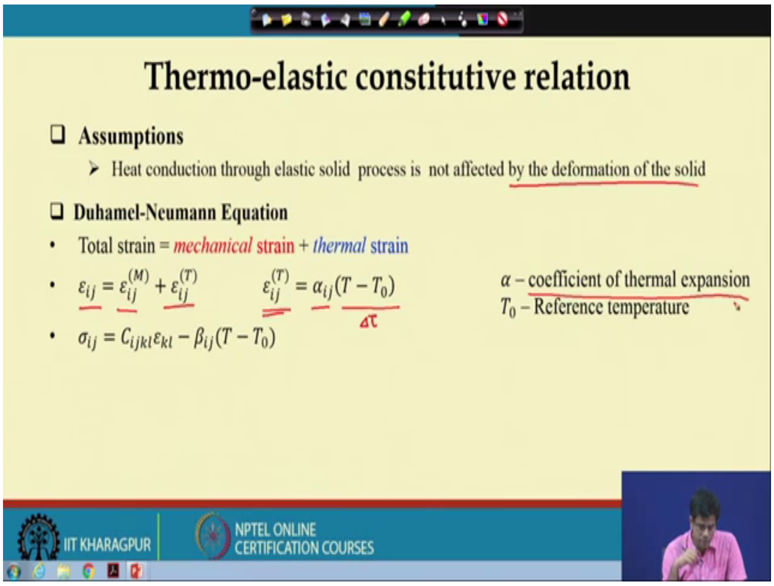
Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 50
Thermoelasticity (Contd.)

Welcome. So, this is the lecture number 50 of module 10, so, where we are basically discussing Thermoelasticity. So, in the last class we have introduced what is the use of thermo elasticity and basically the Fourier law of heat conduction we have discussed. And, then the after Fourier law of heat conduction then we actually described the conductivity of a material and then heat propagation or the heat equation for one-dimension also we have discussed.

(Refer Slide Time: 01:09)

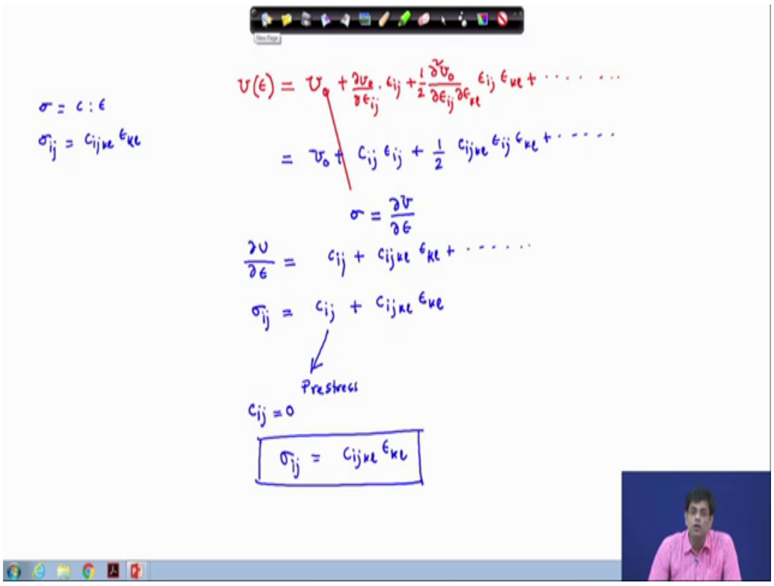


So, in this lecture we will introduce the basic Duhamel-Neumann equation and this equation is actually connected with the total how total strain is related to the mechanical strain as well as the thermal strain. Now, the basic assumption for here is heat conduction through elastic process is not affected by the deformation of the solids. So, this is an assumption even though this is valid for a the structural material that we have used we are using. So, it may not be valid for all material.

So, the material we are talking here is mostly follows this and if this assumption is taken care or if this assumption is taken then the total strain of a elastic body is essentially mechanical strain and thermal strain. So, mechanical strain which it is due to the external loading and then thermal strain is due to the temperature effect or increase or decrease of normal temperature change compare with the surrounding. So, that we know that a epsilon ij is a total strain. So, it has two component epsilon a minus epsilon ij and then epsilon thermal the thermal strain is essentially alpha ij into delta T which is the T 0 is the temperature at the surrounding and delta T is a increment of temperature in the body and T is the final temperature.

