Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 06
Introduction to Tensor (Contd.)

Welcome, this is the last lecture for the module 1 for this course. This is this lecture is about the Curvilinear Coordinate System. So, specially, we will discuss about the orthogonal curvilinear coordinate system. Now, some of the, why; first of all, why do we need to have a different coordinate system except Cartesian coordinate system because, we are very familiar with the Cartesian coordinate system and we learn all quantities in Cartesian coordinate system.

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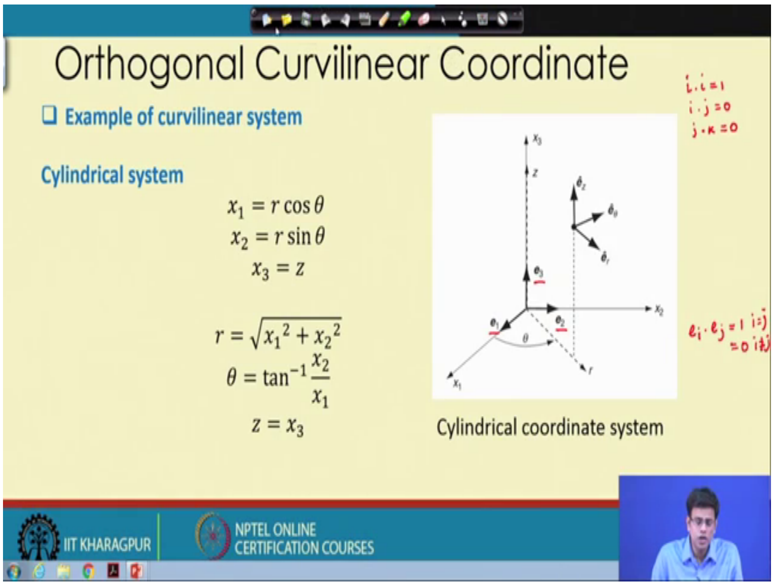

But, due to the symmetry or more specifically, the rotational symmetry of the geometry, we sometime need to use different coordinate system. And, using these different coordinate system, we may get some advantage specifically.

We will see it when we are dealing with the specific geometry for a specific problem for instance axisymmetric problems in elasticity or some cylindrical problems in elasticity, where we will specifically use different coordinate system other than the curvilinear coordinate system. Now, we will not talk about the general curvilinear coordinate system here; rather, we will talk about the orthogonal curvilinear coordinate system.

Now, the this term, the term orthogonal means, all you know that, if there is a basis vector i j k in the Cartesian coordinate system, then I can simply write it as i dot i equals to 1 and i dot j equals to 0 and other for instance j dot k equals to 0. So, these are the orthogonal coordinate system. Now, not only these coordinate system, the Cartesian coordinate system will follow is orthogonal; rather we follow it and we also say these coordinate systems are orthonormal.

What do we mean by the orthonormal? Orthonormal means, this unit vector for instance. In this figure, it is shown as e 1 e 2 and e 3. These vectors, these unit vectors is essentially of magnitude 1. These vectors are have magnitude 1 and similar to the definition of the orthogonal system, e 1 or e i dot e j equals to 1, if i equals to j and equals to 0, if i not equals to j.

Now, this coordinate system, we are very familiar with and this is known as the Cartesian coordinate system. Now, suppose I want to define a another coordinate system, which probably you have seen in some of the courses or some of the lectures earlier, one of them is cylindrical coordinate system. So, where we define the coordinate system as r theta z coordinate.

So, instead of x 1 x 2 x 3, we define that r theta z coordinate. Similar to cylindrical coordinate system, we also have used sometimes the spherical coordinate system.

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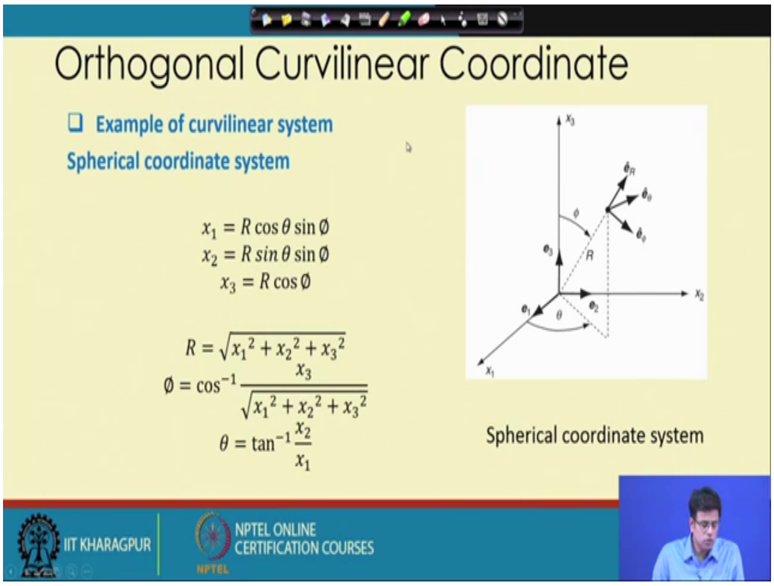

For instance, the spherical coordinate system where you define as R theta phi or R theta x i, whatever it is and these coordinate systems are related with the Cartesian coordinate system like this.

