Theory of Elasticity
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Lecture – 52

Thermoelasticity (Contd.)

Welcome. So, we are in module 10 and lecture number 52 where we are discussing Thermoelasticity or thermoelastic deformation specially we are discussing for uncouple thermoelasticity. So, in the last lecture we have gone through the plane stress form plane strain formulation. In that plane strain formulation, what we have done? We have just use the 3-dimensional thermo elastic equation and specially the Duhamel-Neuman constitutive relation that we have modified for the plane strain case and in doing so we have also done for the displacement potential function which is essentially u is del psi by del x.

And, v is del psi by del y and using these the 2-dimensional governing differential equation is modified and it turns out that after integration, we could convert this to a single differential equation of that displacement potential function psi which is a function of xy. So, for a plane strain case this is this we have done in the last class and in the solution of such differential equation is very general which has been also taught to you in engineering mathematics course or probably the higher mathematics course. The in that the basic way of solving the differential equation is so, you find out the homogeneous solution and you find out the particular solution and then you add those two solution to get the final solution of the differential equation.

So, now, in this lecture what we will do? we will do for the plane stress case in the plane stress case similar to the plane strain case we will also modify it for the using displacement potential function and then additionally we will also discuss the stress function approach which you are already familiar with because we constructed the stress function earlier for the pure elastic deformation in that pure for the pure elastic differential deformation what we have seen is that stress function exactly satisfies the governing equation. So, here also we will check that fact.

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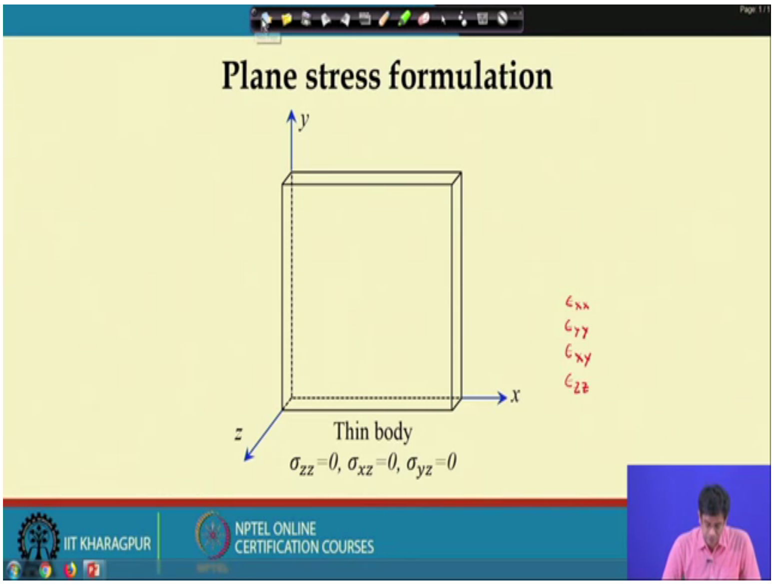

So, to start with theplane stress deformation which is a thin body so, that means, the z direction it is very thin. So, we can approximate as a 2-dimensional body and this plane stress case means there is no stress in z direction and also the shear stresses sigma xz and sigma yz are 0. Similar to that the plane strain case where the strain at the z direction is 0 and shear strain xz and yz is 0. So, in that case for the present case sigma zz and sigma xz and sigma yz is 0.

Now, corresponding to this we have also non zero strain components which are epsilon xx, epsilon yy, epsilon xy and then epsilon zz because epsilon zz will not be 0 due to the Poisson's effect that we know from what earlier discussion.

