Theory of Elasticity
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Lecture – 59
Introduction to Nonlinear Elasticity

Welcome, so, we have reached to the last lectures of Theory of Elasticity. Actually we have finished the course, this part we have kept for the where which we have not studied in this course actually. So, one of the thing is non-linear elasticity which we will discuss briefly in this lecture. So, before starting to this I would like to say that non-linear elasticity or plasticity or continuum mechanics or geometric mechanics these are itself with a course each of them is a course.

So, we will be discussing mostly what is the difference between linear and non-linear elasticity here and what are the measures we need to take care primarily to understand the non-linear elasticity. So, if you remember that when we discussed about the linear elasticity, so now, stress strain relation it is linear that we have understood, but the non-linear before terming the before going into non-linear thing. So, let me first define what is non-linear, which one or which function or which differential equation will say is non-linear.

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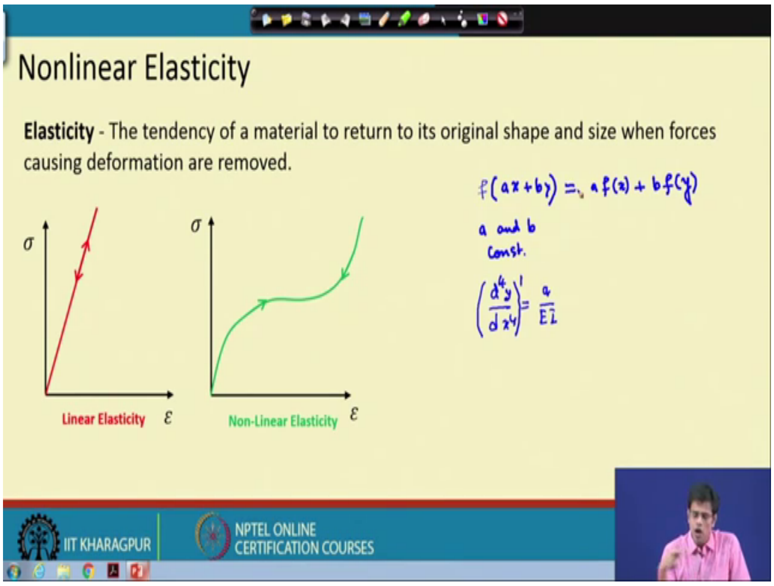

First of all let us see a function is said to be linear or specifically a linear function means, what does this mean? So, suppose there is a function f of ax plus by, where a and b are constant a and b are constant and x and y are argument of the functions. So, now if ax plus by if I can write that a of f x plus b of f y then I say that the function is a linear function. So, what this means? That means, if I can write these functions in this form then I say the function is linear function, if I cannot write in this form then I say it is a non-linear function. So, for instance if you simply write ax square plus y squared I cannot write it in f of x square plus f of y square something like that.

So, similarly this condition is applied to differential equation also. So, if the differential equation if I the highest order even it is 4 for instance the beam differential equation if you have if you remember del to the power 4 y by del x to the power 4 equals to q by EI this is a linear differential equation. So, because this the order of different maximum order of differentiation is 4, but the power of this differentiation that means, if I write it in this form is 1. So, this is a linear differential equation, but if you if this power is not 1, then we say it is a non-linear differential equation similar to the linear functions and non-linear functions.

So, now there is a consequence to this and the consequence is very much will be clear as we proceed. So, before starting with linear and non-linear let me first go back to our first lecture where we have discussed that linear elasticity in is linear stress strain curve. So, this is one type of non linearity or one type of non-linearity if this stress strain curve is non-linear but both of them are elastic. For instance, the this curve is stress strain stress is proportional to strain. So, which is the conventional which we have learnt earlier in this course also is the Hooke's law stress is proportional to strain.

