Theory of Elasticity
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Lecture – 53
Thermoelasticity (Contd.)

Welcome. This is the last lecture of Thermo Elasticity, where we have planned to solve some of the examples of thermal deformation or thermal stress. So, here what we have done is we started with a general thermo elastic deformation or specifically the, we have derived the energy equations and showed that the uncoupled form of the thermo elasticity which again we have extended for plane stress and plane strain formulation.

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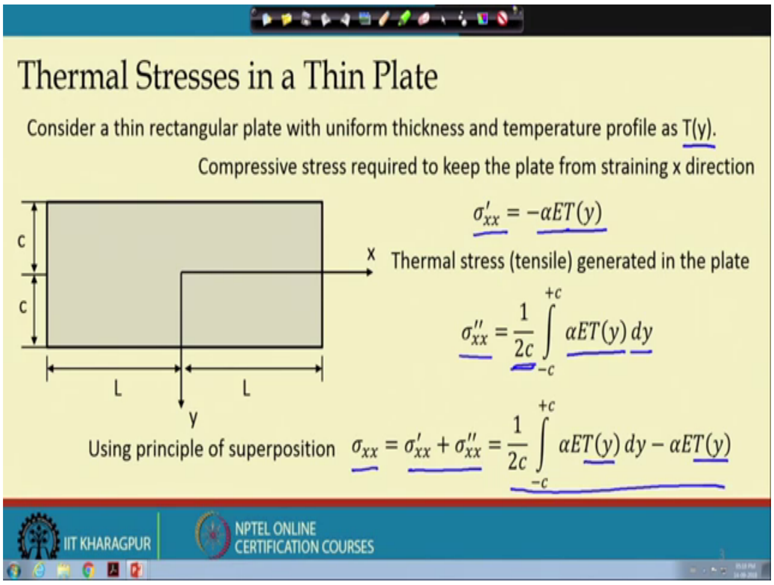

Then we finally, used some well known approaches like stress function and then displacement potential function and then derived the by a harmonic equation for the plane stress and plane stress case.

So, here in this lecture what we will do we will study some of the simple cases of thermo elastic deformation, where for instance let us consider a thermal stresses in a thin plate. So, consider a rectangular thin plate and then the with uniform thickness, but the interesting thing is let us consider the temperature profile is a function of y only, it is not a function of x and z and that leads to actually the thickness direction stress is constant basically we can invoke the plane stress condition.

So, now if you look carefully that if I want to find out the thermal stress along x direction considering that y direction is free. So, if you increase or if your thermal profile is T y then the y direction is free to expand, but if I try to resist in the x direction stress or the compressive stress essentially required to keep the plate from straining in x duration. So, if I just keep this direction if I am not allowing this direction to freely expand then the compressive stress essentially alpha E and T essentially delta T. So, T y is my that temperature variation.

Now, this is this stress sigma xx dash is varying along y direction because this temperature profile is a function of y. Now, similarly the plate will experience thermal stress. So, the thermal stress that is tensile; obviously, the opposite stress. This stress will be generated in the plate has to be integrated from minus c to plus c and then for if we divided per unit thickness, then this stress is integral of this quantity E into dy because temperature profile is variable in y then per unit length unit thickness of the plate or unit width of the plate.

So, now these two stresses xx and sigma xx double sigma xx dash and sigma xx double dash should be balanced. So, that is that we can do in with the help of principle of superposition. So, now, using the superposition I can write this must be the total stress along the x direction. So, now, if I put these two quantities so, this looks like this. So, this is my total stress in the x direction which actually needs to be equilibrated. Now, there is also an effect of you have probably known this thing for a from previous discussion that this is that at the end of the member there is a effect end effect which we called. So, this stress is naturally not valid at the end of the body because we have so called souvenir effect which is the end effect or the boundary effect of the system.