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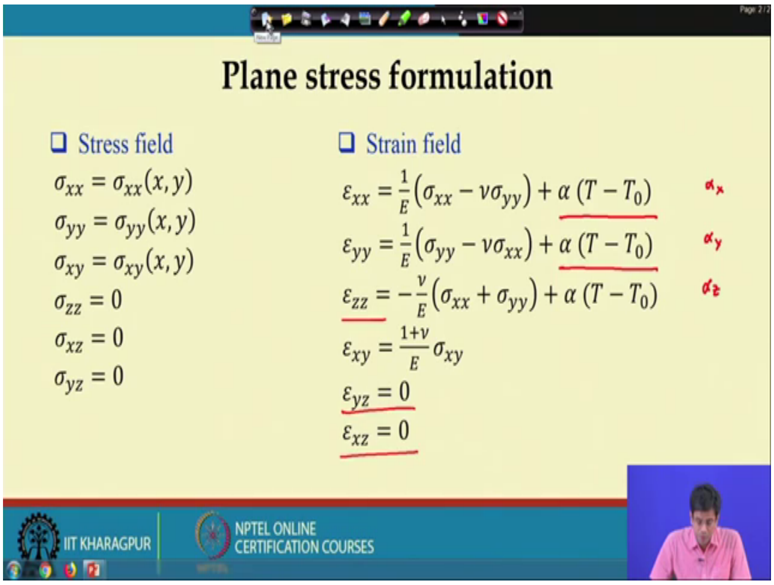

Now, to do this plane stress body under thermoelastic deformation, let us see what are the stress field finally, we obtain we obtain the stress field sigma xx, sigma yy, sigma xy which are all function of xy and the all other three stress components are 0. Compared to that that strain field strain field is epsilon xx is essentially this part of the deformation is due to purely mechanical deformation that we know from our knowledge and then the thermal part which is alpha T minus T 0 and so, epsilon yy is also this thing is nu is a Poisson's ratio and 1 by E and then the thermal part thermal strain part.

Now, here we are assuming the isotropic elastic body. So, there is one alpha and alpha is constant in all direction I mean the alpha is same for the all direction that coefficient of thermal expansion of solid. Now, in case of an anisotropic body where you want you probably want to consider the alpha is different in different direction so, these quantities may be alpha x, alpha y corresponding to that alpha x, alpha y and alpha z you can consider for the anisotropic body. But, since we are mainly interested here the isotropic body E the where the alpha is actually constant now here the epsilon zz will be there because of the Poisson's effect.

And, this Poisson's effect this is how to derive this thing you have already known from your 3-dimensional stress strain relation where probably you have calculated what will be the epsilon z. So, I just put the epsilon xx and epsilon yy in the 3-dimensional strain equation and then finally, we got this now epsilon xy is again the shear modulus and then shear strain how shear stress and shear strain how it is related. So, this is engineering shear since we actually not gamma xy. So, epsilon xy that tensorial quantity and then epsilon yz will be 0 and epsilon xz will be 0 that we know from the earlier discussion.

The only difference here is that you will have this alpha the thermal part of the strain which is additive with the usual mechanical component of the strength. Now, to now similar to the previous case we will see the compatibility and the equilibrium equation.

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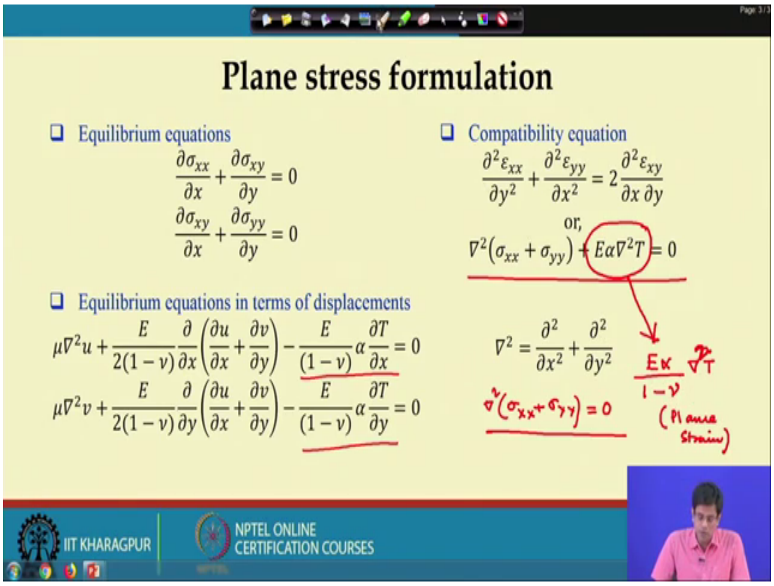