For instance, R is x 1 square x 2 square x 3 square and phi is cos inverse or arc tan of x 3 by root over R and then, theta is tan inverse x 2 by x 1. So, this type of coordinate system we have probably seen earlier. Now, our main objective for this lecture is to define our quantities which we have learned in tensor quantities or vector quantities, we have learned in Cartesian coordinate system.

What we will take? What it will take in the curvilinear coordinate system or specifically, cylindrical coordinate system or spherical coordinate system. In this course, we will be mainly using cylindrical and spherical coordinate systems. So, there are other coordinate system which do exist, but we will not consider these coordinate system. And we will mostly consider only cylindrical and spherical coordinate system.

Now, let us define in general, what is the relation between the Cartesian and any orthogonal curvilinear system. And then, we will go to for a specific case for a cylindrical coordinate system.

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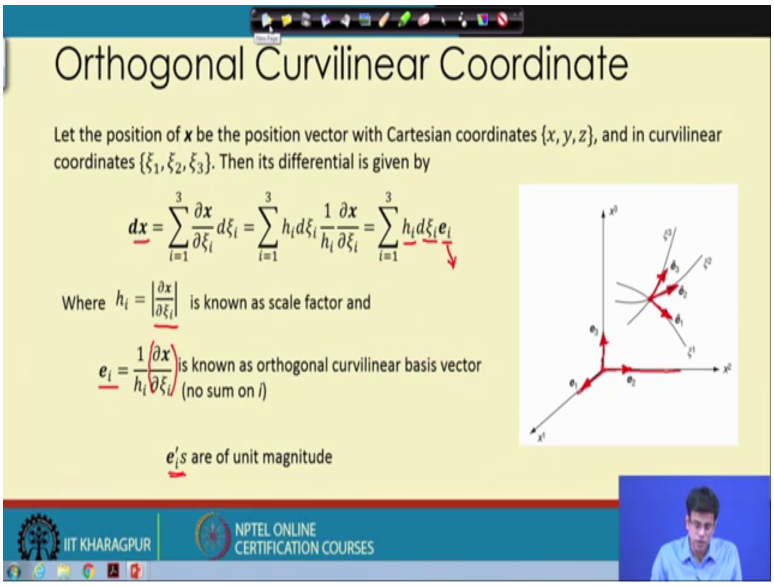

So, let us consider that x be a position vector with the Cartesian coordinates x y z and it is curvilinear coordinate x i 1 x i 2 x i 3.

So, now, if this is the case, then, any differential element, I can define it dx as which is dx by d x i into d x i; so partial of the dx. So, this has to be the summation of i equals to 1 to 3 and if you now expand it, so, if you expand it, I just multiplied a term h i and divided by a term h i here. And then, I define h i as the del x by del x i i. So, which is known as the scale factor of a curvilinear coordinate system.

Now, if I, now define my curvilinear basis for instance e i which is 1 by h i del x by del x i, now, here it has to be a remembered that it is no sum on i. So, now also, it is clear since we are defining this vector dividing this vector with h i. So, we this e i's are of unit magnitude because h i is the magnitude of this vector.

So, these vectors are unit magnitudes. So, my final dx is essentially h i d x i i into the vector unit vector. So, these vectors, I termed as orthogonal curvilinear system. Now, you see there are, one important thing here, in case of a Cartesian coordinate system, the coordinate axis the all coordinate axis is straight line right.

So, the direction in this coordinate system or specifically the Cartesian coordinate system does not change as you move from point to point. But, on contrary, this Cartesian coordinate system, these vectors may change that direction; this is one you have the important distinction between the Cartesian coordinate system and the a curvilinear coordinate system.

Now, so this is the general formula for which we can derive for any coordinate system, whether it is a Cartesian coordinate system or it is a cylindrical coordinate system or a spherical coordinate system.

