Theory of Elasticity
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Lecture – 05
Introduction to Tensor (Contd.)

Welcome. This is the 5th lecture of Module 1. So, in this module, we are covering Introduction to Tensor. So, in the last lecture, we have discussed the tensor calculus and previous to that tensor algebra. Now, there are two, mainly two topics left and one of them is integral theorems. So, integral theorems, the purpose of learning integral theorem is that when we will use or when we will try to understand the boundary value problem of elastic solids, we need to use this theorem to solve problems and specifically for constructing the weak forms. So, first what we will do, we will first describe the integral theorems.

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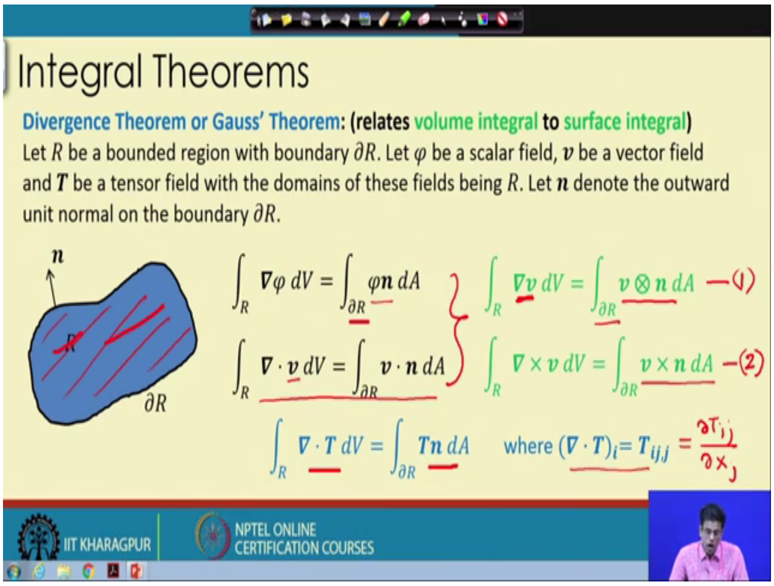

So, these theorems I hope all of you know from your vector calculus or vector algebra course which covers these theorems. So, we will be modifying these theorems for tensors. Now, first theorem is the divergence theorem or also known as the Gauss theorem which relates the volume integral to surface integral.

So, this theorem is already some of you have already studied. So, consider a simple body which could be boundary, could be multiply connected. So, let R be the bounded region with boundary del R. So, R is the domain of interest and its boundary is denoted by that del R. Now, let phi be a scalar field, V is a vector field, T is a tensor field with the domains of these fields and n is specifically the outward normal which is actually acting at the boundary because this normally is outward.

So, now divergence theorems states that divergence of these scalar field phi over this inside of the domain that is your this domain should be this domain, should be equal to phi of n integral over the boundary. That means, this boundary, the flux or the quantity del phi should be integrated over this volume should be equal to phi of n, right and this integral look carefully this is on the boundary. Now, similar to that this also if phi is a scalar field here, now if we consider a vector field V, then divergence theorem states that the divergence of V over the volume or over the region R should be equal to V dot n over del R.

Now, these two theorems are I will assume that all of you know how to prove these two theorems. So, if we consider a tensor field now instead of a vector field, then del dot T which is if you write it in a component form which we have seen in the last class which is essentially the del of T ij del of X j which we have seen in the last class. So, if we take the divergence of these tensor field and then, it should be equals to T of n over the boundary. So, this theorem will prove or this expression will prove with the help of these two or specifically this expression.

So, there are two corollaries also which are useful if we know these theorem or if we know this expression. Now, if we have to apply divergence theorem over a gradient of a vector field or del V, then what should be the form. So, this states the first corollary which is del V over R should be equals to v tensor product n del A over del R. Now, similarly called of a vector field that is del cross v if I have to apply divergence theorem on to that, then del cross v over the integral of over the volume should be equals to v cross n dA.