Now, in the equilibrium equation that remains same for a 2-dimensional body we are considering here the body force is 0, because that will not generally affect the even if it is a constant body force we have seen that compatibility equation does not change at all. So, but if the body force is a function of x and y; then obviously, the compatibility equation will change and that discussion has also be done in earlier.

So, now once we get the stress differential equation in terms or the equilibrium equation in terms of stresses we can substitute the previous expression for the stresses and then we can put it the strain expressions. So, strain expressions if we put the, which is a strain displacement relation and then we get the differential equation in terms of strains in terms of displacements u and v. So, which I have done it and you can also try it there and then this equation looks like this.

So, it is two differential equation for two unknowns u and v and then the important part is that this part of the differential equation is containing the partial derivative of temperature that is T; T is a function of xy. So, that is the only difference. Now, again similar to the previous case, the compatibility equation which is important, So, in the stress base formulation we need compatibility equation to satisfy and finally to calculate the displacements. So, here the compatibility equation is same, but once you substitute to the strain components which is discussed in the last slide then you get the compatibility equation in this form.

Now, if you remember that compatibility equation for the general plane stress case which was del square sigma xx plus sigma yy equals to 0 at there is a constant body force or no body force constant body force or no body force. So, now, this equation only the modification is the thermal part of the strain thermal part of the deformation. So, this quantity is additive with the, this quantity. So, this is for thermoelastic deformation. So, now, you see this equation becomes a non-homogeneous equation and this is the homogeneous equation.

Now, if you also remember that in case of a plane strain deformation we have also deduced it in the last lecture which is in case of a E alpha we had E alpha into 1 minus nu. So, this quantity for the plane stress plane strain condition is E alpha by 1 minus nu del square T. So, this is for plane strain plain strain condition. So, only difference with the plane strain and plain stress is this is actually E alpha for the plane stress and plane strain is plane strain is so, this is del square T. So, plane strain is E alpha into 1 minus nu.

Now, that is the only difference that we know even from the constitutive equation the constitutive equation for 2-dimensional plane stress and plane strain. So, that difference is popped up here in case of a thermal strain.

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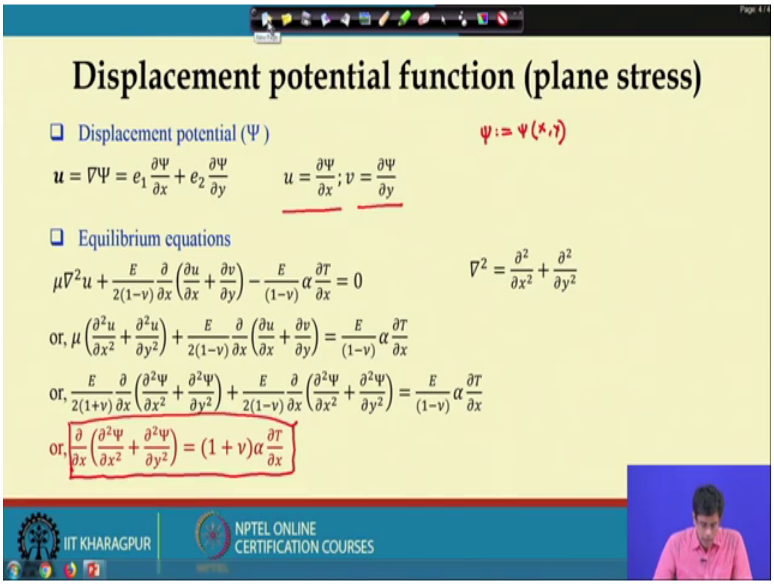

Now, once we have this plane stress plane stress d formulation so, we can now move to the displacement potential approach which is essentially writing the displacement in terms of a single displacement potential function which is your psi. So, psi is a function of a similar to the earliest case psi is a function of xy. So, psi I can write is a function of xy and then this psi E and this psi is how it is related with the displacement is del u del x and del del psi del x and v is del psi del y.