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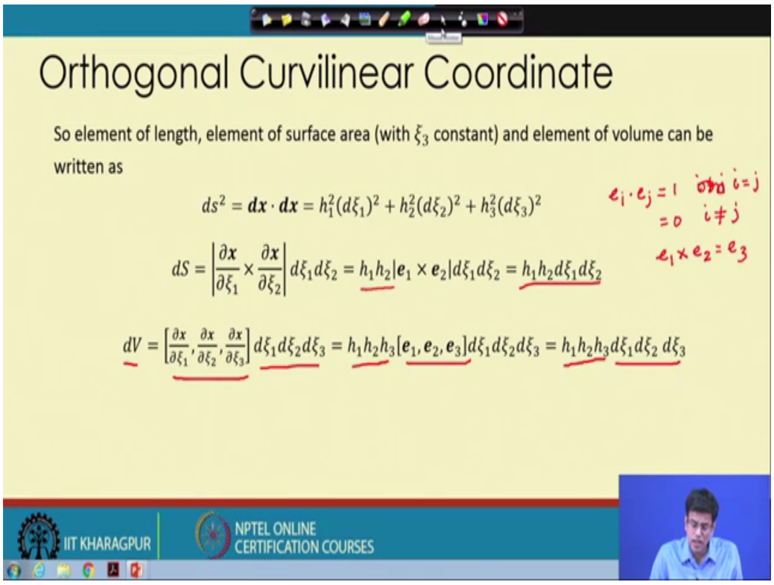

Now, for instance, if we if I want to consider a length element or a the length of a vector, I can just simply take the dot product of a differential this dx dot dx and which by this formula, we can just calculate the what is the length of a vector or specifically the length of length squared.

So, which is essentially h 1 d x i's square. So, h i square and d x i square which is a simple dot product. Now, here it, it is used that these e i's are since these e i's are orthogonal to each other, so, I use this relation e i dot e j is 1 when i equals to j and equals to 0 sorry, when i equals to j and equals to 0, when i not equals to j.

So, this relation I have used in deriving this in quantity. So, this become, the length have become a scalar product. Similarly, the surface of a body so that surface of a body, we can define as a cross product between the coordinates or the basis vector along as x i 1 and x i 2. So, which is essentially del x by del x i 1 and del x by del cross product of del x del x i 2 and then it is magnitude.

So, if you write this specifically in orthogonal coordinate system, so, you will simply just h 1 and h 2. It will come out from and these vectors will be e 1 and e 1 cross e 2. Now, even cross e 2 is essentially your e 3. So, this is the relation we know from our orthogonality relation. So, e 1 cross e 2 is essentially e 3 and mod of e 3 is or the magnitude of e 3 is essentially 1. So, this quantity becomes the surface area.

So, now you see carefully, that h 1 and h 2 is getting multiplied with the usual d x i and d x i 2. So, this is the actually the difference the scale factor which is essentially the term you need to consider when you talk about the general curvilinear coordinate system. Similarly, the volume of an element which we know from our knowledge of Cartesian coordinate system is, it is the scalar triple product or dx d x i d x x i 2 d x x i 3. So, this is the scalar triple product formula.

And, so, if I write it in this form, then this scalar triple product will similarly h 1 h 2 h 3 and this scalar triple product will becomes 1 because, it is an orthogonal with a or a specifically orthonormal. So, it is determinant will be 1. So now, this quantity will be just this thing. So, this is altogether the general curvilinear coordinate system.

Now, if you remember that, we talk about the transformation of the coordinate system. So, when we transform from one coordinate to another coordinate, we really use the transformation matrix or the rotation matrix.

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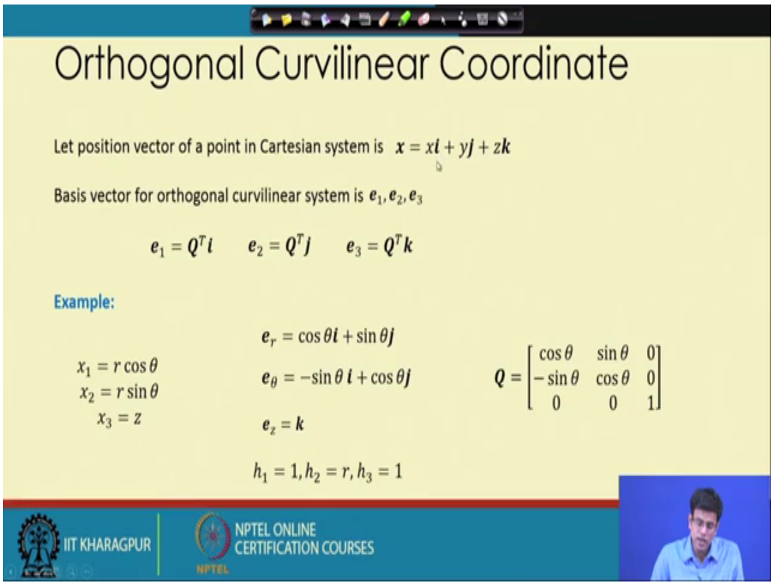

Similarly, in this case, when we transform from a Cartesian coordinate system to a cylindrical coordinate system, we use the rotation matrix.