So, these two corollaries these are not called divergence theorem, but these are the results of divergence theorem which is these three expressions. Now, let us first prove this tensor field case. So, if divergence theorem is applied for a tensor field, then what it looks?

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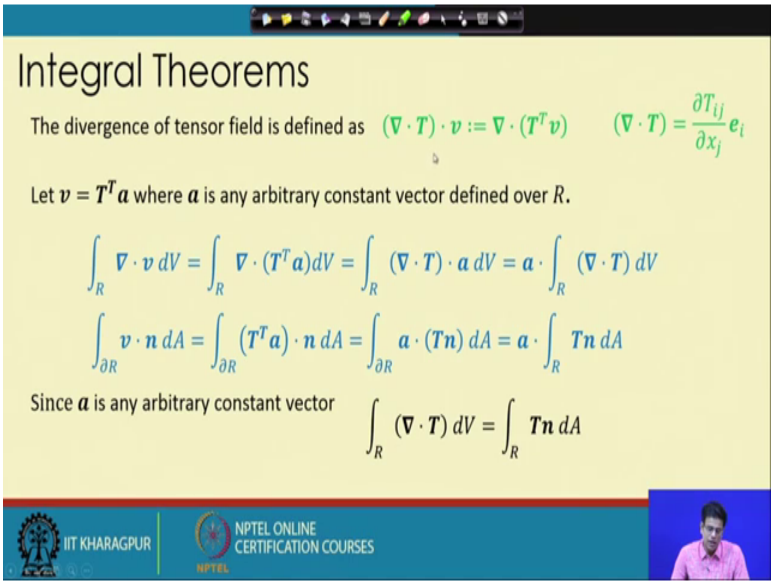

So, if you remember the divergence of a tensor field is defined in this way. In the last class, we have defined it the divergence of a tensor field. So, del dot T dot v, v is any other vector or should be del dot T transpose v.

Now, it is in indicial notation with the vectors it is with the unit vectors e i e j where e i's like this. Now, let us consider any vector let us consider a vector v which is T transpose e, where e is an arbitrary constant vector. Now, this arbitrary constant vector could be anything, ok. So, if I can write v as T transpose a, that means these T transpose these tensor transform these vector to v. Vector A to v if we assume this v can be represented in this manner, then del dot v can be written in this form.

So, del dot v first we substitute T transpose a and then, del dot T dot a because I just take the transpose here which is coming from this formula. So, now if we apply this formula here, you will get this expression and then finally, since this is a dot product, I can take n a is a arbitrary constant vector. So, I can take a outside. So, a dot integral del dot T d V. Now, similarly v dot n d A which is the left hand side of the divergence theorem, so v dot n if I now substitute T transpose a and then, after manipulation, I just take t transpose here so it becomes T n so a dot T n.

Now, similar to the previous case I write a dot T n d A. Now, since a is an arbitrary vector, I can say these two quantities are equal which proves the theorem. So, this is the proof of divergence theorem in case of a tensor field. Now, as you have seen and now we will prove the other two corollaries which are probably will be using in this course. So, let us see the first corollary.