Now, E 1 and E 2 are the unit vectors along u direction and v direction displacement that is the x direction and y direction. So, now, u vector is written in this form. So, this del psi means if I take the del operator or then a nabla operator outside. So, now we know the equilibrium equation in terms of displacement from the previous slide. So, now, I substitute this psi in the displacement the displacement potential in the governing equation. So, once I substitute that I just need to rearrange it and then after rearrangement I finally, get this equation. So, this is for the u this first differential equation which is of this form.

So, now a similar to that we will do the second differential equation and this second differential equation again combining these two we can integrate and then form a single differential equation compared to the previous case. Now, here let us do the second differential equation.

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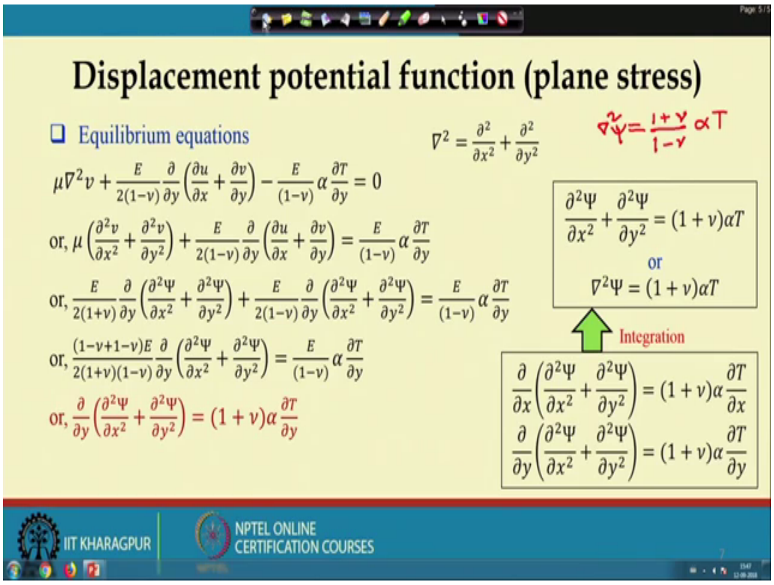

So, in the second differential equation similar to the previous one I just substitute the displacement potential function and then once we substitute the displacement potential function it looks in the final form. So, this manipulation you can also do it on your own and then check that this is satisfied.

Now, here once we get the two potential equation; that means, one of them is partial of x with this quantity and one of them is partial of y this quantity and this will be x for the previous case. Now, this is the two equations. So, in a final form; now, once we integrate these two equations we get again a Laplace ma Poisson's equation of this form. So, del square psi equals to 1 plus nu into alpha T.

Now, this equation if you remember for the plane strain case there was the difference here. For the plane strain case what we have observed is that due to this difference in the constituency metrics we had del square psi for the plane strain case we had del square psi 1 plus nu by 1 minus nu into alpha T. So, this was for the plane strain case and this is for the plane stress case. So, the only difference lies in the, this coefficient of this thermal part. So, other part is exactly remains same. So, this actually helps you to remember the equation itself.