So, now this relation is specifically known as the Duhamel Neumann equation and this alpha, alpha ij is the coefficient of thermal expansion which all of us know that from our knowledge of mechanics basic mechanics or the first year mechanics we know that what this expansion of thermal expansion coefficient of solid means. So, now we will derive this Duhamel-Neumann equation and to do that actually we need to have some understanding of the usual strain energy function and how we can derive that.

(Refer Slide Time: 03:59)



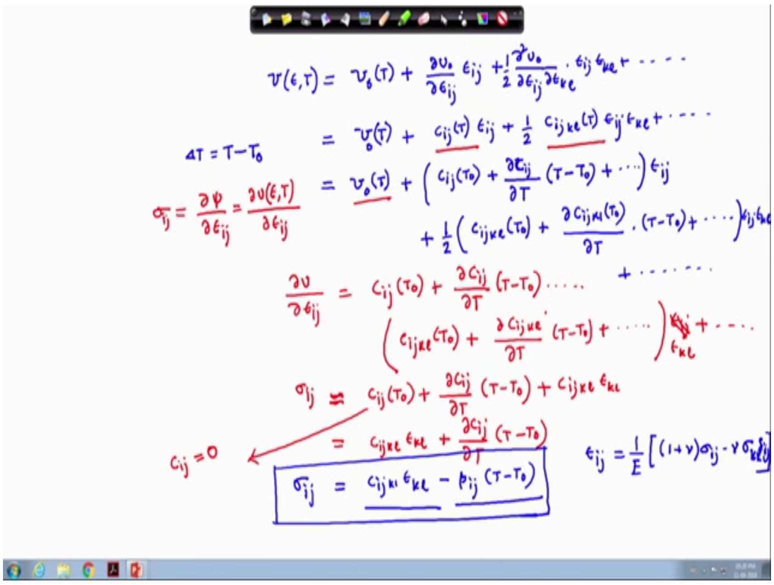
So, for instance the if you remember the Hooke's law then this strain energy function is a function of strain only the epsilon. So, now this we know from the our previous elastic deformation only elastic deformation case where strain energy function U is dependent on the strain only. So, now, if I expand this strain energy in terms of Taylor series then what it looks like this strain energy is if I can write or if I can expand in Taylor series. So, which is U 0 plus U 0 or del U 0 del x that is del epsilon ij into epsilon ij plus del square U 0 into del epsilon ij del epsilon kl into epsilon ij epsilon kl and then half so, then other third order derivative and so on. So, this is a Taylor series.

So, this is the first constant term and then the this constant term becomes the first derivative of this thing and then epsilon ij and then half of a this half is coming due to the factorial and then epsilon ij and epsilon k l. Now, if I now write this as this U 0 is my constant and then this del U 0 del epsilon ij, I write c ij right epsilon ij then half of c ijkl that also we know and then epsilon ij epsilon kl and then this becomes this.

Now, the stress strain relation for 3-dimensional body following the Hookean material or the linear elastic body that also we know that sigma equals to c colon epsilon or in a indicial form sigma ij equals to c ijkl into epsilon kl. Now, how sigma ij is defined? Sigma ij is essentially sigma is del this strain energy by del epsilon. So, if I now take the derivative of this expression. So, del U by del epsilon if I [ni/now] now take. So, this quantity goes off. So, and this becomes c ij c ij plus this is a quadratic function. So, and the derivative will be c ijkl, c ijkl into epsilon kl. So, now, this half will cancel the derivative and then so on.

So, now if I neglect the higher order terms or the third order tensors or mixed form of tensors means the term after these then this is essentially my Hooke's law, that is sigma ij equals to c ij plus c ijkl into epsilon kl. Now, this is essentially the prestress the prestress or predefined stress that we may or may not consider the prestress. So, if the body does not have the prestress then this is my this quantity c ij becomes 0. Now, if c ij is 0, that is prestress is 0 then my constitutive equation is essentially sigma ij is c ijkl epsilon k l. So, this is my original Hooke's law that I have derived from the Taylor series expansion of the strain energy. Now, using these thing actually I will now using this thing I will derive the strain energy for the thermo elastic material.

(Refer Slide Time: 09:07)



Now, if I now consider a thermoelastic material where the strain energy function is not only dependent on strain, but also dependent on the strain energy the temperature; so, U is not dependent on strain it is dependent on the temperature also. So, now, then if I expand this thing U 0 it will be function of temperature and then I first explained in terms of strain or it is a the Taylor series I am first writing in terms of strain.