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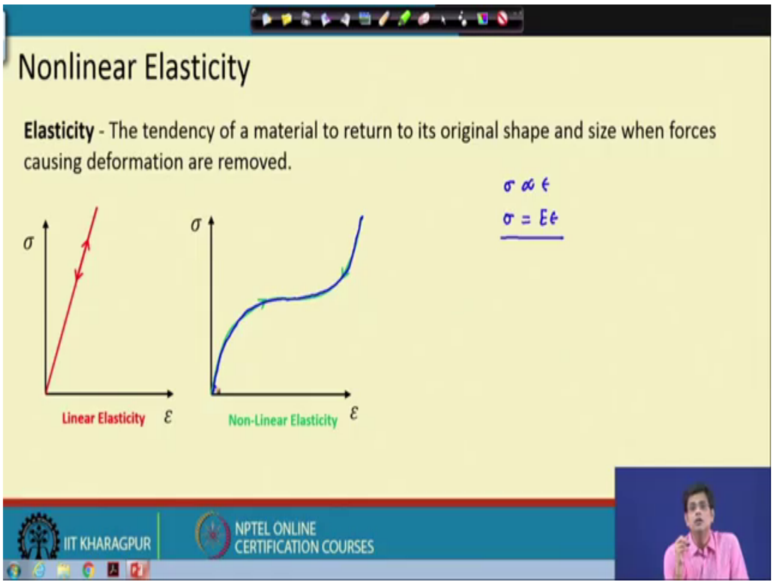

Now, that means, stress is a linear function of strain. So, I can put a constant here now, so which is sigma equals to E into epsilon for one dimensional stress strain law this is a linear function. So, now, if this thing is not valid then we say is it is a non-linear function that means, stress is not proportional to strain. So that means, the increment of stress is not equal not same as increment of strain so, that means, the increments are not constant.

So, then we say it is a non-linear elasticity, but again both of them are elasticity elastic. What does this elastic means? The elastic means that the tendency of a material to return to it is original shape and size when force causing deformations are removed. That means, that even though the stress strain curve if we plot if we start from here then it goes here and it reaches here and once I drop the load it follows the same path not a different path same path and reaches here.

So that means, the path is also important so that is why we call it non-linear now that is what you call it elastic. Now, then this type of elasticity is known as materially a non-linear elastic. That means that material parameter or the material behavior is non-linear. So, now, or what does this mathematically means? The constitutive behavior is non-linear. Now, let us see what are the basic equations that we have seen in your non-linear elasticity? In case of let us see the procedure.

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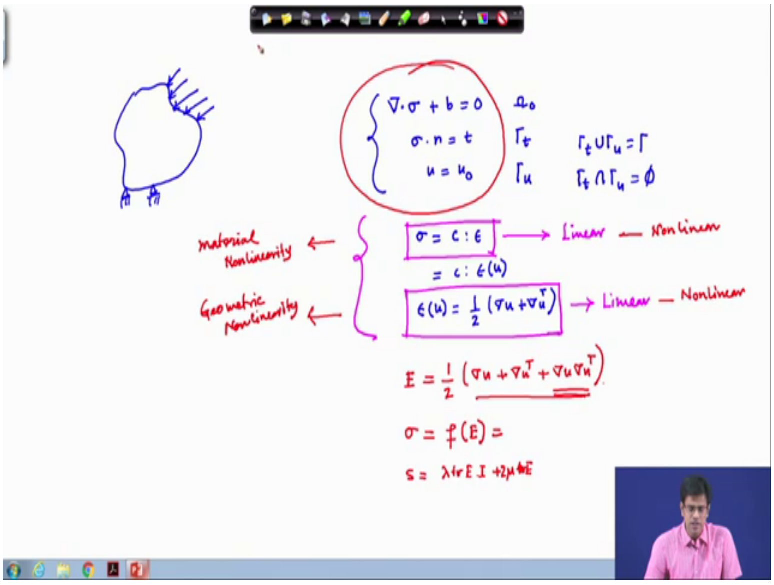

So, we have we know that our differential equation is del dot sigma plus b is the body force equals to 0 and then the boundary condition is sigma dot n is t that traction it is coming from the Cauchy's law and then u equals to u 0 at the boundary. So, if there is a domain here so, this domain there is a part of the domain is a traction boundary and part of the domain is the declare boundary.