So, if it is sufficiently away from the end of system and end of the body then we can express this sigma xx as this. Now, it is interesting to know if you have this alpha E T y. So, if you know the temperature variation how T y is varied along y and so, you can find out the stress and you can compute this stress at any cross section. So, now, let us see let us assume some of the some stress some profile of the temperature are essentially and then let us see how it looks like.

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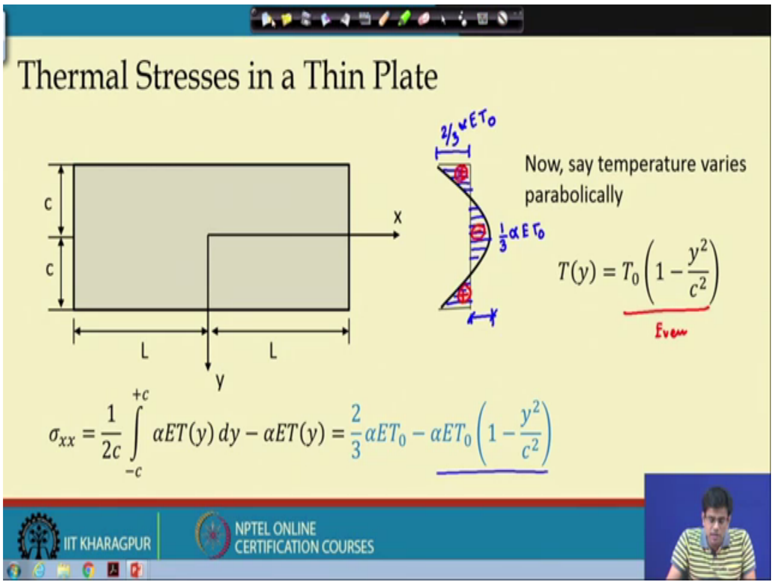

So, let us now consider in the temperature varies parabolically. So, it is a parabolic equation or something like that. So, now T 0 is the constant temperature and then how it is varying along y. So, if you now put this T y and integrate into this equation, then you get this expression; so, which is again a means variable in y square. So, if you see; that means, the temperature the stress along this body varies is along y square. So, this from which, we can calculate where the maximum stress occurs and where the minimum street occurs. So, suppose that y equals to c; that means, at the surface then this, the value if I write, so, this quantity goes to 0.

So, this stress is actually the positive stress or the tensile stress here in the body. So, this is two third two third alpha ET 0. So, this stress we know now if at y equals to 0 then this is two third alpha ET 0 minus alpha ET 0. So, this is now again the compressive stress. So, because this quantity is greater than the later quantity is greater than this quantity. So, this is essentially one third alpha E T 0.

So, now the, this part. So, this is this part essentially and this is this part. So, now, we can see this is the compressive negative parts and this is the positive part of the stress. So, this way we can find out the, what are the stresses in the body. So, if we assume the temperature profile in this parabolically, what will be the stresses in the body? So, this is very simple example by which we can understand the how temperature creates actually the variation.

Now, here it is important to know that this when we apply the principle of superposition we have applied only for the stress or the force actually, because this is the temperature profile if you look it is symmetrical about the x axis. So, that means, this function is a even function. So, now, if this is unsymmetrical now this temperature profile is unsymmetrical then you have to consider the couple also the couple balance also you have to form and then you can equate with the equilibrium and using principle of superposition then you can find out the what are the stress.

So, I recommend the, this part you can also go through in the Timoshenko book where the thermal stress chapter this is already derived in there. So, this you can follow later. Now, let us consider some of the other problems where actually we need to understand or the problems are little more complicated.