So, again this del U del U by del epsilon ij then epsilon ij in del it is a half of del squared U U 0 del epsilon ij del del epsilon kl and then epsilon ij epsilon kl and so on, right. Now, you see this U 0 is a function of temperature only. So, what I did is here first we expand in terms of Taylor series with respect to only strain now you see for each paramet[er]- each term is temperature dependent term. So, first let me write it in this way that U 0 is a function of temperature then this is c ij it is also function of temperature and then epsilon ij is my strain and then half of c ijkl it is also a function of temperature epsilon ij and epsilon kl and so on.

Now, again if I expand this thing in terms of temperature for instance c ij; so, c ij I am expand in it again in terms of Taylor series with respect to a temperature T 0. So, we the temperature change is now delta T is actually T minus T 0. So, T 0 with respect to T 0 if I can expand now so, which is this c ij and T 0 plus delta T delta c ij of T 0 delta c ij of T 0 with respect to T into T minus T 0 and then so on, this is the first term epsilon ij. Now, similarly the second term plus half the, this term I have to this term I have to now expand. So, this term and this term actually I am expand in in terms of temperature.

Now, to do that c ijkl T 0 it is a function of T 0 then del c ijkl of T 0 into del T into T minus T 0 and then again second order terms. So, this multiplied by epsilon ij epsilon kl and then again all other third order terms, right. Now, if I now use my original definition of stress that is sigma is the first derivative of the strain energy function which is sigma ij is del psi by del epsilon ij, if I now use this definition for an hyper elastic body, but now here my psi or my psi is my strain energy here which is here I am representing in terms of U. So, del U is my epsilon comma T so, which is a function of temperature as well.

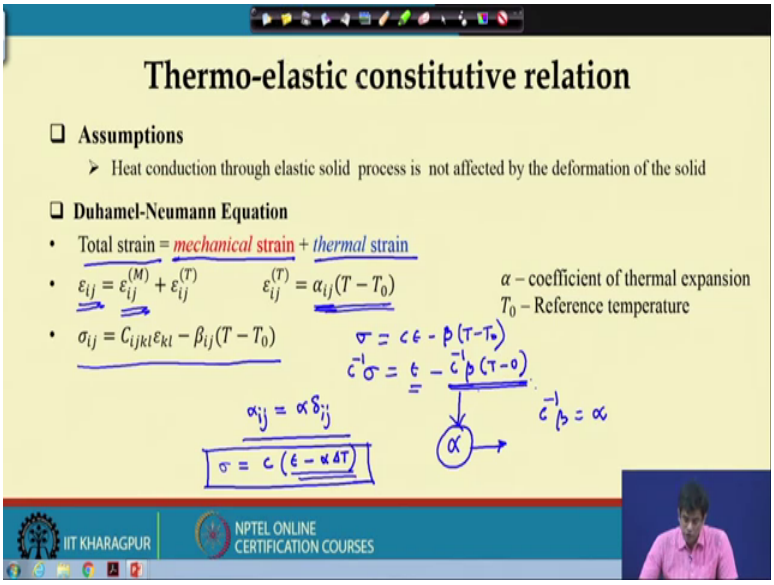
So, now if I now take the derivative of this expression with respect to epsilon ij then del epsilon del U by del epsilon ij is my this term goes you cancels out and this term only this c of ij T 0 plus del c ij del T into T minus T 0 then other terms and other terms will be here and then from here again the this is the quadratic function so, half will be absorbed in the derivative. So, the first term will be c ijkl it is a function of T 0 plus this then the second term del c ijkl by del T into T minus T 0 and then, but this will be with epsilon ij right now and then the other terms as well.

So, now here if we neglect if we considered the small deformation assumption and then if I considered that del T is less than comparatively less than T 0 then I can neglect the higher order terms in these Taylor series expansion and basically I can write this the sigma ij which is coming from here. So, sigma ij is my c ij T 0 plus del c ij by del T into T minus T 0 and then c ijkl of T 0 and this is also I am neglecting. So, now, here this I can say approximately equal for small deformation now this is very similar to the pre stress of the previous case.