Now, suppose, there is a position vector in Cartesian coordinate system, here I am denoting the basis vector for Cartesian coordinate system as i j k and for the curvilinear coordinate system, e 1 e 2 e 3. Now, as we know from our transformation rule, so, a vector is transformed with a matrix. So, now, e 1 can be simply transformed Q transpose i e e 2 can be represented at Q transpose j and so on.

So now, for example, if you see the cylindrical coordinate system, which is x 1 x 2 x 3 is related as r cos theta, r sin theta and z, then the basis vector again we computed from our previous formula which is essentially del x by del r. So, cos theta i and then sin theta del x 2 by del r which is sin theta j and del x 3 by del z which is del x 3 by del r which is 0 here. So, similarly, e theta we can compute and e z we can compute as k only and, but it is note notable that h 1 h 2 and h 3 is like this.

So, h 1 equals to 1 h 2 equals to r and h 3 equals to 1. Now, if you use this formula, so, what will be the Q matrix for the orthogonal cylindrical system? So now, this Q matrix turns out turns out to be this. Now, if I want to have a 2 Cartesian coordinate system, this Q matrix is essentially identity matrix. Now, since this is a cylindrical coordinate system, so, we will have this Q 1 these elements are not identity; rather it is cos theta sin theta and 1 for the z coordinate.

Now, our main objective here is to define the quantities which are in the Cartesian coordinate system we learn for instance gradient, divergence and called all those quantities how we can derive it through the Cartesian. Derive it in the Cartesian coordinate system. Now, we want to see the form how it takes in the general cylindrical coordinate system for instance.

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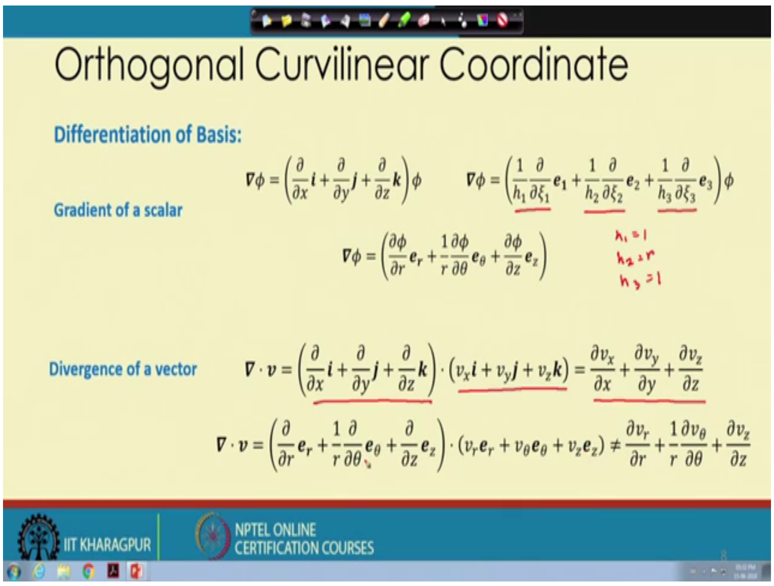

Let us talk about the gradient of a scalar. So, scalar we know in the Cartesian coordinate system which is del x i del del y j and del del z k with scalar function phi. So, del phi is essentially del phi del x i del phi del y j and del phi del z k. Now, in a cartel, the in an orthogonal curvilinear coordinate system, these quantities will be like this because, we will have this h 1 h 2 h 3 will be non 0. Now, if we now specify it for the cylindrical coordinate system, it is essentially we know h 1 equals to 1 for cylindrical coordinate system, h 2 equals to r and h 3 equals to 1.

So, it will be del phi del r e r 1 by r del phi del theta e theta and del phi del z e z. Now, let us consider for a divergence of a vector. Now, divergence of a vector in a Cartesian coordinate system, what we do? We will take the del operator or the nabla operator del del x i del del x y del del y j y j and del del z with the dot product of a component of a vector v x v theta v v x v y and v z and we take the dot product like this.

Now, if we do so divergence of a vector field is essentially a scalar. So, this is how we compute it. Now, in case of a cylindrical system, if we do the same thing, so, it turns out to be this because, since this is a orthogonal and a orthonormal basis, so e theta e r dot e r and e theta and dot e theta will be 1 and if r naught equals to theta, then it will be 0.