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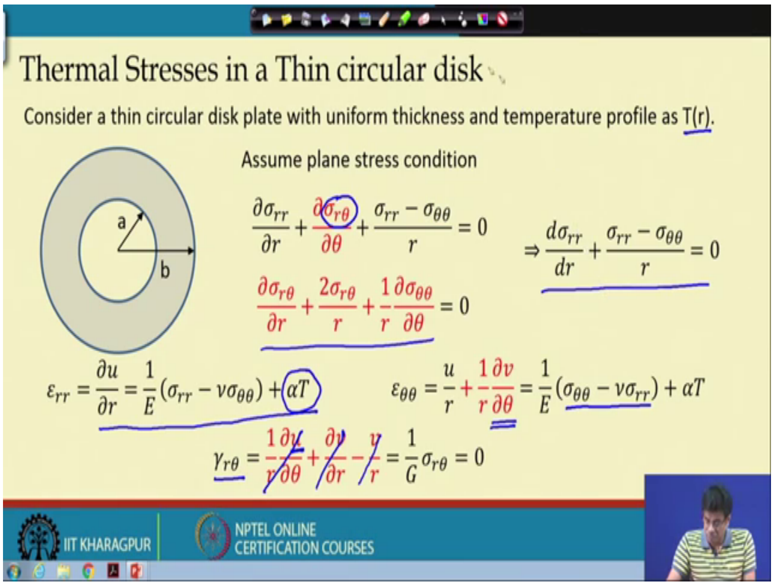

So, say instance for a thermal stress in a thin circular disk. Now, this disk is centered at the inner radius say it is r equals to a and outer radius r equals to b then in the again if I assume that temperature profile is a function of r only. So, there is no theta dependence on the temperature. So, then again since the disk we can assume again plane stress condition and then if we see that only sigma rr is the quantity that will actually be the nonzero, but since the deformation will be symmetric this quantity that is sigma r theta will be 0 essentially. And, then sigma theta will also be there, but sigma theta will be a function of r only because there is no variation of the temperature in the theta direction; so, because of this assumption.

So, now, if you will assume in the polar coordinate system what is the governing differential equation or the Navier's equation then you get these quantities, the red quantities are already zero. So, only this quantity and these partials becomes a total derivative. So, this becomes this equation.

So, now similar to our previous case we know the strains in polar coordinate system, so, this there will be u the radial displacement and the theta direction displacement is v which will be 0 because the v is not a function of theta. So, if I write now this quantity so, this will be the standard compliance relation that we know. So, only the thermal strain part is added. Now, similar to since the r theta is 0 so, and so, this quantity will be 0 and this quantity again u is u will be function of r only because from here we can see that u is a function of r only. So, this quantity goes off this quantity also goes off and this quantity also goes off. So, essentially shear strain is also 0 because it is a symmetric deformation. So, there will be no shear strain generated.

Now, if we now put this compliance relation that is epsilon rr and epsilon theta theta and then find out what is the constitutive relations or the stress strain relation and then again we can sub substitute, this sigma theta theta and sigma rr into the into our governing differential equation then we can get a final differential equation which is of in terms of u and v only in terms of u.

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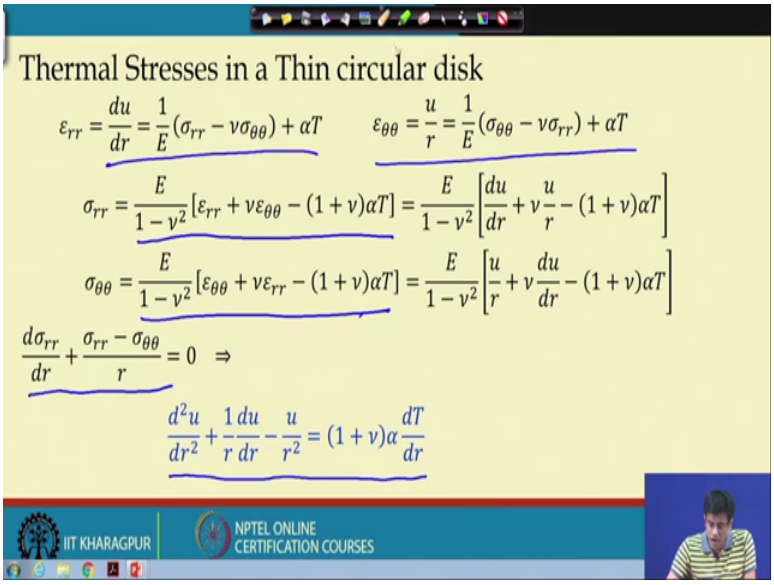

So, let us see what is that. So, this be[ome this does not become partial now because u is a function of r only. And, then we write in the final expression for the strains in radial direction and then theta direction and then we obtain the constitutive relations as you make that plane stress condition. So, this quantity is my final stress expression. So, you see this is very similar this thermal strain that we have discussed that will come into the stress expression.