So, now or the part of the domain is that Norman boundary and part of the domain is the declare boundary. So, now, this field if I define it as omega is my omega 0 is my the domain and the traction boundary is lambda t, gamma t and declare boundaries gamma u. So, we know that gamma t and gamma u union of this represents the total boundary now, gamma t and gamma u is our total boundary. So, lambda and gamma t intersection of gamma u is non-set so that we know so, now this is the differential equation in terms of stresses.

Now, here in this differential equation there are two things comes into picture, first thing is what is the this different differential equation you do not directly solve actually. We solve it for the in case of a stress based formulation that we have learned. So, this stress based formulation when we learned then we will directly work with the stress function approach and all those things so, that is a separate issue.

But, when we substitute the stress strain laws into this differential equation then essentially we use a constitutive law. What is that constitutive law? That constitutive law is for a linear problem that we have seen is sigma is c colon epsilon or c inner product with epsilon, so that is a c is a fourth order tensor and so that is a contraction with second order tensor to become a second order tension tensile stress.

So, here this c is a constitutive matrix that we know and epsilon is a function of displacement. So, again we introduce another kinematic quantity that is displacement. Now, this displacement again is related with the strain so, strains are functions of displacements. So, what is that? So, we know that is del u gradient of u plus gradient of u transpose so that also we know.

So, there are two relations one is constitutive relation that is the stress strain relation this is this relation and another one is strain displacement relation. So, this stress strain relation is known as the cause of constitutive relation and this is the strain displacement relation. So, once you have essentially substitute this epsilon to this equation and then with this equation you substitute here the strain you get finally, that displacement yes formulation. So, it is remember, it is to be noted here that even in stress based formulation if you want to calculate the displacement you need to have a constitutive relation.

So, now then this problem we say it is a linear problem. Why we call this linear? Because this relation is linear, so, first let us understand this relation is linear, this relation is also linear. So, now, that we have seen now if these two relation one of these relations is non-linear then our problem becomes non-linear.

So, what can be non-linear? So, the non-linearity comes into two way, one is this could be non-linear, so this could be non-linear and this could also be non-linear, right. So, if this is non-linear then we say it is material non-linearity. What does this means material non-linearity? And if this is non-linear then we say geometric non-linearity, right.

So, a problem can be geometrically non-linear and materially non-linear. So, if we say if stress strain relation is non-linear that means, this relation I cannot just write it in this form. So, for instance the strains, the strain measure in non-linear elasticity is done through that you bring Lagrange strain.

So, this green Lagrange strain can also be represented through this displacement gradients and displacement gradients can also be written in the non-linear terms. So, half of del u plus del u transpose plus del u del u transpose. So, this can be non-linear so, this term is essentially non-linear term.

So, now, this if this relation is a non-linear function of displacement. So, strain if it is a non-linear function of displacement we say it is a geometric non-linearity. Now, if the stress strain relation for instance it is not necessary that sigma is proportional to strain rather sigma is a non-linear function of strength so, sigma is function of strain.

So, it can be for instance if this equation can be written for a different constitutive model for instance Neo-Hookean model (Refer Time: 13:49) model or any other model for Ogden type of model. So, these type of models is essentially stress strain relation is not a linear functions and so for instance func type model, the exponential type of model. So, exponential stress strain relation which is a not a linear function of strain so that means, the stress strain relation is non-linear so, this gives us materially non-linear problem.

Now, one of you have asked that whether a material can be materially non-linear, but geometrically linear. Yes, theoretically it is naturally possible, but most of the material does not show this behavior, because the once the deformation becomes large our strain becomes large material behavior is essentially become the non-linear.

Now, but the other one is possible that means, if a material is linear, but a geometric non-linearity in the material presents of the behavior of the body is geometrically non-linear, but material behavior is linear so, that is all special. For instance the equivalent of the linear elastic Hooke's law is solved in a Kirchoff model which is essentially if I write it in terms of Piola Kirchhoff stress second Piola Kirchhoff stress say it is which is lambda trace of E; trace of E into I, so plus mu trace of mu trace of E square sorry mu tracer mu E, so mu E.