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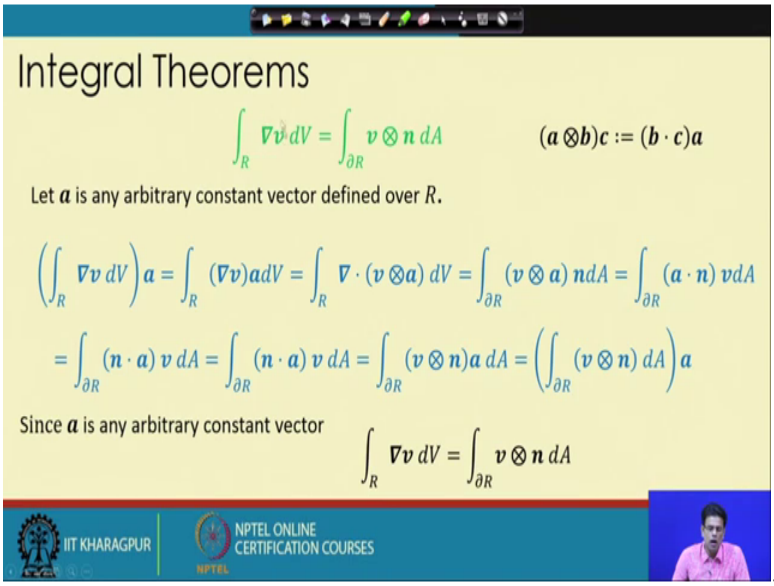

If I take the gradient of a vector field over R or over the volume, then it should be equal to v tensor product n, where n is the outward normal. Now, if we similar to the previous case if we choose a as an arbitrary vector again if a is an arbitrary vector again, then we can write, we can post multiply with this expression with a and then after some manipulation I obtain this del dot v tensor product a.

Now, we know this quantity is a tensor. So, v tensor product a is a tensor. So, this tensor again I apply the divergence theorem for tensor field which is del dot T is T n. So, again v tensor product a is the tensor and n is an outward normal. So, now from here to here I use the divergence theorem for tensor field. Now, if I just use this identity or this definition of dyadic product, so a tensor product b multiplied by c is essentially b dot c into a. So, if I do this, then I obtained a dot n into v d a. So, similarly since this is a dot product, I can interchange.

So, since this is a scalar, so n dot a and then n not a v, then similarly I can write it again this quantity. This quantity I write it in tensor product form which is v tensor product n a d A. So, again I can take a out because a is a arbitrary so I write this into a. Now, since a is arbitrary, I can say this quantity must be equal to this quantity which proves this corollary. So, this is with the help of the divergence theorem over a tensor field, I can prove this. Now, the second one is very easy.

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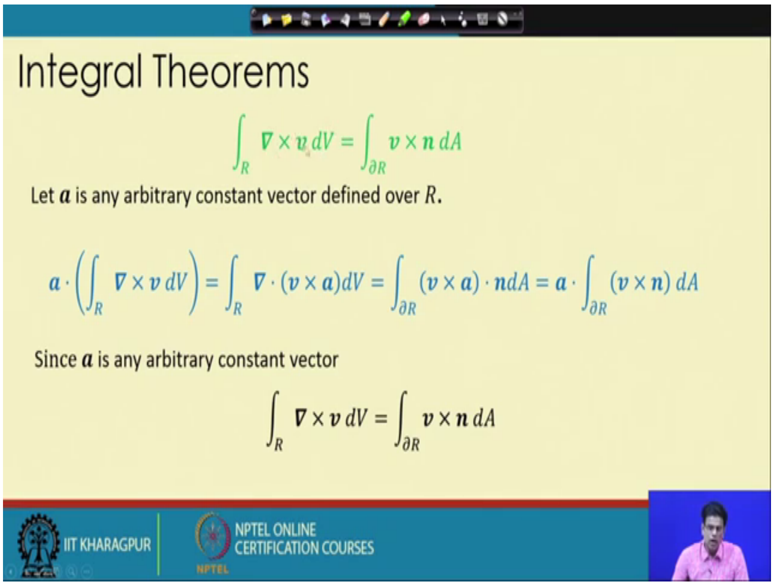

Let us see the curl the del cross v over the volume should be equal to v cross n over the area or the boundary. So, again if I take the dot product, so a dot del cross v. So, I take a inside. So, I can write after some manipulation I can write this and then, again I apply the divergence theorem over vector field and then, I get this v cross a dot n and then, finally I take a out and write v cross n del A. So, since a is again a arbitrary vector, I can write this equals to this. So, these two corollaries are worth to remember because probably we do not need directly to prove these corollaries, but we may use some times when it is required. So, these are two important corollaries of divergence theorem.