So, now, if I substitute epsilon r r and epsilon theta theta here the stress = components look like this. Now, if we again substitute here this equation then this equation becomes a second order partial differential equation. Now, if you look carefully this type of differential we have studied earlier. So, this is the simple case, but this type of deformation we have seen in terms of displacement potential function.

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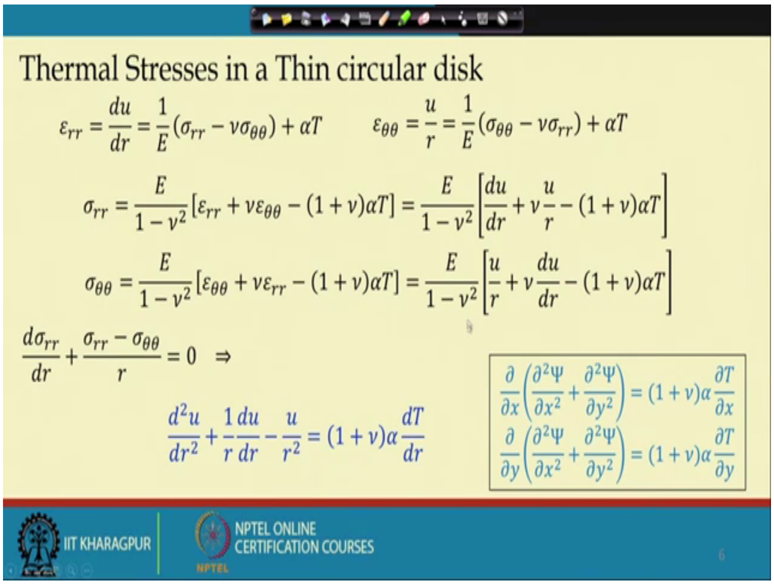

So, if you remember that this displacement potential function this del psi this is very similar this equation is very similar to this equation you see in a plane this is for plane stress condition. Now, what we did actually there we integrated this equation and to and make it one single equation. So, here before that integration this equation is very much similar in the polar coordinate system, this is in a Cartesian coordinate system.

Now, so, the fact is we are following the same approach here. So, now how to solve this equation? So, the solution of this second order differential equation we can easily obtain by this.

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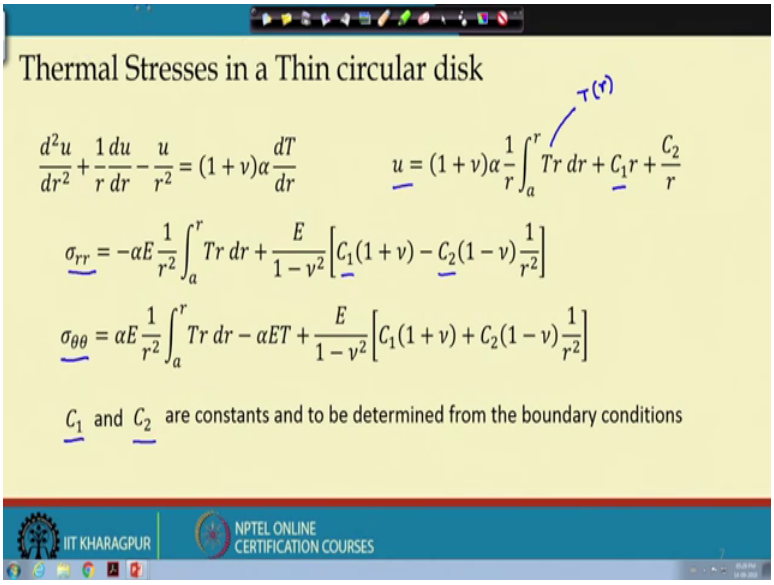

And, the solution is of this form. So, here the displacement is a function of r only since we do not know what is the profile of T. So, that integration we cannot perform so, it is a function of T r only. So, now, there is two integration constant and C 1 and C 2. So, now, if I write my stress quantities sigma rr and sigma theta theta then these stress quantities will be in terms of C 1 and C 2.

