 you need to finally, know the u 1 1 and u 1 2 and u 2 2 so, all those things you need. So, once you find out the u and l the solution is very simple.

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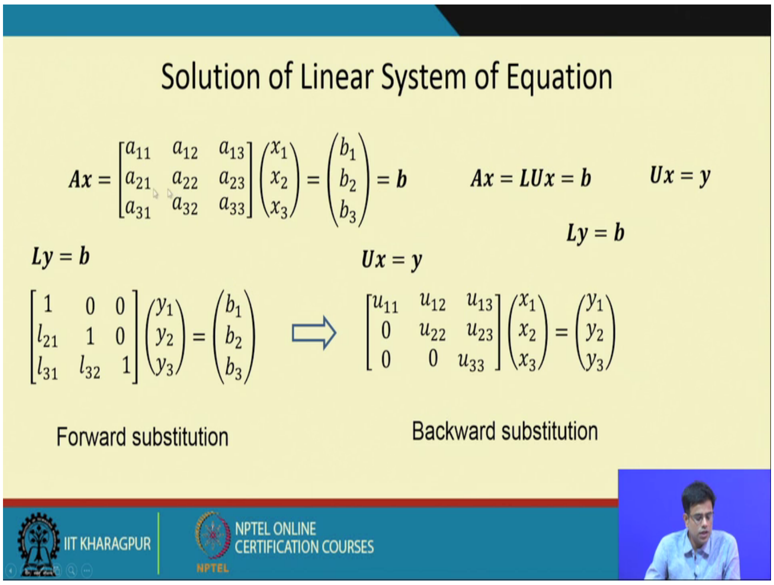

Let us see what is the solution, the solution is. So, we decompose this matrix a in LU form. So, if I write the matrix equation A x is LU x. So, LU x will be b. So, my new equation is LUx equals to b. Now if we assume Ux is y then L y equals to b because if I substitute Ux equals to y here and then L y equals to b. Now L y equals to b means we have to find out the y.

So, again you see this is a lower triangular system, where we use the forward substitution to find out the y 1 y 2 and y 3. So, once we know the y 1 y 2 y 3 the second part ux equals to y, the ys become goes to the right hand side. So, our job is to find out x 1 x 2 and x 3. So, once we write once we find y 1 y 2 y 3 again this is an upper triangular matrix. So, we use the backward substitution to find out x 1 x 2 and x 3. So, in this way the LU decomposition can be used to find out the solution of system of equation.

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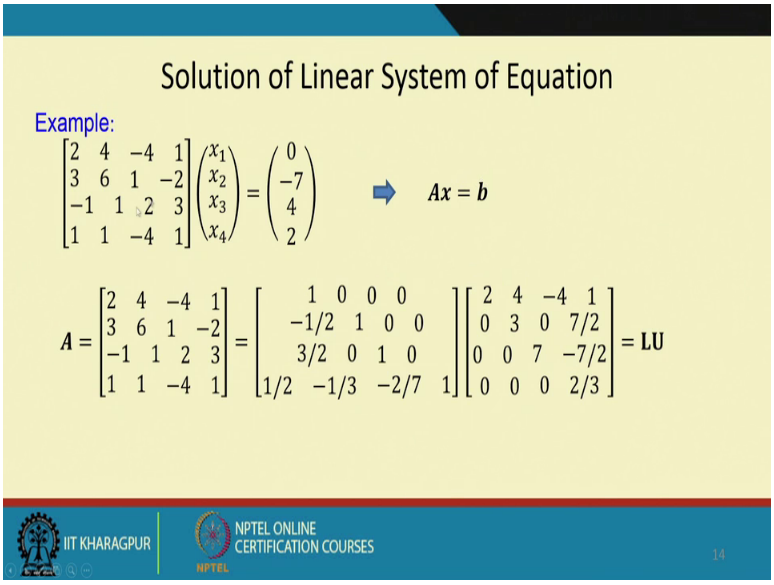

So, next let us see an example; so, if there is this is a the just the previous example. So, this is just the LU form of the coefficient matrix A.