So, this is the most of the most of us do mistake by doing this because, this is not true actually. Why it is not true? We will see in a next slide. So, most of us does this mistake just it represents in a Cartesian system, we present it the way we represent it in a Cartesian system in a cylindrical system we also represent it in the same way, but it is not true. Let us see why it is not true and what is the difference.

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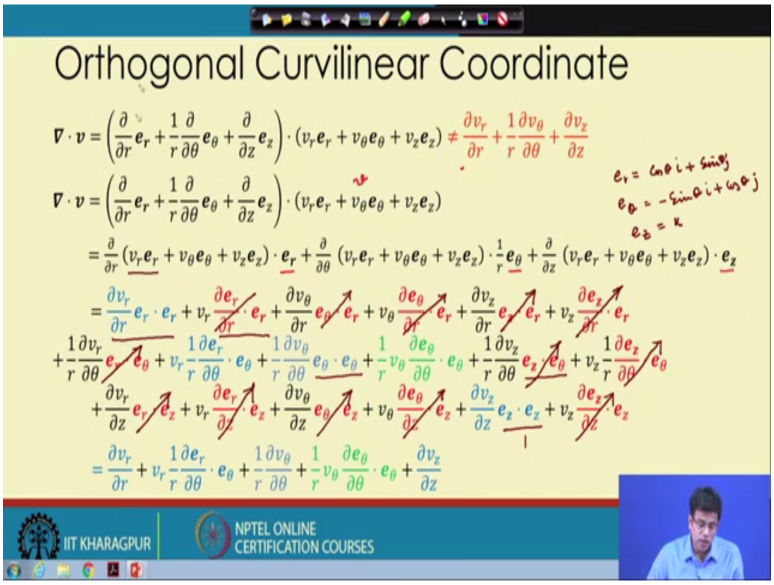

Now, let us expand del dot v. So, del dot v is this and which is I have explained why this is not true. We will see it now if you carefully do this dot product, so, which is essentially del del r of v r e r. So, this vector comes here; this vector v, this comes each of the term here and it is dot product with this. So, similarly, del del theta of vector v with dot product of 1 by r e theta.

Similarly, del del z of vector v with dot product of e z. Now, if we expand it, so, in a each term, we will expand it very in a religious manner. If we expand this each term, so, del del r v r e r dot e r so this is the first term we have. So, the second term del del r of del del r of e r v r. So, this is a product of 2 functions.

So, essentially v r could be a function of r and e r could be a function of r. So, in case of a Cartesian coordinate system, we do not see this difference because, in a Cartesian coordinate system, i j k's are constant essentially are not the function of the coordinate variables. But here, e r could be a function of r. So, that is why, the derivative with respect to r may not vanish. So, this quantity comes here. So, each term essentially is differentiated 2 times because, it is a product of 2 functions; one scalar function, one vector function.

Now, similarly, v theta e theta we can expand it d v theta d r e theta dot e r and v theta d e theta d r dot e r and v z d r e z dot e r and this e z d r dot e r. Similarly, the second term we will expand 1 by r del v r d theta e r dot e theta and so on. So now, similarly, the third term we expand in this manner.

Now, we see what is the consequence. So, since this is a orthogonal or orthonormal curvilinear coordinate system or the cylindrical coordinate system, so, e r dot e r will be 1. So, del e r by del r. So, del the first quantity del e r by del r. If you remember, our e r is essentially cos theta i plus sin theta j. So, it is not a function of r. So, that is why, this quantity will be 0. So, this quantity goes off.

Now, similarly, e theta e r will be 0 because, it is a orthogonal system. So, this quantity will also goes to 0. Similarly, del e theta del r so, e theta is also not a function of r because, we know e theta is essentially minus sin theta i plus cos theta j. Now, if we see e theta is not a function of r, so, this quantity will be 0. So, this goes, this cancels.

So, now e z dot e r. So, e z dot e r is will be 0 because, it is a orthogonal system so these cancels. So now, del e z by del r. So, e z we know for our cylindrical system which is only k. So, this quantity will again be 0; because, it is not a function of k. Now, the next term e r e theta again it will be 0 because, this term is since it is an orthogonal system, so, this term will be 0.

Now, del e r del theta. Now, if you remember carefully that e r is a function of theta, so, it is derivative need not be 0. So, let we will investigate later. So, similarly, del v theta d theta. So, e theta dot e theta. So, which will be 1 and del e theta e theta again, this will be not 0 because, e theta is a function of theta and e z e theta. This will be 0 because it is an orthogonal system.