Now, since this C 1, C 2, C 1 and C 2 we have to find out from a appropriate boundary condition. Now, based on certain boundary condition we can find out C 1 and C 2 for instance if there is a solid disk the disk is solid then there is a boundary condition and if there is a disk with a whole that is r equals to a there are some stresses and r equals to say r equals to b the outer radius of the disk then there are certain boundary conditions. So, if we substitute those boundary condition then we can get the appropriate C 1 and C 2 value and then we can calculate the stresses.

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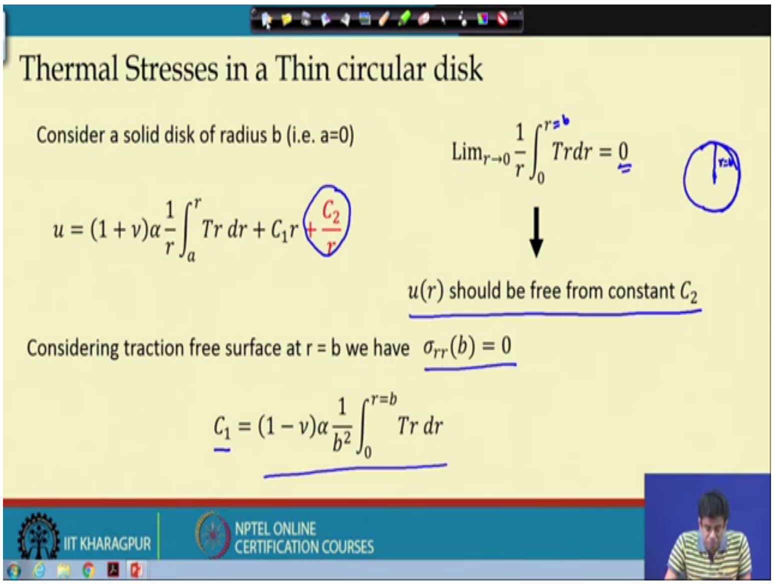

So, for instance that if that if we consider a simple solid disk then that is a equals to 0 and then b is the radius of the disk then if you look at this quantity. So, this quantity 1 by r 0 to r; r equals to b here. So, this r equals to if I take b then Tr dr this quantity if I tends r tends to 0 then this quantity becomes 0. So, this implies that this cannot be in the displacement solution or C 2 can be 0. So, u r should be free from the constants. So, this limit you can evaluate and then see that these goes to 0.

So, now if there is a solid disc there is no surface boundary condition surface traction here. So, only the temperature rise. So, there is no external force then sigma rr b must be 0, so, that is at r equals to b. So, this is r equals to b. So, if we put that then we can find out C 1 what is the value of C 1. So, now, then we can substitute this in the previous stress expression to get the surface temperature what is that and that leads to finally, there is a constant temperature at the centre. So, this constant temperature can be 0 also. So, this way we can solve some of the systems analytically.