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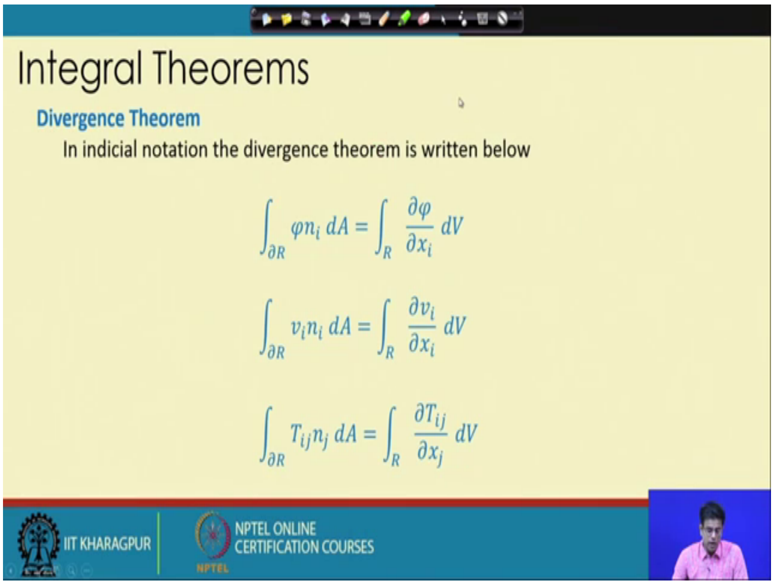

Now, another important before that divergence theorem if I want to write in initial notation, it is very simple. So, del phi del x i is equal to phi n i d d A. So, similarly divergence theorem for vector field is v i n i over the boundary is equal to all over the surface is equal to del v i del x i over the volume.

So, similarly for the tensor field T ij n j over the surface is equals to del d ij del x j d V over the volume. So, these are the initial notation.

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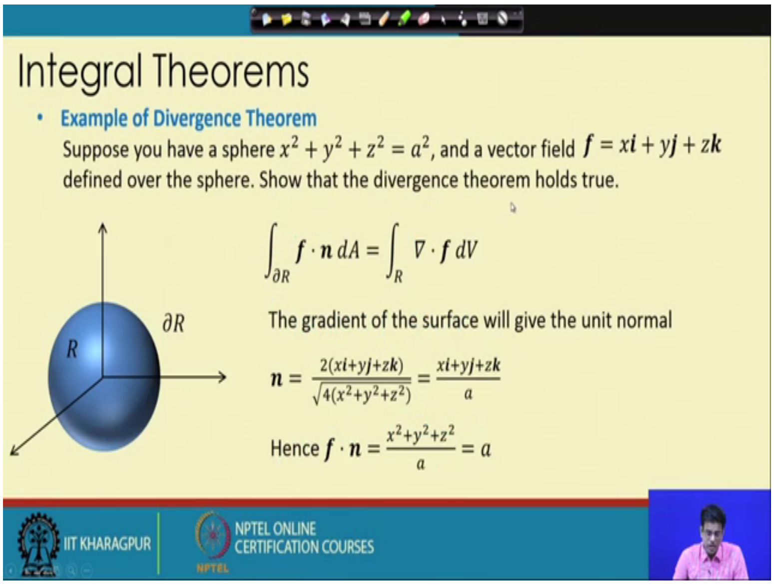

So, now if we want to see some application or some examples of divergence theorem, let us see one application or do one example where suppose this is a sphere which is characterized by the equation x square plus y square plus z square equals to a square and its boundary is denoted by del R. So now consider a vector field over this sphere. So, the vector field could be anything so let us now not bother what this vector field could be rather let us see where the divergence theorem holds true here or not. So, the divergence theorem is f dot n d A is equals to del dot f dV. So, if we first we have to first calculate n so how to calculate the outward normal because it is simple.