So, now this can be solved in a Kirchoff model so, this is again if we think of that is a linear elastic that means, linear material behavior, but E itself is a green Lagrange strain, which is essentially the non-linear. So, I think this will be two so, now you see even for the non-linear elastic problem this quantity remains same. So, this quantity or the basic differential equation remains same but, what will be the different? Difference will be in terms of constitutive equation, difference in terms of stress strain relation which defines two characteristics of non-linear elasticity which is materially non-linear or geometrically non-linear problem.

Now, before we start again what is materially non-linear or geometrically non-linear let us investigate what is the linear behavior means.

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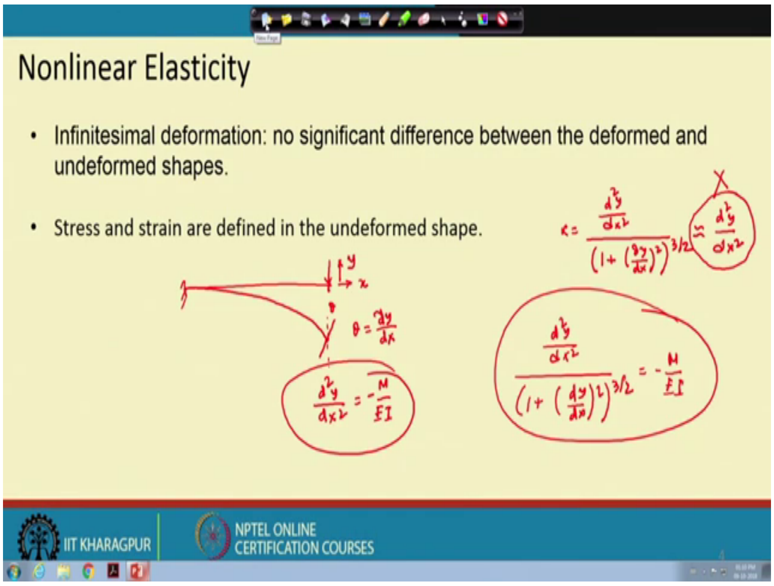

So, in a linear behavior we often said that stress strain relation is linear and the strain displacement relation is linear so, now what this mean? This means that in the case of a small strain problem or small deformation problem no significance difference between the deformed and undeformed configuration. So, if you remember that in a cantilever beam that you have solved that if you have a load, so if you take the displacement that means, the point here in the x direction if it is x direction and if it is in y direction. So, point this point does not move in x direction.

So, for instance the point here which was originally a undeformed state of the beam does not really move from here. So, what it changes actually it rotates so, it was which is perpendicular to this the fiber perpendicular to this it actually rotates little bit. So, this rotation is again we assume it is small so, we can characterize it in terms of del w del x or del y del x. So, this rotation is actually characterized if the rotation is theta then this theta is del y by d y by d x.

Now, how this has come this is comes from the bending theory pure bending theory where we have seen that relation that 1 by r equals to m by I, m by E I and all those formula and so we know what is 1 by r, 1 by r is the curvature. So, curvature if you remember the expression for curvature which is Kappa this Kappa is actually d 2 y by d x 2 by 1 plus d y d x whole square to the power 3 by 2 this is the exact curvature formula.

So, what we have taken? We have taken this is d 2 y d x 2. So, we substitute in the m by EI equation that bending equation. So, what we need get is d 2 y by d x 2 equals to minus m by E I. So, this equation becomes the differential equation for the cantilever beam.

Now, suppose I do not approximate this curvature. Now, this relation does not it is not true. Then naturally this equation the curvature formula if I write, so this Kappa will be in terms of this will be changed to d 2 y by d x 2 and 1 plus d y by d x whole square to the power 3 by 2 equals to minus m by EI.