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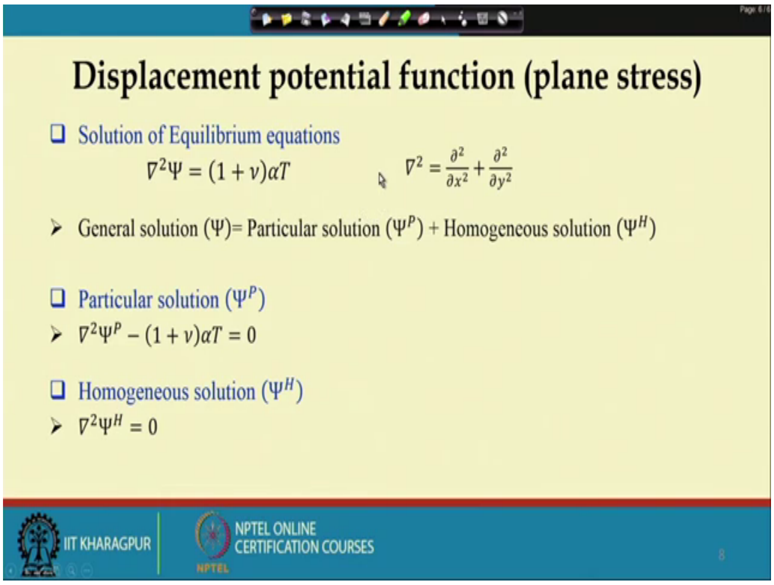

So, now once we are done that then the solution of this differential equation is similar to the earlier case like we can just have two part of the equation two part of the solution which is the particular solution and the homogeneous solution.

So, homogeneous solution we know how to do it and then particular solution also we have to find out. So, this is the basic theory of solution of partial differential equation. So, now, this after this actually we want to discuss the stress function approach because stress function approach you have already seen. So, this stress function approach can also be applied to the thermo elastic deformation.

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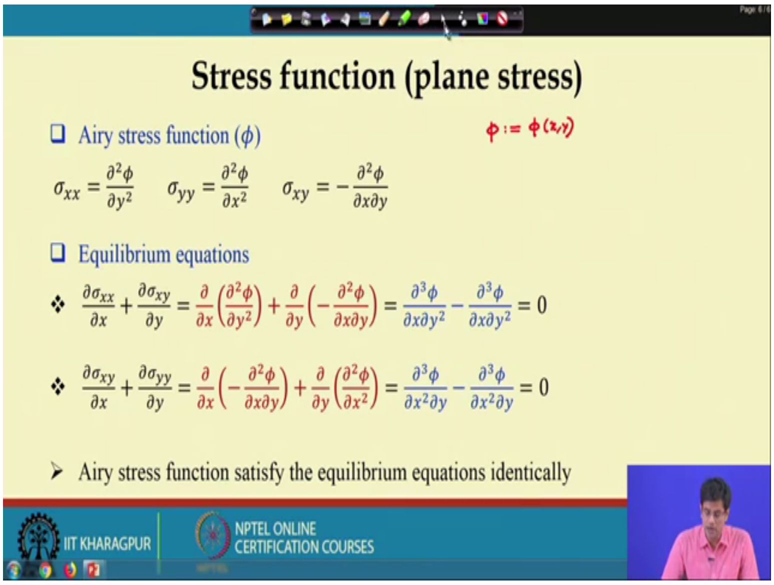

So, here if you see that stress function that we have defined earlier. So, this stress function can also be defined for where the nonzero body force is there. So, that also we have seen earlier. So, here the sigma xx is del square phi del y square and sigma yy is del square phi del x square sigma xy is minus del square phi del x del y. So, now, if I substitute this in the differential equation or the governing equation of equilibrium then it exactly satisfies.

So, that property you have also earlier I have seen we have seen earlier. So, this stress function phi is known as the airy stress function. So, now, once we are equipped with the stress function and this stress function is actually function of xy, so, phi is actually the function of xy. So, phi is defined as phi of xy. So, with this boundary condition also has to be satisfied that also we have seen earlier. So, now, once we know the stress function then what will be the form of the final equation?