So, if we take the derivative and then takes its mode, so I think all of you know how to calculate the normal vector which is essentially x i y j plus z k and since, x square plus y square plus z square equals to a which satisfies the equation of the sphere. So, this I can substitute so my normal is this. Now, f dot n if I write carefully, then it is simply x square plus y square plus z square because f vector field itself is x i plus y j plus y j plus z k. So, now x is f dot n is essentially a because x square plus y square plus z square is essentially a, right. So, now I have to prove this a f dot n over the whole boundary or whole surface should be equal to integral of r delta t over d V. Now, let us see.

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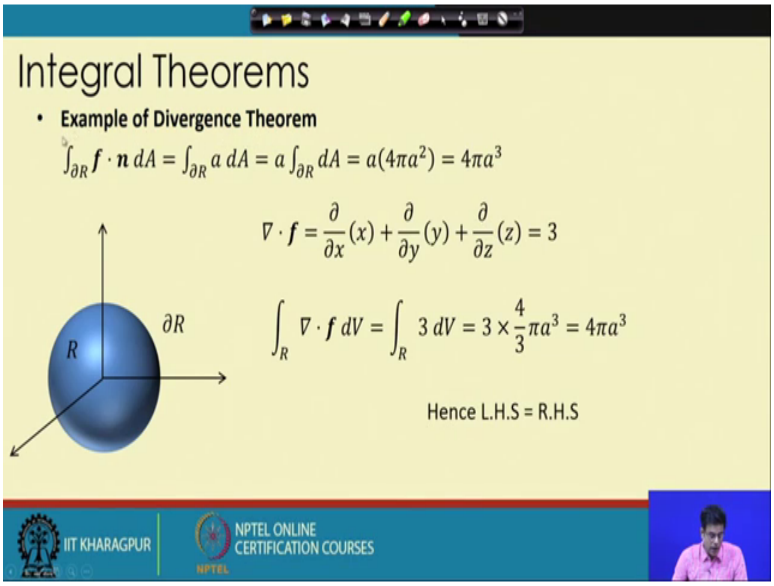

So, del R f dot n d A, so area of a sphere we know which is 4 pi a cube. So, finally the answer becomes 4 pi a q which is f dot n over the surface. Now, similar to that I have to calculate del dot f which is del del x del del y and del del y or del del x of x del del y of y del del z of z which is 1 plus 1 plus 1, so 3. So, del dot f d V is 3 d V so volume of a sphere is four-third pie a cube which becomes 4 pi a cube. So, if you see this quantity becomes this quantity, so for a vector field defined over a sphere divergence theorem holds true. So, this is one example of how divergence theorem could be useful for us.

So, let us now go for the next theorem that we would like to recapitulate or no, that is Stokes' theorem. So, Stokes' as the divergence theorem connects the surface integral to volume integral, Stokes' theorems connect the contour integral to surface integral.

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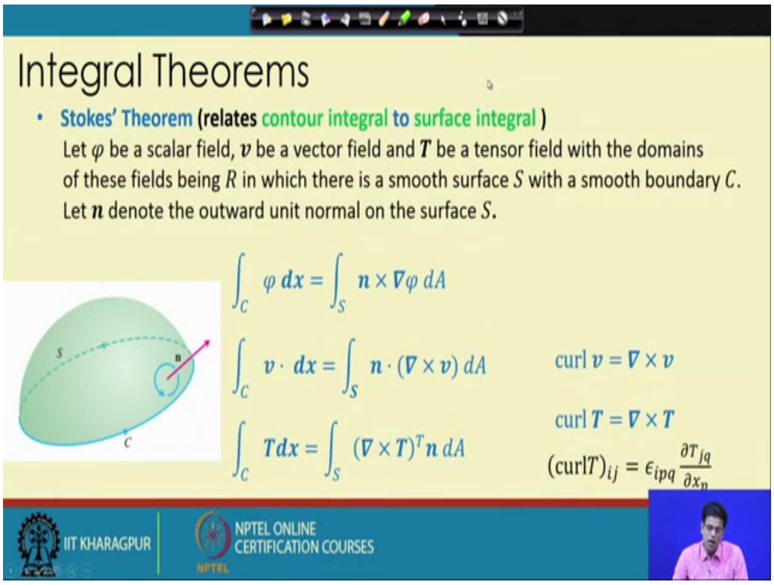