So, now this is a essentially a non-linear equation because you see this has powers of 2 and 3 by 2. So, essentially this is a non-linear equation so, this means this equation the differential equation for the beam will not be a linear equation. So, now, why it will happen? Because when that means, if this point does not move for instance the in case of a elastic a problem where you have a differential the beam moves actually here so that means, this point is actually moved here. So, it is not necessary that this point will be staying here so, there is an a amount of shift delta.

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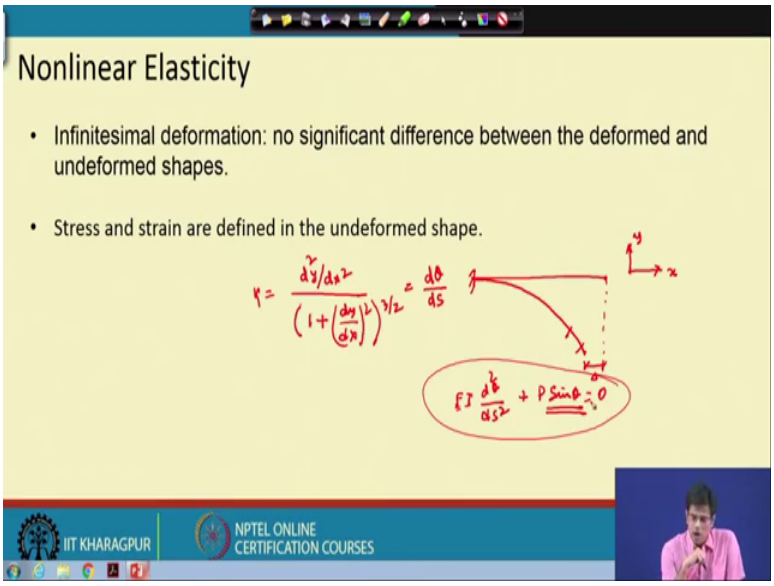

So, there we will actually write the differential equation not in terms of y and x rather we write in terms of the rotation itself that rotation of the beam which is if you write the cape Kappa there. So, which is earlier the Kappa formula is this and that is exactly by the way. And then you write this d y by d x whole square to the power 3 by 2 that is essentially d theta by d s. What is d s now? d s is the arc length of this beam.

Now, if you now we cannot write it in terms of y and x rather we can write it in terms of theta and s. So, finally, the differential equation will change so, EI d 2 theta by d s 2 plus p sin theta equals to 0 so, that becomes the or q x or if there is a load, so q x. So, this equation becomes an equation of elastica which is essentially non-linear function of sin so that means, sin theta is a non-linear function if I write it in terms of theta in series. So, theta sin theta is a non-linear function so, this equation becomes the non-linear differential equation.

So, now this will happen if I cannot approximate the deformed configuration and the undeformed configuration as same. So that means, that there is no significant difference between deform and undeformed configuration when it will happen if the displacement is small, right. So, then only it will happen so, if the displacement is very small or rather the displacement gradient is small then it will happen.

So, at which condition it will happen? Yes, for certain load, certain material, certain boundary condition, some displacements are small, but it is not natural for the all cases where the displacement will be only small. So, displacement can be large and certainly strain can go large, but not necessarily if displacement is large strain need not be large. So, strain can be small again displacement can be large so that is also known as large displacement, large rotation displacement larger rotation problem and small strain problem for instance Von Karman plate theory that is a large rotation problem essentially.

So, now another intrinsic characteristics are that stress strain relations are defined in the undeformed state. Since, the deformed and undeformed state is essentially have no significant difference we can approximate as the same configuration that means, the body which was undeformed and deformed state remains in the same position mathematically. So, even though it deflects, but deflection is small or the displacement and deflection gradient is deflection gradient is small. So, stress strain is defined on the undeformed configuration naturally. Now, if this is the basic characteristics of the linear elasticity that we have seen.