So, if I take this c ij is 0 pre stress then my this will be epsilon kl. So, this will be epsilon kl. So, now, if I considered the pre stress term is 0 that is c ij is if I take 0 here then my stress strain relation for the thermoelastic material is c ijkl epsilon kl plus this quantity. Now, del c ij by del T into T minus T 0. Now, this quantity if I say this quantity is my beta ij or m[inus]- specifically the minus beta ij. So, c ijkl epsilon kl is minus beta ij T minus T 0. So, this is my the thermo thermal stress actually and this is my the total stress. So, this is essentially coming from the this is known as the Duhamel Neumann equation.

Now, Duhamel-Neumann equation of this form; now once we know this form then we can easily convert it to the other means that we can write the stress strain in terms of thermal and the mechanical strain. So, now if I write similarly the compliance relation; so, where I write epsilon ij in terms of sigma ij which is essentially 1 by E into 1 plus nu into your sigma ij minus nu it is a sigma kk delta ij then this if I substitute here what we get is something like this that we will see now. So, now, here what I have written is your this quantity. So, now here what I have written is this quantity.

(Refer Slide Time: 18:43)



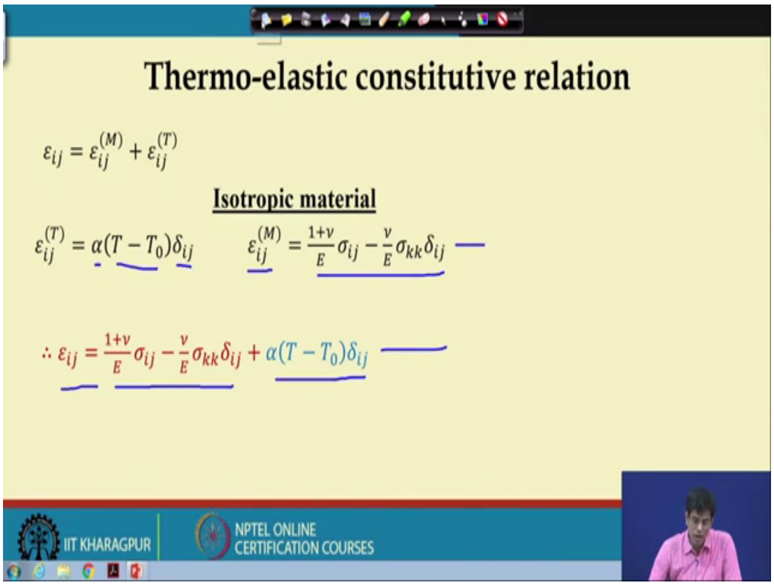
So, here my so, here this quantity is essentially coming from the Duhamel Neumann equation. Now, if I if I write this sigma if a in a matrix format if I write sigma is essentially C into epsilon here C I am writing in a wide notation and then this beta I can write it in a matrix form also I can write beta or the beta matrix beta and then T minus T 0.

So, if I now write C inverse sigma is essentially epsilon minus c inverse beta T minus T 0. Now, this quantity I am saying alpha alpha matrix is the coefficient of thermal expansion solid. Now, this coefficient actually then if I write this is my total strain and this is my thermal strain so, and this is equals to the mechanical strain which is this quantity. So, now, the this actually proves my mechanical strain plus thermal strain is my total strain. So, now, how does this alpha is derived that also is pillar.

So, C inverse beta if I write which is the alpha. Now, for an isotropic material alpha ij alpha ij a this quantity is constant which is alpha delta ij. So, that means, the the three direction it is alpha that is x direction coefficient of thermal expansion of solve it easy alpha y direction is also alpha and z direction it is also alpha. Now, this thing you can be use for an an isotropic material also alpha 1 in the one direction or x direction, alpha 2 in the y direction, alpha 3 in the z direction this can happen. But, so, this if we if I write it in a stress strain relation the stress strain relation becomes now this sigma is becoming C into epsilon minus alpha into T minus T 0 which is essentially delta T. So, alpha delta T. So, this is my final constitutive relation for the solids.