So, what is contour? So, let us consider a simple geometry which is a sphere half sphere kind of thing. So, its surface is s and its volume is hard which is inside the total volume and surface is s and then, its contour, this contour is denoted by the blue line here is denoted by contour c, right. So, the purpose of the Stokes' theorem is to connect this surface integral or over s or the field over s to field over contour c.

Now, let phi be similar to the previous case. Let phi be a scalar field, v a vector field and T be a tensor field. So, with domain of these fields defined is over R which is a smooth surface and then, there is a smooth surface and this smooth boundary is c and n denotes the outward normal unit normal on the surface x. So, what these Stokes' theorems tells if I take the phi as a scalar. So, phi dx should be equal to n cross del phi d a. Now, this phi dx is over the contour the line c and n cross del phi is over the surface s. So, this is for a scalar field. Now, if this is a vector field, so the vector dot product that is v dot dx should be equals to n dot del cross v dA. Again I will assume that these two proof of these two things you already know from your basic vector analysis course and then, if I want to use these Stokes' theorem for a tensor field, so I can write del cross T transpose n d A is equals to T dx over the contour. So, curl of a vector we know which is del cross v. So, similarly curl of a tensor which is del cross T, we can write it in indicial notation in this form, right.

So, let us again similar to the previous case, we will prove this theorem from with the help of the vector quantity. So, let us see how we can proceed.

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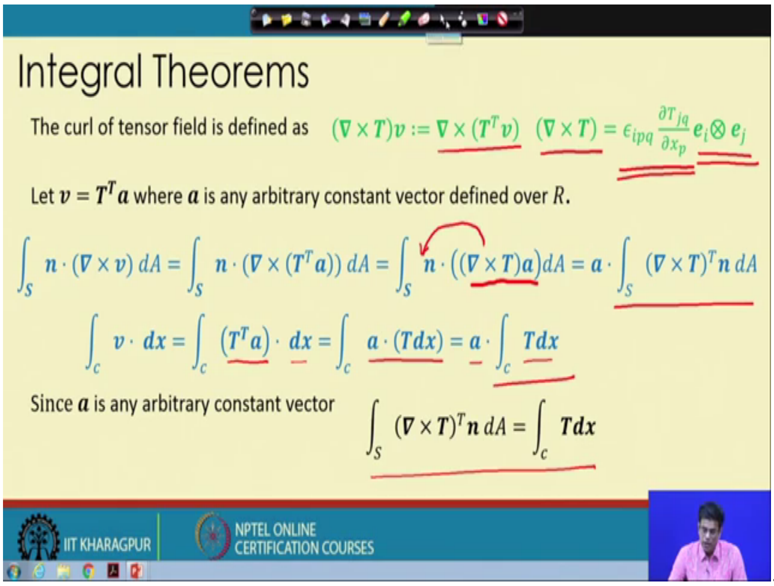

So, curl of a tensor field if you if you know the definition of a tensor curl of a tensor field, so it looks like that del cross T over v is actually del cross T transpose v so similar to the previous case.