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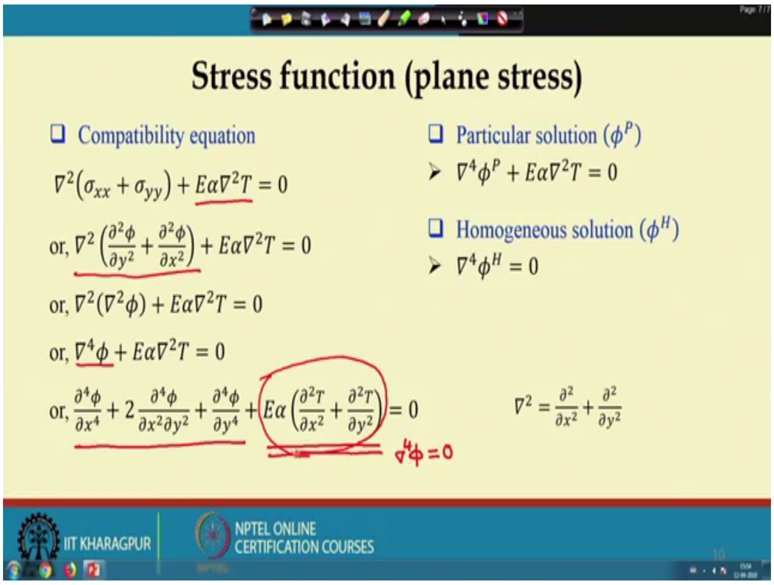

So, the compatibility equation again we know the compatibility equation from our previous discussion that only this term the thermal part will be there. So, now I substitute the stress function that is sigma xx and sigma yy. So, finally, it gives me another del square. So, which is now this quantity becomes del to the power 4 that is bi-harmonic component. So, we have seen that bi-harmonic equation in case of a purely elastic deformation where this part only is0. So, the bi-harmonic equation was for without the thermal deformation is del to the power 4 phi equals to 0 that was the for the purely elastic deformation.

Now, except this now this only the difference is here now is that the thermal part of the deformation. So, this is now is a non-homogeneous equation. So, this was a homogeneous equation. So, now, this becomes a non homogeneous equation. So, you will have the, a two part solution again the particular solution and then the homogeneous part of the solution. So, this two equation similar to the previous case we can now solve this differential equation. So, essentially the stress function approach if you look carefully the compatibility equation plays an important role.

So, compatibility so, the reason already you have stated earlier that again I am re iterating that that when you need to satisfy the when you approach from the stress function approach or stress base formulation or then finally, you want to have the displacement solution. Now, displacement cannot be obtained unless you have the compatibility equation. Now, you have the from the stress once you solve for the stress you get from the constitutive by plugging the constitutive or the compliance relation essentially you get the strains. Now, from that strains actually you have to find out the displacement.

Now, if you look carefully that strains are there are six number of independence strains that is for 3D and for 2D it is three. So, now, from 3 in case of a 2D three strain components essentially you need to have a two displacement components. So, that is overdetermined system kind of thing and this actually you need to so, there will be non uniqueness in the displacement. So, for the uniqueness of u and v you need additional equation that needs to be plugged into and that additional equation is compatibility equation. So, this compatibility equation along with the strain definition you need to satisfy and then finally, you get the u and v which is unique.

Now, the in the on the other case where you actually formulate the displacement base formulation that is you start with the displacement itself and then you calculate the you finally, obtained u and v and then you finally, calculate the strains and stress finally. So, in case of a thermo elastic deformation the procedure is similar very similar the only thing is that the thermal strain part which will act as a body force term in the differential equation which we have seen here. So, even in the bi-harmonic equation this term will act as a body force term here. So, this term is essentially the non homogeneous part of the equation and is has it has to be added with the governing differential equation. So, in a nutshell this is the procedure.

Now, this kind of equation solving this kind of equation there are two approach; one is analytical approach another is numerical approach. Now, analytical approach can be can also be subdivided into semi analytical approach and an a purely analytical approach. So, purely analytical approach is very restricted for the domain size and the distribution of temperature and all those things and then semi analytical approach we relax some of the, we can alleviate some of these disadvantages, but again that is also very restricted. So, finally, it boils down that you enter into a numerical approach and that is the universal approach because you can solve for any geometry any type of]dis/distribution] distribution of temperature and any type of boundary condition.