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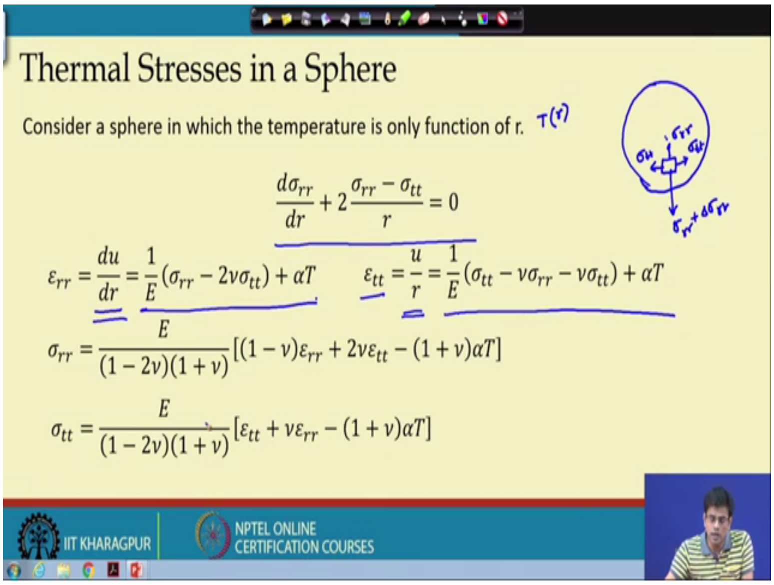

So, for instance, if we look another problem say a sphere of a thermal stress in a sphere; so, which is a 3-dimensional body. So, if I look in this way; so, now, from here there is a if I take a element. So, there is a tangential stress sigma tt in this direction and then sigma tt in this direction. So, this has to be balanced and then this quantity if I take sigma r here sigma rr and then this is again sigma rr plus delta sigma rr. So, this is the only stresses we will observe. So, again if we put this thing in the differential equation of 3-dimensional elasticity then we get this simple formulation simple differential equation of this from similar to the previous case.

Now, here again you I have we have a, we are assuming that T is a function of temperature is a function of r only. So, there is no tangential dependence of the temperature now similar to the previous case we get the du dr is radial direction strain and then epsilon tangential direction strain will be only u by r. So, finally, you get this form of the compliance relation.

And, then again likewise in the previous case if we substitute these compliance relation to the or if we find out the stress strain relation then this looks this form. So, now again if we solve this if we put this equation here then we get a differential equation in terms of u and then we can solve this differential equation and again there is a boundary condition by which we can find out the constant like previous case.

So, let us see how it turns out.

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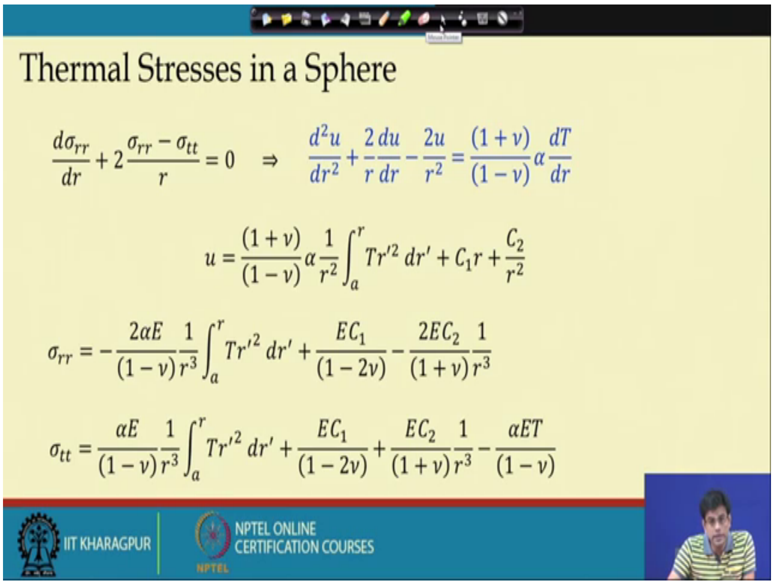

So, for instance if I just substitute that, so, I will get a differential equation which is also if you remember for a plane stress a plane strain case we get similar type of equation. So, now, even though this is a 3D problem. So, this is if I solve this so, I get a solution in terms of two constant because it is a second order differential equation. So, now again if I put in terms of stresses in terms of stresses so, this C 1 and C 2 will appear in the stress expression.