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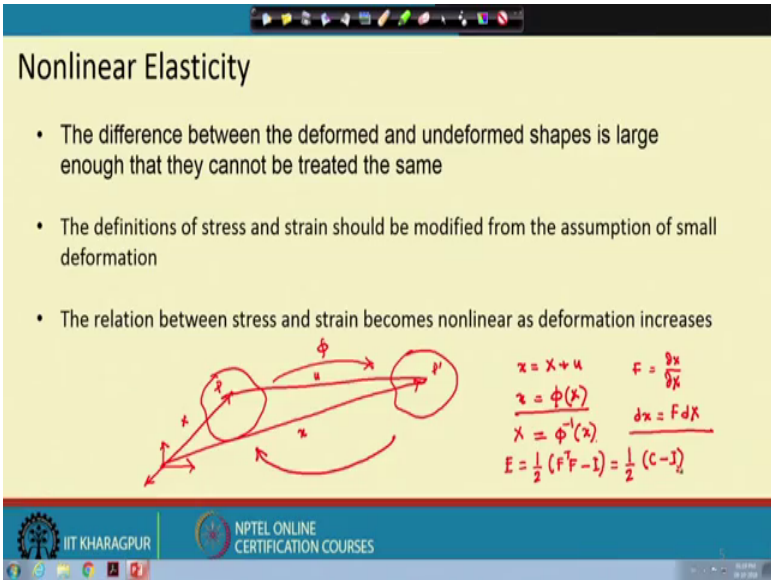

Now, in case of a non-linear elasticity all these all these deformations need not be true. So, the difference between the deformed and undeformed shape is large enough that they cannot be treated as same. So, what does this means? This means that if a deformation goes from suppose this is a body and this body goes here to here. So, with a point p it was earlier p and it is now in the p dash. So, now, this is a deformed configuration undeformed configuration and this is a deformed configuration.

So, I can just simply write this in x y z inertial coordinate system if I write then if this is my the position vector of this point is capital X and this is my position vector for the deformed configuration is small x then my displacement is essentially this. So, I can easily write the displacement u is small x equals to capital X plus u. So, this relation is essentially giving the displacement and the relation between the deformed and undeformed space.

Now, if you think of this, so there is a mapping between deform and undeformed space and this mapping is actually I can write that x is essentially phi of capital phi is a mapping is capital X and which is x plus c. Now, similarly I can write I can go from deform space to undeformed space which is the inverse of this phi or phi inverse that is cap capital X is phi inverse is small x, right, I can write it in another function also.

So, what this means? That means, that phi and phi verse is actually unique so that means, there is a one to one mapping that is the basic fundamental assumptions here so that means, the Jacobian of this phi is essentially should be greater than 0 that means, if there is no if the Jacobian is negative that it is non-unique or it is not possible the deformation is not possible

So, Jacobian of this mapping from deform to undeformed configuration is has to be written 0 and that gives us the concept of deformation gradient, that is how the deformation is happening. So, if I write that del, so if I write which is del x by del capital X that defines my deformation gradient and if a small element d x in the undeformed configuration. How it is related with the deformed undeformed configuration? With capital del x.

So, this gives me the deformation s`o, to summarize this that is the mapping from deform to undeformed configuration is unique and one to one that means, inverse mapping exist which actually defines the Lagrangian description of the system and the Euler in description of the system. So, it is a just a brief introduction because we cannot really cover these things in detail in this course.

So, now this any length element, we can find out the length element, we can find out the change in length, we can find out change in area, we can find out change in volume. So, how these things are calculated for instance the area transformation we can understand the Piola transform and all those things. So, now the strain measure similarly is defined as E the green Lagrange strain is half of F transpose F minus I so which is I can again write it in terms of Cauchy green write Cauchy green strain and then sees the right, Cauchy green strain tensor. And then I can also write the left Cauchy green strain sensor which is b so, that in terms of b also I can write, which is F transpose.

So, that is how we define the Lagrangian description of the system and the Euler in the description of the system. For instance another distinction is the definition of stress and strain should be modified from the assumption of small deformation that means, the stress strain relation is essentially non-linear and the stress and strain will be defined on the both the configuration, can be defined on both the configuration.