So, all those things are numerical approach is essentially useful for engineers how like us that we finally, need a need some solution. So, then this plane stress condition actually this stress function approach can also be done for the plane strain condition which you can appropriately modify.

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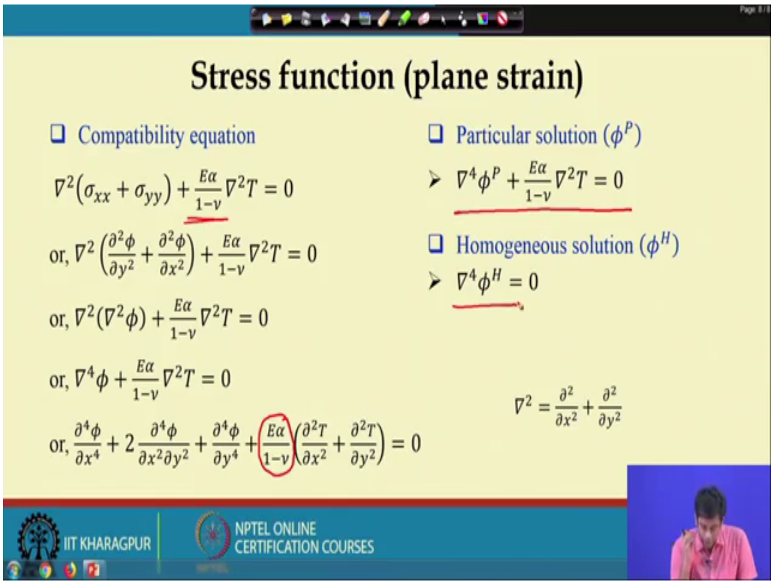

So, this is for plane stress function approach if you, so, if you remember my earlier discussion this equation will have this 1 minus nu term here. So, this term the bi harmonic equation remain same only this factor is change which is divided by 1 minus nu.

So, similar to the previous case of plane stress case again we will have this bi harmonic equation we will have two part of solution one pat is a particular solution and other part is homogeneous solution and some of these two solution is actual final solution. So, this is in a nutshell of a plane stress and plane strain formulation for the thermo elastic deformation.

So, we will solve some of the problems and mostly the analytical we will try to give the analytical solution, but I must mention here that the solution of such analytical problems are very limited and the if you get some analytical result that is good. But, unfortunately due to the complicated geometry boundary condition temperature profile it is mostly impossible to the find out a exact solution or the analytical solution of a of such differential equation. So, in such cases we follow basically numerical methods like finite element, like mesh free methods, finite difference methods, collocation methods all other methods we follow.

So, you must remember studying these subjects or studying the objective of this material or the lectures are is not just to throw you there that you finally, try to solve analytical equation, but essentially you need to know the differential equation how it comes from then actually the numerical part by which you will finally, able to solve this equation must be consistent with the theory.

So, the theory part of this equations are very important in that sense and also we will show some of the analytical example by which you can solve this differential equation and the solution is somewhat exact. But, it is very difficult to get some results at as we have seen in case of a flexure that equations are very big and then the stress and strain components are very difficult to find out. So, we use principle of superposition and all those things to finally, get the analytical solution.

So, the in a in brief the objective of this module was actually to throw some light how the thermal elastic deformation happens and how these deformation is actually creating the differential equation or the governing differential equation which finally, you need to solve and what are the assumptions behind it and how it is derived from the other the theories.

So, in the next class, so, I stop here today. So, in the next class what we will discuss is we will show some of the examples that we have the theory for which we have studied in this whole lecture and basically, we will solve some simple problems for instance strip problem and circular disc problem those problems we will try to attack and get some analytical results. But, again I am saying that objective is not to find out the solution analytically because it has been it has been proved or it has been found out that finding out the analytical solution is very difficult for a general class of problems. So, I stop here. So, in the next class we will discuss some of the example for thermo elastic deformations.

Thank you.