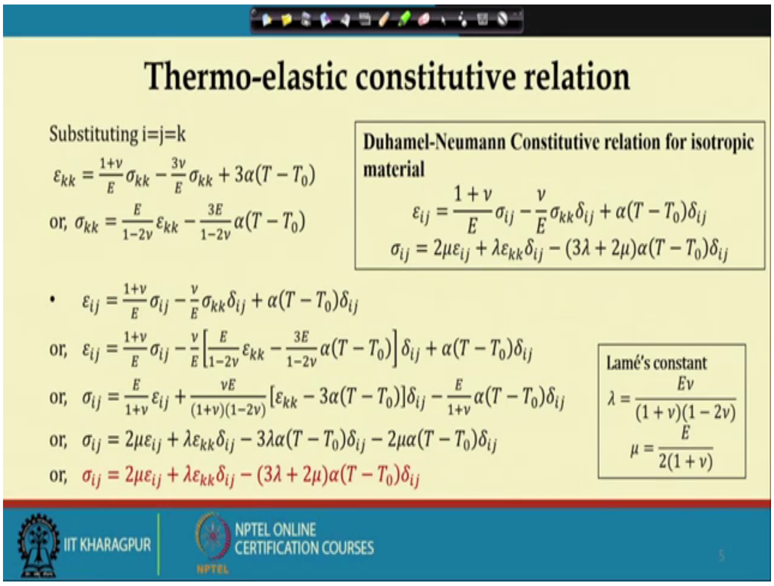
So, you see this total strain epsilon is a total strain here this is the total strain and alpha delta T is my thermal strain. So, this quantity is essentially the elastic strain so, or the mechanical strain. Now, for an isotropic material this can be easily observed this can be easily observed.

(Refer Slide Time: 21:53)



So, for instance here we can see this isotropic material which is alpha into delta the thermal strain is alpha and delta ij. So, this is delta T and. So, I can write the compliance relation that is strain versus stress relation and then from here actually we can get the compliance relation for the thermo elastic material this is for the mechanical part or the elastic part because we are essentially considering elastic material. So, this is the elastic part of the earth strain and this is the mechanical total strain which in composed of mechanical strain plus thermal strain. So, now, this is the basic introduction of the Duhamel Neumann constitutive relationship of the thermo elastic material.

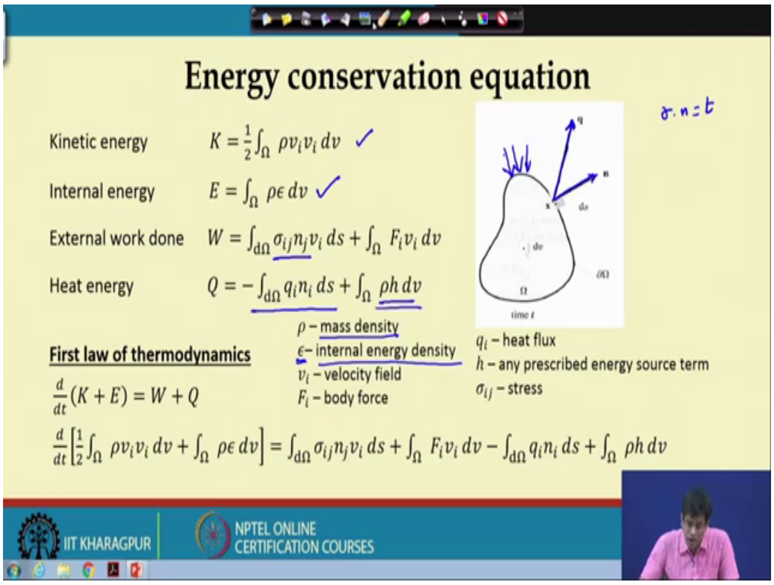
(Refer Slide Time: 22:47)



Now, if you can you can modify this so, this is with this. So, if I substitute this ij k with in the previous relation or this relation. So, epsilon kk I can get the expression for epsilon kk which is this and then if I want to find out sigma kk so, I transfer it. So, once I transfer it, it looks like in this form then epsilon ij also I can do it and I can after some manipulation after some manipulation I can convert it to the first Lame constant and second Lame constant these I think already we have seen, that is mu and lambda; that mu is the first Lame constant and mu is the shear modulus of the second Lame constant.

Now, once this is done then Duhamel-Neumann relation for isotropic constitutive material or isotropic material is this. So, this is the compliance relation and this is the constitutive relation. The constitutive relation you see in the thermal strain part there is a 3 lambda plus 2 mu, because this is how it has been derived. So, because this is this quantity is essentially the beta that we are discussing in the previous slide and this is alpha is the thermal coefficient thermal expansion coefficient. Now, this is the for the isotropic. So, these can be modified for the an isotropic material as well.