But, essentially the stress in the undeformed configuration has no physical meaning. For instance if I do if my and the stress cannot be there in the undeformed configuration because the body is not deformed how stress will produce, but in a mathematical sense it can be defined because, we have a correlation between deformed and undeformed configuration through the mapping f or the phi.

So, the stress measures which defined in the undeformed configuration is known as the Piola Kirchhoff stress, there are two type of Piola Kirchhoff stress, which is first Piola Kirchhoff stress and the second Piola Kirchhoff stress. And the undeformed configuration and the deformed configuration the stress is Cauchy stress; the Cauchy stress that we have usually seen.

So, that is why the differential equation that we write is on the deformed configuration in terms of Cauchy's stress. So, in the non-linear elasticity this distinction is not there in the linear elasticity because we really cannot distinguish between linear and means deformed and undeformed geometry, so that is a one aspect. And the relation between stress and strain becomes non-linear as deforming deformation increases.

So, if you remember the stress strain behavior of a steam for instance. So, if you have seen this stress strain behavior of steel for instance that there is a linear part of this and then there is an in point and then it is going like this. So, there is some part of this, this is actually the non-linear elastic part.

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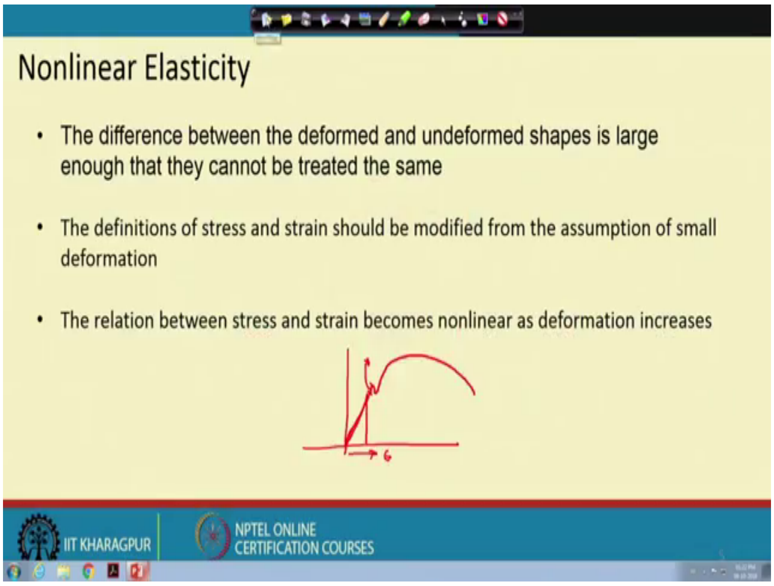

So, and this part is essentially the limit elastically limit or the linear elastic limit or the up to which the material behavior is linear or the stress is proportional. So, as the deformation increases or the strain increases this axis is strain so, as the strain increases the deformation becomes non-linear. So, and the stress strain relations now is non-linear so, this is one aspect of the non-linear elasticity.

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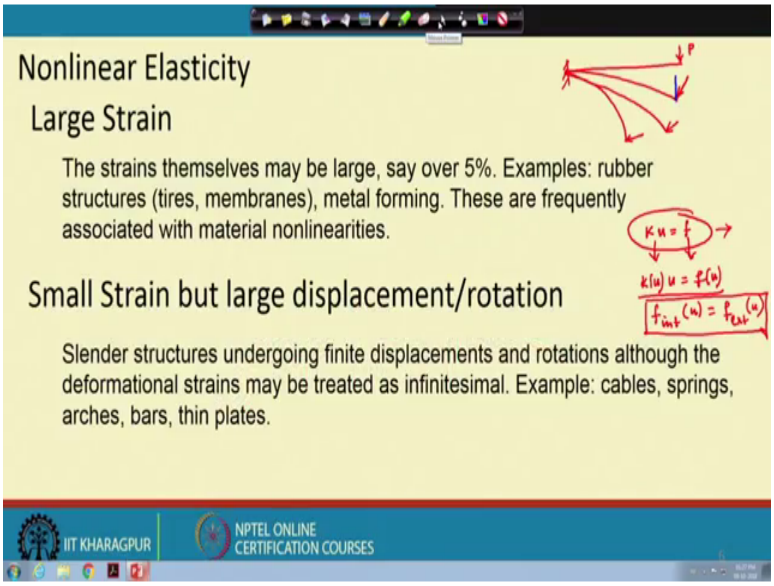