Now, similarly del e z d theta which is not a function of theta. So, then this will term will be 0. Now, again the last part the e r dot e z will be this will be 0 and del e r by d z. Since e r is not a function z, so, this quantity will be 0 e theta e z will be 0. So, and d e theta d z will be again 0 because, e theta is not a function of z, but e z dot e z this will be 1. Now, del e z by e z. Again, it is not a function of z. So, this will be 0.

Now, so finally, what are the terms we will have? We will have del v r del r e e r dot e r will be 1 and the next term will have del v r 1 by r del e r by del theta dot e theta and 1 by R del v theta d theta, right. Because, e theta e theta will be e theta dot e theta will be 0 and similarly, 1 by R v theta del e theta d theta dot e theta and this will be 1. So, this will be your the last term.

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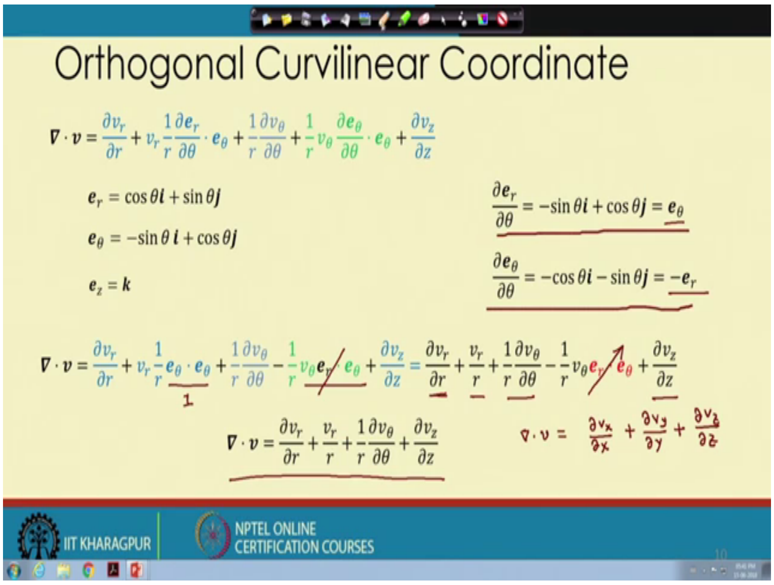

Now, let us see what happens to the others. So, now so our most non 0 terms are all non 0 terms are these. Now, since we know the basis e r, e theta e e z, so, d r d theta is essentially minus sin theta plus cos theta j. So, which is the same again, it is an e theta and d e theta by d theta is it is minus e r. So, in this way, actually we can substitute it.

So, we if we substitute it in place of d r, d of e r d theta; so, which is again e theta. So, e theta dot e theta and then this quantity 1 by r del v theta d theta remains unchanged and then 1 by r v theta del e theta by d theta which is minus e r. So, I substitute minus here. So, it becomes minus e r dot e theta and del v z d z. So now, this quantity is again remains same and since this is 1 e theta, e theta is 1. So, this will becomes v r by r and 1 by r del v theta d theta and then the last quantity this e r dot e theta is essentially 0 because, it is an orthogonal system.

So, when r naught equals to theta, it will be 0. So, this quantity again vanish. So, this quantity goes to 0 and del v z d z. So, finally, our non 0 terms will be this plus this plus this plus this. So, this is the gradient divergence of a vector field. So now, you see, there is a in a Cartesian coordinate system. The del dot v is we represent del v x del x plus del v y del y plus del v z del z.

Now, if we explain it, if we explain it clearly, why this change is here because, derivative of the Cartesian derivative of the curvilinear coordinate system are not necessarily 0. So, this is proved from by this fact. So now, the curvilinear case the expression for the gradient or as the divergence will be different.

Now, similar with this logic, we can extend it for the gradient of a or the curl of a vector field. So, that is cross product.

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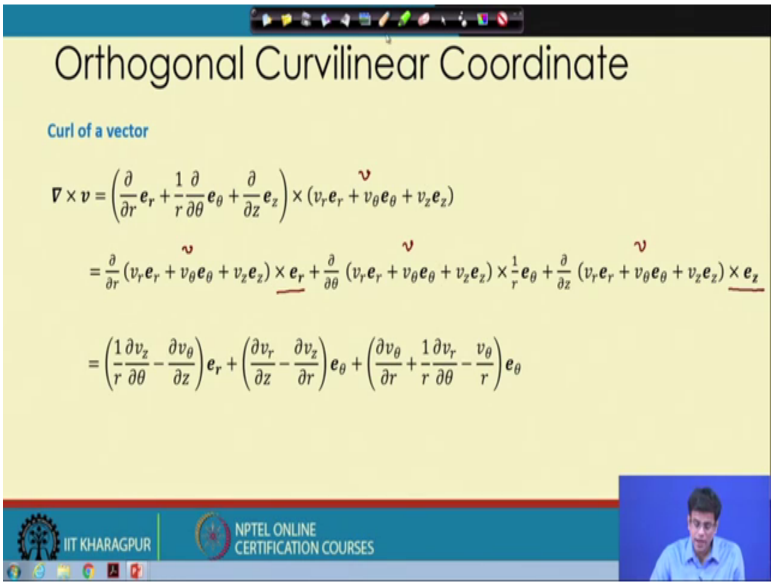