(Refer Slide Time: 24:21)



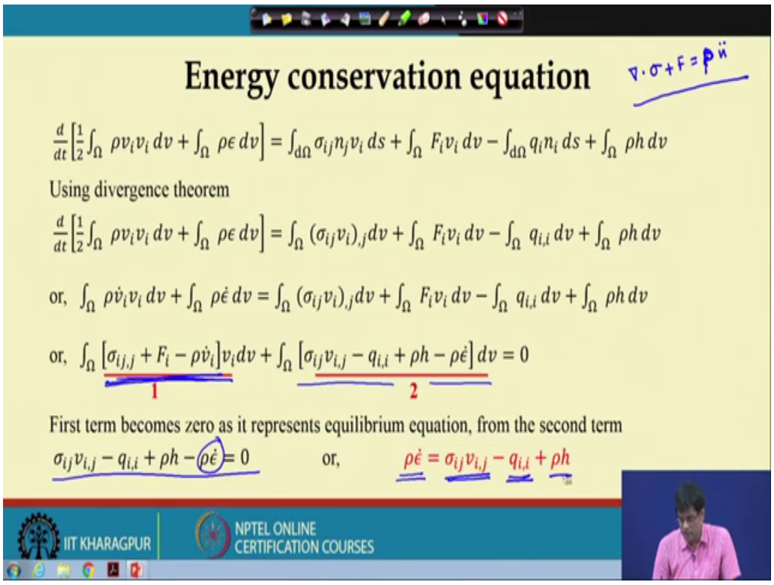
Now, if I now considered the basic energy conservation equation that is for a body with thermo elastic energy is required. So, first starting point is the first law of thermodynamics. So, all of you must have studied the last of thermodynamics which is the in a basic elementary notation will follow here and which is the d dt time derivative of kinetic energy and the internal energy is equals to work done plus the heat flux q.

So, this or heat energy capital Q so, now, if the kinetic energy I know if I consider body here these body and these is suppose d omega is the boundary and omega is the body and then there is a normal here this normal and there is a heat flux the heat flux is also there. So, now in a purely elastic place you do not have small q. So, there is a heat conduction is happening. So, there is a q here and this body is with the mechanical loading some mechanical loading is there and there is a heat flu[x]- heat conduction is also happen.

So, now these if I write the expression of the kinetic energy which is we know half of m v square. So, which is may m is essentially we are consider in the mass density rho. So, this is my kinetic energy and this is my the internal energy going to internal energy density small epsilon is here internal energy density. So, this is integrated over the total volume. So, this is my internal energy density and this is my the work done. Work done is due to that could be a two force. There is a traction force here may be there is a traction force here.

So, which is this traction we know that sigma dot in is t. So, this sigma ij n j into v is my total external work done due to the purely mechanical ruling and then this is my the volume volumetric force due to the body force done and then the heat energy is essentially this we have seen from the previous lecture that q n i is the heat flux in the boundary. So, and rho h is the source term in prescribe source term within the body or not heat maybe h maybe 0 h may not be 0. So, if there is a heat source then there is a heat energy inside the body so, which is over the omega or capital omega inside the body. Now, if I substitute these quantities here so, this looks like this. Now, if this will now this will now see how this looks in a system.

(Refer Slide Time: 27:45)

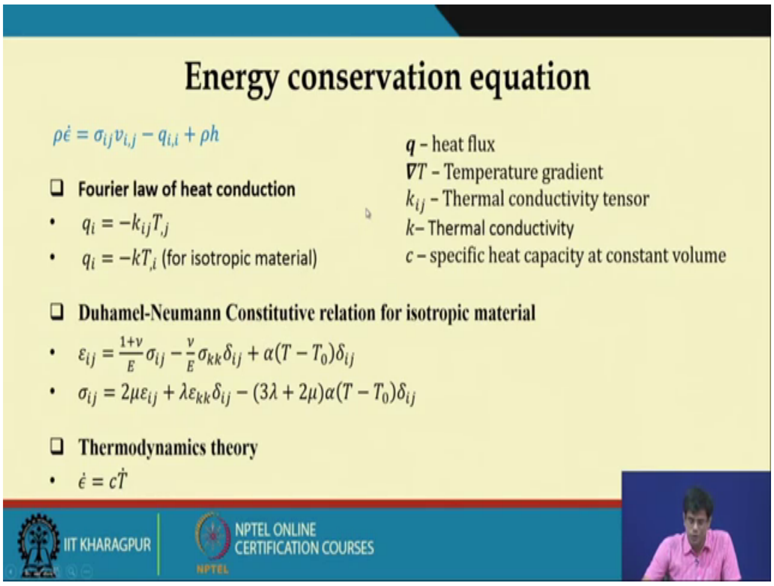


So, in a if I know do this operation and using the divergence theorem I can change this to the volume; that means, the body. So, divergence theorem we have studied earlier which actually converts the surface integral to a line integral or the line integral to a surface integral and a volume integral to a surface integral. So, here a what we are doing is the volume surface integral actually we are converting into a volume integral. So, now, this if I use divergence theorem here and similarly that divergence theorem if I use so, I can compute the volume integral. So, this is the expression for volume integral this all of you know.