So, del cross T is actually I can write it this is the component and this is the vector tensor e i tensor product e j. Now, similar to the previous case if I assume that vector v, I can write it as T transpose a, where a is constant arbitrary constant vector over R. So, I can just use the vector Stokes' theorem for the vector field which is n dot del cross v over s. So, n dot del cross T transpose a i substitute v T transpose a. So, again I can write it this quantity I can write it from this definition del cross T into a. Now, since this quantity if you see carefully, this quantity is a vector. So, this vector or I can this vector is dot product with n. So, easily I can take a out, but when I take a out, this tensor quantity has to go here. So, it will become transpose. So, delta a dot del cross T transpose n. So, now, the contour part of the vector field which is v dot dx, so I substitute T transpose a here. So, T transpose a dx. Now, a dot t dx because I take T transpose with dx. So, as in the previous case, transpose will again transpose which is T.

So, a dot T dx now since a is arbitrary. So, I can say this quantity and this quantity is equal. So, this is Stokes' theorem for the tensor field. So, in this way we can use our knowledge of Gauss and Stokes' theorem over scalar and vector field to the corresponding tensor field. So, the purpose of this expression is not just to prove it how whether it is true or not. The purpose is efficiently use these expressions to simplify our problem which we will see as we proceed further.

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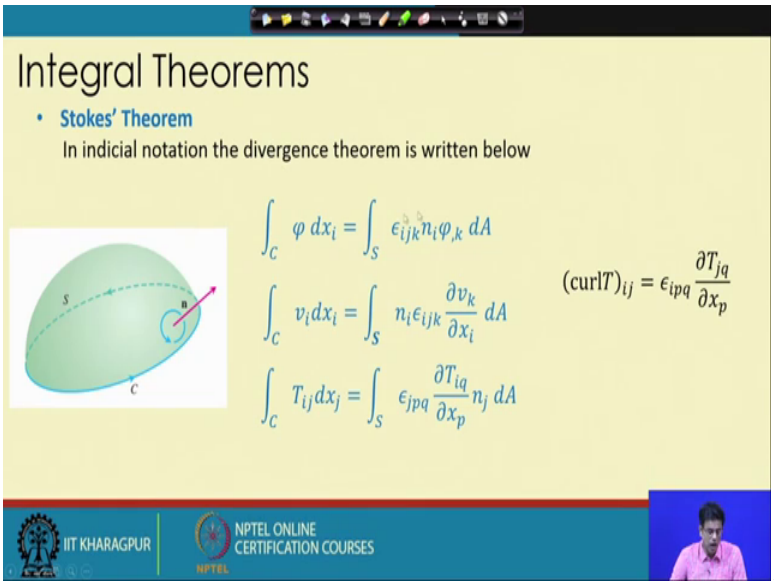

So, let us see through, let us first see the indicial notation how it looks. So, similar cases since phi is a scalar, so phi dx i which is I can write with the christoffel symbol epsilon ijk n i phi of, j that is del phi by del k over a.

So similarly for a vector field v i d x i n i again using Christoffel symbol so you can see that we are using these expressions this Christoffel symbols because the sin and all those things is taken care by these symbols. So, if you remember these epsilon ijk is a Christoffel symbol of third or Levi Civita symbol which is a third order tensor which will have a 6 non-zero component. So, other all components are actually 0. So, we can use efficiently to write these equations in initial form similar to that this contour integral over tensor field which is T ij d xj with epsilon j p q t l del of T jq by del of x p n j dA. So, these are the indicial form. So, now let us see an example of Stokes' theorem.