So, similar to this, we can just expand it with the cross product. So, individually you have to take cross product with r. So, del del r of that vector v. So, this is vector v. So, cross product with r and then del del r vector v cross product of 1 by r e theta. So, this is again vector v so, this is again vector v.

So, in the cross product of e z, so now, if you expand and use those quantities the derivatives with respect to the coordinate variables, so we will we come out with this formula. So, which is pretty difference from the Cartesian system; so, this we can achieve.

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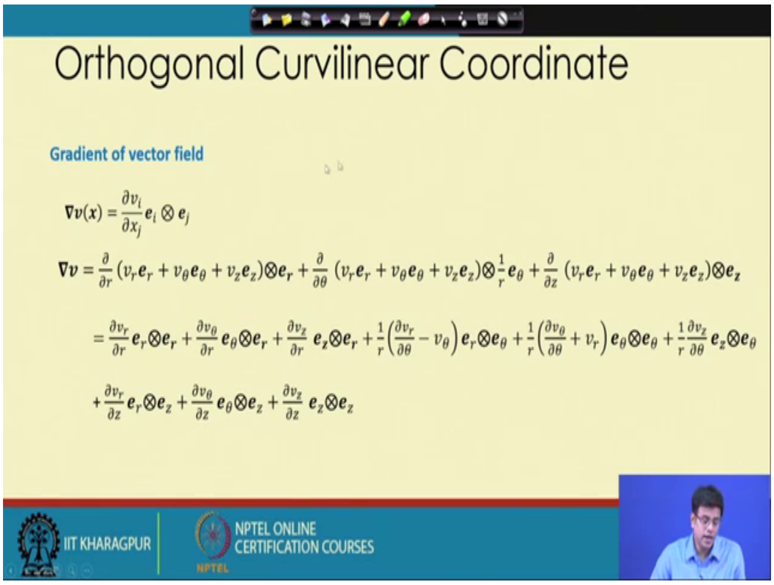

Now, for instance, the gradient of a vector field; so, we know that a gradient of a vector field is del v i del x j e i tensor product e z.

Now, if this quantity we have to evaluate in a Cartesians, in a cylindrical system, similar to the previous process, the vector field is essentially the v and this v has to be tensor. Take a taken tensor product with the each of this basis vector in the cylindrical system which I have done here. And so, if you do some manipulation, you come up with this formula. This is a long formula.

So, you can just try it and use the derivatives of the basis vector with respect to the coordinate variables to come up with this formula. Now, if suppose, I we can also find out the different formulas which is related to the orthogonal coordinate system or cylindrical coordinate system.

For instance, if I want to find out the Laplacian; for instance, if I want to find out the Laplacian or which is essentially the del square phi.

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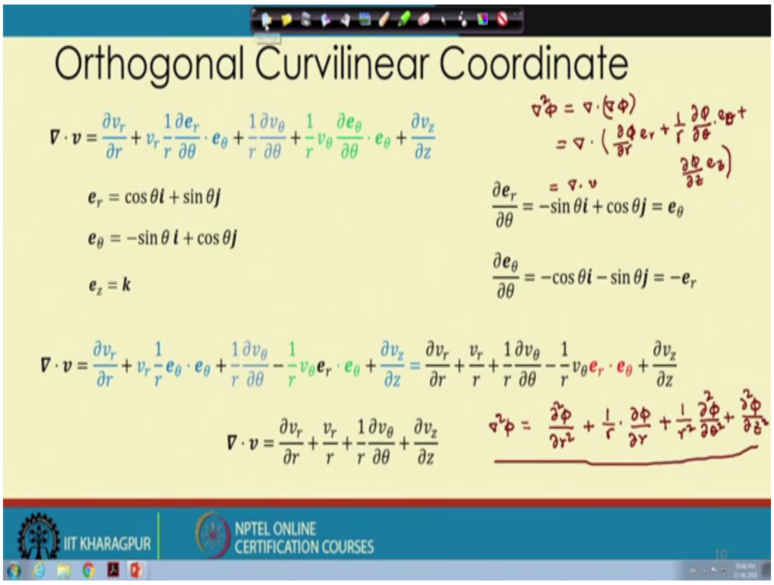

So, we know the del square phi is like this. So, del square phi is essentially del dot del phi so this we know. Now, del dot this del phi we know from our previous expression how to do it for the scalar quantity. A scalar function which is del del r of phi e r plus 1 by r del del phi of d theta into e theta e theta plus del del z of phi and e z.