And, now you after some manipulation I just take the time derivative here. So, if I thus take the time derivative so, it will do one of them will appear v dot and since there is a half so, that will be observed and then the internal energy density will be time derivative. So, this if I after some rearrangement I just form this. So, if you look carefully this first term is essentially this quantity we have already seen. So, this is the equilibrium equation that we have observed. So, this is nothing, but that sigma divergence of sigma plus F is the body force is equals to rho rho u dot or u double dot or if u is the displacement then acceleration. So, since v is a velocity. So, it is v dot. So, this is essentially your equilibrium equation or the force balance.

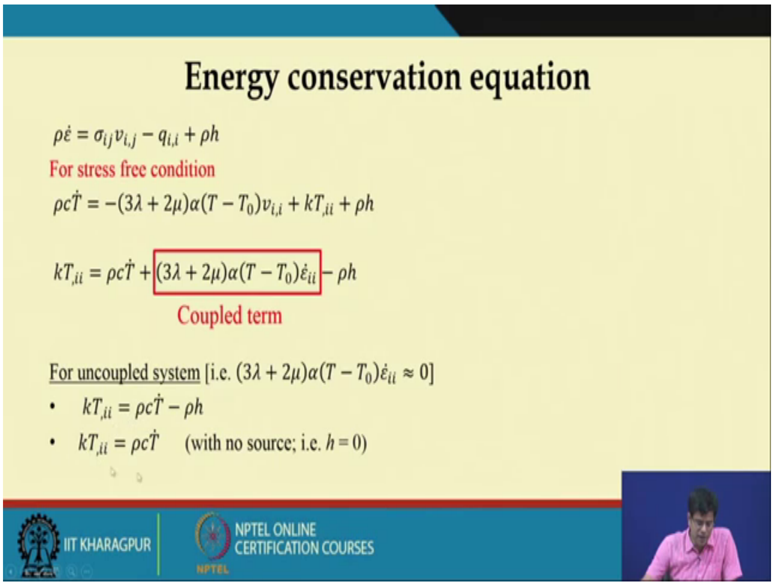
So, this equation we have seen we have put 0 for static is here now this is the since we are considering for general dynamical case. So, this will be the inertia term will be there. So, now, this equation is essentially 0. So, this becomes so, now what is left is essentially this quantity this quantity now. So, this quantity first term is 0, because it represents the equilibrium equation and then the second term is essentially this quantity. So, after some modification I can just now write this quantity that is internal energy derivative with respect to other quantity. So, you see the first term is due to the your stress and then this is due to the heat flux and this is the source term.

(Refer Slide Time: 30:33)



So, now if I little bit modify this. So, this is my the final expression and now we have the Fourier law of heat conduction which also you know we know which is for an isotropic material heat flux q i is k into T comma i for an isotropic material. So, now, this Duhamel Neumann relation we have studied just previously and from the basic thermodynamical theory where considering the ideal gas approximation we can have this epsilon dot equals to c is the specific heat is the temperature derivative with respect to time.

(Refer Slide Time: 31:15)