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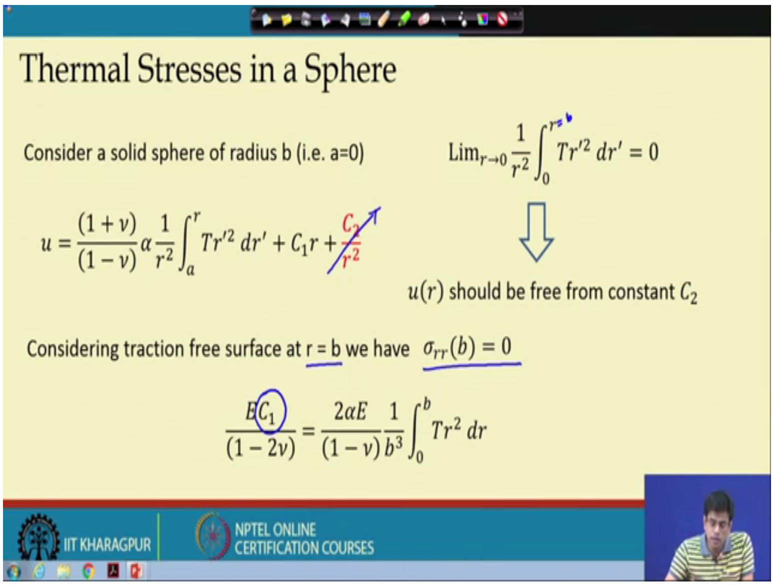

So, now, again likewise in the previous case if I assume it is a solid disk. So, then my this limit goes to 0, then again C 2 can be 0. So, this quantity I can remove from the body. So, this quantity I can remove from displacement expressions. So, now, once if I assume there is a solid disk. So, there is that means, a equals to 0 essentially so, here if I put r equals to b or there is a that outer radius is b and then the outer radius or the radius of the sphere is b, then considering again likewise in the previous case in the traction free boundary r equals to b that C r sigma rr has to be0. So, this gives me the constant C 1. So, now, once we know this one and then we can write the stress expression what will be the form of the stress expression.

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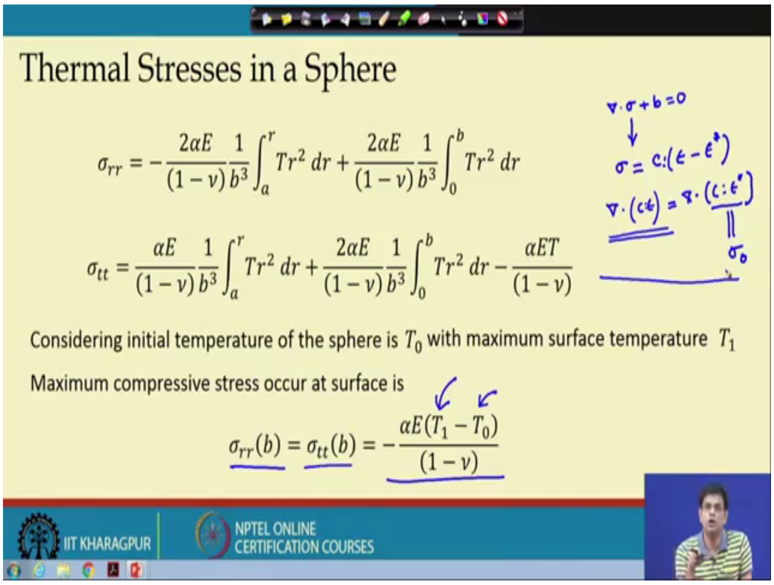

So, stress expression will be like this and tangential and radial stress expression. Now, suppose we have a sphere which is uniformly heated from T 0 to T 1. Now, this sphere this can be liquid or anything means the heating through the liquid or any external heat source. So, if the temperature is maintained T 1 degree then we can show from this expression that my stresses varies sigma rr sorry, the maximum stress can be this and which is the compressive stress maximum compressive stress occur at the surface. And, so, T 0 is my initial cold temperature and T 1 is my the temperature which is higher of the higher than T 0. So, this gives me a compressive stress at the maximum compressive stress at the surface.