So now, you see this is a unit vector. So, this quantity is again a vector. So, I can write out write it as del dot vector v or the let us assume now this vector is v now. So, see if we use the divergence formula here, we can compute the Laplacian. So, or so now, my del square phi will be del square phi will be using this formula. So, del del v r by del r. So, the first v r will be the del phi del r and then e r.

So, the first quantity will be del square phi by del r square right and similar to this, that v r by r. So, v r we know. So, which is 1 by r del phi by del r and then 1 by r del phi by del theta are del v theta by del theta. So, 1 by r del square phi by del theta square 1 by r square del square phi by del square del theta 2 and then the last term is essentially del square phi by del z.

So, this way we can del z 2. So, this way we can compute the Laplacian in the curvilinear coordinate system. Now, for instance, if we want to have, suppose if we want to have a cylindrical coordinate system where we essentially know the variable.

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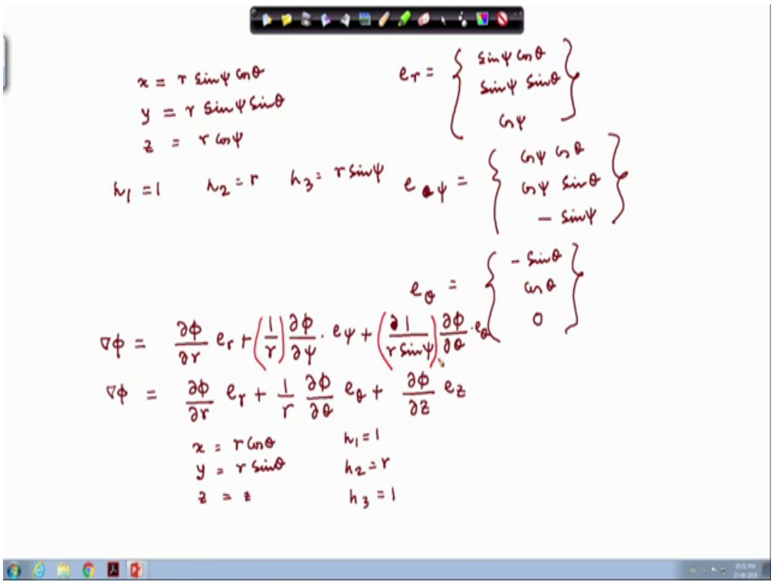

For instance, if we want to have this x is r sin psi cos theta, y is r sin psi sin theta and z is r cos psi.

So, if this is my relation for the spherical coordinate system, so, we can easily compute the basis vector. So, basis vector e r is essentially your sin psi cos theta sin psi sin theta and cos psi. So, this is my e r and then simply e theta. If I write e psi, if I write e psi, if I write so which is r cos psi cos theta cos psi sin theta and minus sin psi.

Similarly, e theta if I write, which is minus sin theta cos theta and 0. So, once we know this, we can compute the other quantities for instance the h 1 is or h 1 is 1 h 2 is r and h 3 is essentially r sin psi.

So now, if we know this, we can compute the gradient of a scalar function which is or the grad phi which is again del phi by del r e r plus 1 by r del phi by del psi into e psi plus del 1 by r sin psi del phi by del theta into e theta. So, you see the formula remains the same.

So, if you remember, for the cylindrical system what was the case for the cylindrical system? The cylindrical system, the case was del phi was del phi by del r e r plus 1 by r del phi by del theta e theta plus del phi by del z e z. Now, where e x is related with this r cos theta y is r sin theta and z is only k or z is z only.

So, if you relate this with the spherical system, you see only the h 1 h 2. If you remember, the h 1 h 2 h 3 for the cylindrical system, which is 1 h 2 is r h 3 is 1. So, you see the only h 1 h 2 h 3 here changes. So, which is essentially your 1 by r and 1 by r sin psi. So, this way, we can compute any quantity in a cylindrical or spherical system.

So, this is just a exercise keeping in mind that so with h 1 h 2 h 3 and the basis vector e 1 or the e r e psi e theta. So, all the quantities which we can calculate in the cylindrical system or Cartesian system can be converted to the spherical system as well.

So, now so, this actually helps us in doing some of the problems. So, now, in the next module, we will be discussing about the concept of stress and strain. And this module, actually with this lecture, this module we finish the introduction to the tensor.

Thank you.