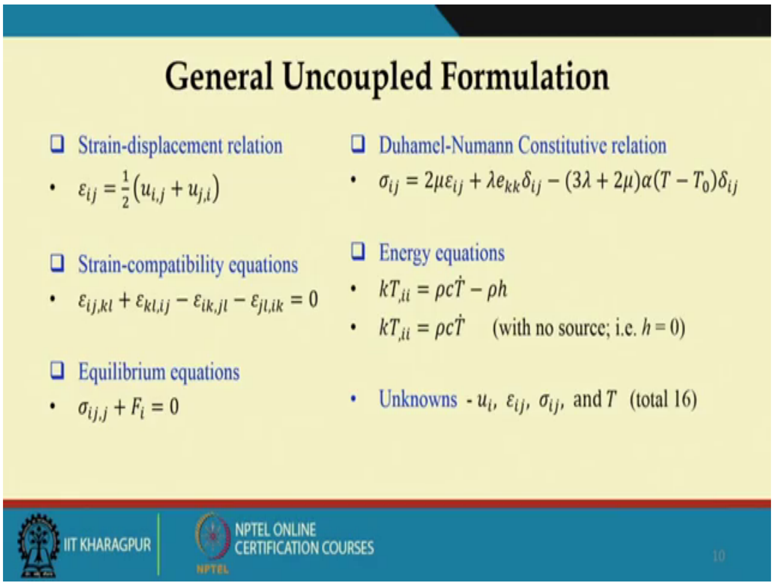


Now, if I just write it little for a stress free condition what does this stress free condition means? That means, there is no sigma ij this term is not there. So, now, there this term is there, but there is no mechanical strain so, only the thermal strain part will be there. So, the which is coming from the Duhamel integration if you look from the previous case this term this part will be 0. So, this part will be there. So, this there is no mechanical strain path.

So, now then this is coming from the heat conduction equation that is Fourier law. Fourier this is k is the coefficient of conductivity and then rho h is the source term. Now, if I after some rearrangement if you see these there is a couple term here and this is coming this. So, where the alpha and epsilon strains even though there is a in the stress free conduction there is a strain which is coupled with the coefficient of thermal expansion coefficient.

Now, here is the departure for our theory. What is that departure is that, we want to study the uncoupled thermo elasticity here. Now, here this coupling is actually ah creating a couple form of the thermo elastic equation now for a uncoupled system this quantity must go. So, now, this if this term becomes 0 then we get the usual uncoupled elasticity. Now, is if there is a no source term inside the body then this is might the energy equation or the stress state. So, this is this a epsilon is actually the rate of internal energy which is coming which is essentially c T dot from the basic thermo dynamical theory and then this becomes the my final energy conservation equation.

(Refer Slide Time: 38:29)



Now, so, in a general uncoupled formulation what are those things? The things are strain displacement relations are these which is usual that we know from the elastic deformation theory. Strain compatibility equation which is also we have studied earlier. Equilibrium equation for a static case this is the equilibrium equation and that for the here comes the speciality of thermo elastic constitutive relation. You see this quantity is due to the mechanical this thing. So, this will be epsilon kk. So, and this part will be due to the thermal stress part.

Now, the energy equation if there is a no source term so, if h equals to 0 which also we have derived for a uncoupled form what will be the energy equation. So, as a whole we have three displacement unknowns six strain components and six stress components and the temperature unknown. So, total in a elasticity case there is another unknown which is coming as temperature.

Now, for a uncouple thermo elasticity temperature will be generally given to you because you are not actually solving the heat conduction equation or as if you solve the heat conduction equation; that means, the thermo elastic or the Navier's equation of elasticity actually you solve separately and heat conduction equation or the heat equation you solve the separately. From the heat equation you get the T that is temperature distribution of the body and once you plug into this T into the Duhamel-Neumann constitutive relation which is delta T which can be space dependent; that means, which the xy variation can be accommodated here. So, this T once you get it from the heat equation you can have the usual elastic problems usual elasticity problem, but thermal strain as an additional strain there.

So, now this as a total you have 16 unknown where in the elasticity case you have total 15 unknown so, temperature unknown was not there. Now, in case of a it is important to remember here in the couple thermo elasticity case you solve both the equation thermal heat equation to find out the temperature as well as the displacement thermo elastic displacement you solve both the equation together to get the distribution of displacement and temperature.

But, in case of uncoupled elasticity you solves you solve sequentially or you get the temperature distribution or that means, the temperature our first assumption in the beginning of the slide we have discuss that heat conduction does not change the properties of the material or does not affect the response of the material. So, if we follow this thing then this term basically gives us the uncoupled elasticity, uncoupled thermo elasticity.

So, finally, this becomes a 15 unknowns for the elasticity and one temperature unknown. So, generally these temperature will be given to you and otherwise you can solve heat equation in a body to get the temperature profile in a body. So, once you are done with that then you solve the usual thermo elastic problem where temperature is an input to the system and then you get the thermo elastic response in terms of stresses strain and displacement which is usual as per the ah previous our previous discussion.

So, I stop here. So, we will continue in the next class, where we will discuss some of the specific cases for plane stress plane strain thermo elasticity.

Thank you.