So, in this way we can solve some of the simple problems of thermo elasticity analytically. Now, again I want to say that we have done the theory of thermoelasticity what are the governing differential equation, what are the plane stress governing differential equation, plane strain governing differential equation and we used stress function likewise for the purely elastic case we use ela[stic]- stress function approach. So, in the thermo elastic case also we have shown displacement functional approach. So, this can be assuming some displacement potential functions like psi we can solve those differential equation.

But, most of the solution even for the simple cases will be very analytically complicated. So, for ours this our objective is not to form this analytical expression rather we want to solve this equations in a numerical setup, where we use our numerical tools for instance the finite element method finite difference method or even mesh free method or even other collocation type of method where we just discretize those differential equation in space and then we solve the differential equation like in the finite element method we solve for a Navier's equation of elasticity.

Now, there are two – three things that has to be remembered for the thermoelastic deformation that we are following a uncoupled thermoelasticity, where actually we do not solve the heat equation with the naviers equation of elasticity rather we follow the Duhamel Neumann Duhamel Neumann constitutive relation and we separately get the temperature distribution and then put those temperature into the Navier's equation to solve the system.

Now, there are problems or there are type of problems where this uncoupled form of thermo elasticity is applicable, but most of the civil engineering structure and that linear regime; linear regime means where the deformation is small. So, this uncoupled form of the thermo electricity is applicable, but since the differential equation or the analytical solution is very difficult we go for the approximate solution through different approximate methods. So, that is another part of that.

Now, the other part of the thermo elastic deformation is actually we can even model the residual stresses of a body through the thermo elastic formulation. For instance, if you when the steel members we receive it consists from the mill it contains the residual stresses essentially. So, what is residual stress? Residual stress or essentially the initial stress in the body. B ecause the milling process gives you some permanent deformation which is embedded into the members.

So, this this residual stress can be very important in a member where actually if you have a crack or something some damage in the life span of the in the lifespan of the member then this damage and crack propagate very differently if it has inbuilt residual stress while coming out from the mill. So, this thermo elastic deformation can also model this residual stress model. So, this goes to a again the advanced topic which we will not discuss here so, but it is important it is important to know here that even though residual stress that can be modeled with the same equation of this thermo elastic deformation the equations which is essentially applicable for the thermo elastic deformation.

And, another important part is that even if you look this differential equation this goes to this acts as a body force in the body. So, if you if you if you remember the Navier's of Navier's equation of elasticity which is sigma del divergence of stress plus b or f equals to 0. So, this if this contains a thermo elastic deformation which is in terms of alpha delta T for an isotropic material. So, this essentially goes to the this side. So, if I again write this if this if my thermo elastic constitutive relation is this c is a my constitutive matrix then epsilon is my total strain then epsilon is my thermal strain.

So, then this if I substitute and if I take body force is 0, then we can easily show that this term c epsilon, so, divergence of sigma c epsilon is essentially. So, this is inner product of 2 tensor. So, this goes to this, so, divergence of c into epsilon star. So, this epsilon star is the thermal strength. So, these this if I say this is my residual stress or sigma 0. So, which is if I know then this problem can be so, residual elastic strain is developed in the body. So, essentially this part is the body force which acts on the differential equation. So, that is all.

We have solved our objective of this module was to solve some of the to know the first the theory of thermo elasticity and which is the uncouple thermo elasticity specially and then we did some of the plane stress plane strain formulation and then we have solved some of the analytical examples by which we can solve some simple problem with simple geometry and boundary condition. But, major emphasis is actually the solution of this equation are mainly done with the approximate methods because analytical methods we have several difficulties in obtaining this analytical functions of the displacement.

So, that is all for this module. In the next module we will start another aspect of the elasticity which is very similar to the experimental stress analysis is known as the photoelasticity.

Thank you.