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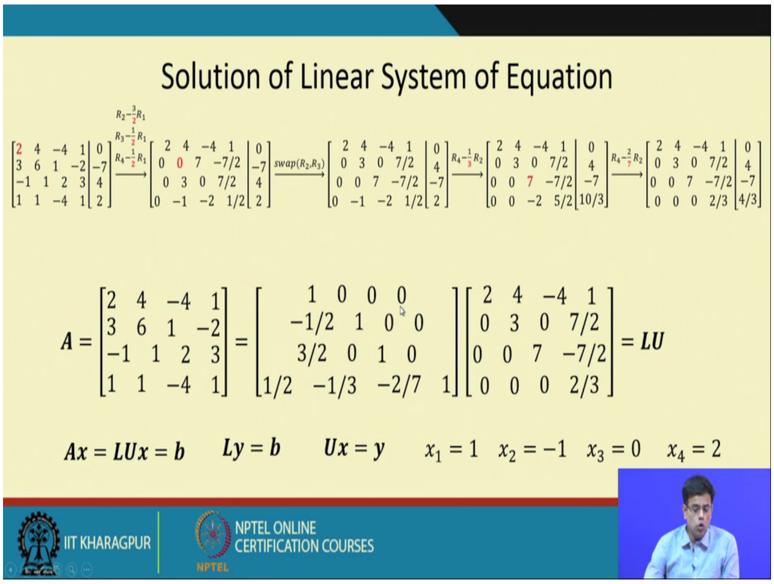

So, if you look carefully that in the Gaussian elimination, we were obtaining the same form of the triangular matrix through the using pivots and row operations.

So, if you look now these 2 procedure that this l matrix is essentially consisting of these pivot elements, the pivot multipliers the multipliers which we subtract or which we use to make the row operations and finally, making the matrices upper triangular. So, if you see these l consists of all such factors, and u consists the matrix which is finally, we obtained. So, but we do not change the right hand side here. So, we keep the right hand side original with the b. So, again this L y equals to b and Ux equals to y. So, the right hand side is actually L y this y is the change right hand side, which will be obtained by solving the L y equals to b.

So, there is a connection between the Gauss elimination and LU decomposition. The Gauss elimination is actually using the row exchange and pivoting operation, but in an LU decomposition, you obtain these pivoting through a decomposition procedure so or the factorization procedure. So, there is a connection between these 2 method.

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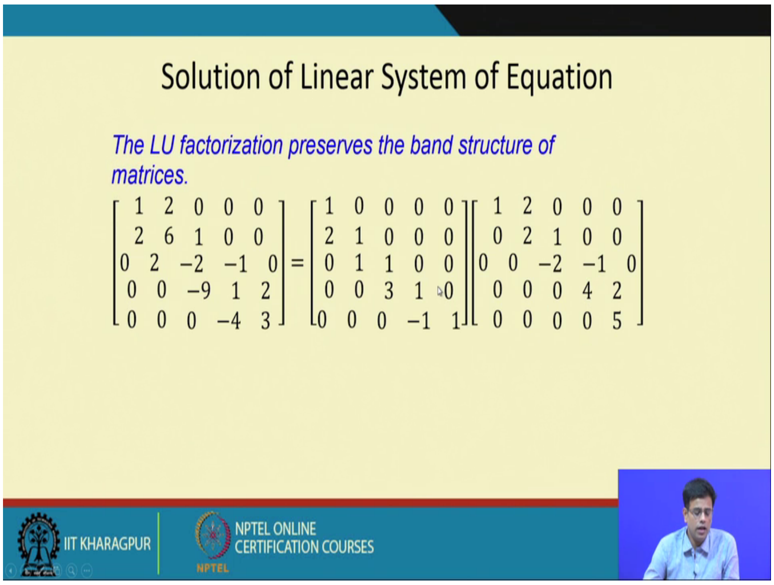

Now, another important thing is that LU factorization actually preserves the branch structure of the matrix. So, this is very important because we will be using the banded matrix mostly for the structural analysis, because on the stiffness matrix is when we will be assembled it will be in a band format.

So, if the banded matrix if we do the LU decomposition, it is banded form is preserved in L and U matrices. If you look carefully this is a banded matrix and if this the diagonal elements remains one in L and the first band is nonzero the other part of the matrix or the other elements of the matrices are 0.

So, keeping in mind that this band form is very similar to the original matrix because since this is a lower triangular matrix. So, upper part has to be 0; so, only the first band or the first diagonal form is the important here.

So, similarly in the upper triangular matrix, the diagonal matrix a diagonal elements are nonzero and the upper part only one entries in the band. So, this structure preserving LU decomposition is very important for us because as I have said already that we will be using mostly the banded matrix. So, that is all for today.

Thank you.