Now, another aspect I would like to see what is large strain? For instance which one I will say it is a large strain, generally if we it is we say it is more than 5 percent of the, 5 percent strain is we consider large strain for instance rubber metal forming these are essentially a large deformation problem. So, rubber elasticity, metal forming these are large deformation elasticity.

In case of a small strain especially for a slender structure for instance cable springs, and then plates thin plates, arches, bars, these are the slender structure essentially which can go for large deformation, but strain remains small. So, these are large strain large displacement rotation, I give an example of a Von Karman plate so that is a large rotation problem. So, this way we can distinguish between linear and non-linear elasticity.

So, the basic to summarize the basic thing for non-linear elasticity is that there are two source of non-linearity, one is actually the geometrical nonlinearity where, the strain displacement relation is non-linear and another is material nonlinearity where the stress strain are the constitutive behavior of the material is non-linear. So, these two aspects can happen together or it can happen individually. For instance geometric nonlinearity with small or the linear material behavior, but the non-linear material behavior with the linear geometry behavior generally does not happen because at the large deformation the material starts behaving non-linearly.

So, this another important thing is that that we need to understand that the deformed body and the undeformed body the configurations are not equivalent, there is a difference in the deform, undeform configuration which is actually not there in the linear elasticity.

So, for instance given another aspect is the load is deformation dependent which is actually for instance the follower forces. If I give you an example of follower force, for instance this if there is a cantilever beam and this beam if you load it after some time it goes here, but this load does not become this rather it becomes normal to it. So, this earlier in the linear elasticity it was this; it was this. Now, it becomes again perpendicular to the point of this so, the angle between them remains 90 degree so, this is one aspect.

So, as we move this beam this angle remains same this angle between these two remains 90 degree so, this is an example of follower force. That means, the function the load is P is dependent on the deformation it is dependent on the deformation. So, this type of loading is known as the follower force.

So, for instance, the example of it if you give a water into your garden the water hose pipe as the water pressure increases the pipe bends and then the but the force the water that is coming out is continuously changing it is direction so that is an example of follower forces.

So, this that means, your force is not constant. So, essentially your stiffness and the force vector that finally, you get it through finite element solution or any other discretized version of the differential equation is essentially dependent on the displacement or deformation. That means, stiffness of the body does not remain constant on earth so that means, the stiffness is a function for instance in case of a linear problem if you do finite element analysis then you get a k u equals to f this equation you essentially solved.

Now, this equation what does this mean? This is an internal force this is an external force, so the discretized version of the governing equation. So, this is an internal force and this is an external force. Now, in case of a non-linear problem this way we cannot really write so, stiffness is a function of u only. So, that means, I can write stiffness as a function of u into u is f of u this is one form. However, the most general form is that internal force that f internal is function of u and if external is function of u. So, that is essentially the non-linear equation for the non-linear elasticity.

Now, this equation since it is non-linear the solution method is again we just cannot solve like this that is k matrix vector solve, it is not that. So, we need to have a solution methodology is also has to be different. For instance we can use any non-linear solver for instance one of them is mu action which is doing successively linearization of the non-linear equation.

So, that Newton reaction we can use and form the non-linear and solve the and this non-linear equation incrementally so, that is the basic change in the solution methodology. So, that is that is what we want to give you an idea what could go different in non-linear elasticity. So, here I stop today so, in the next lecture what we will discuss, we will summarize the whole thing what we have learned in this theory of elasticity course.

Thank you.