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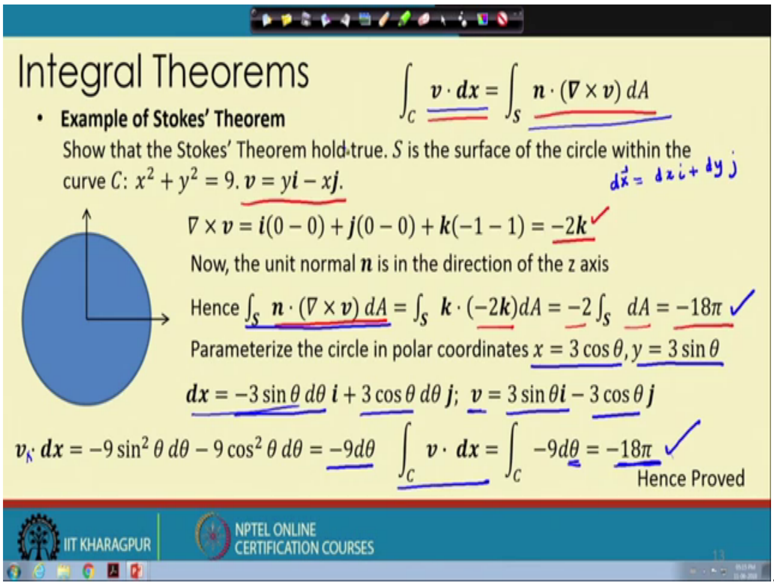

So, what it says is that show that Stokes' theorem holds true for Stokes' theorem holds true for the case. This case this S is a surface of a circle within the curve C.

So, x is the equation of the circle is very similar, very simple which is x square plus y square equals to 9 and there is a vector. So, this vector is y i x j, right. So, now if you remember what is the Stokes' theorem for this case which is this v dot dx is n dot del cross v. Now, first we find out del cross v which is we know the del operator or the nabla operator which is the determinant. So, if you compute the determinant, it comes minus 2 k. Now, similar to that I think all of you now can efficiently calculate the del cross v, right. So, now the unit normal in the direction of z axis because unit normal is along the z axis which is perpendicular to the slide which I can calculate very efficiently because n is actually z direction remains x 3 direction. So, n dot del cross v over d A which is since this is the dot product and this is these contents only, the third unit vector. So, it will be k dot minus 2 k d A which is essentially minus 2 d A. Now, integral over the surface d A is essentially your the area of the circle.

So, area of the circle we know which is pie r square which is 9 pi. So, if you multiply, so it is minus 18 pi. So, we have found out this portion this portion of the quantity. Now, we will need to find out this quantity. So, to find out this quantity we have to find out first what dx is. Now, dx is essentially, to find out dx we have to know parameterize x and y because it is in the cartesian coordinate system. So, if I write as we know for a parameterization of a circle which is r cos theta or sin theta foam which is very simple. So, x equals to 3 cos theta y equals to 3 sin theta. So, similarly if I can write dx or dx, this is a vector so these vector I can write dx component of i.

So, i is the unit vector and d y component of j so and k component will be essentially 0 because it is a circle. So, if I now find out d theta for if I find out dx, so dx is essentially 3 minus sin theta d theta which is minus 3 sin theta d theta i and similarly, y d y which is 3 sin theta, its derivative is 3 cos theta d theta j. Now another for the vector given v which is if I write, if I substitute 3 sin theta and 3 cos theta which is this, right. So, now what we have to calculate is that v dot dx. So, v dot dx is essentially dot product between this vector and this vector. So, if I do this, so this is essentially a scalar which comes out to be minus 9 d theta. Now, if I want to integrate over this minus 9 d theta over this contour or the boundary of this circle which is essentially minus 9 d theta and minus 9 d theta is essentially 2 pi r. So, r is actually that radius which is essentially 18 pi. So, how it comes 18 pi? Because it is 2 pie and the angle is integrating the angle which is 2 pi so it is minus 18 pi.

So, if you can see is that this quantity becomes equals to this quantity which is 18 pi here ao hence it proves. So, this is one application for divergence theorem that or one example of Stokes' theorem. Now we need to know this much of thing for this course, but this can be extended for some other things which we will not consider in this course. So, this sends off the Integral theorem in the next class actually we will consider general curvilinear coordinate system which is also very important and one of the example of such general curvilinear coordinate systems are one of the special cases or orthogonal curvilinear say coordinate system and then, its different type of curvilinear coordinate system. For instance, spherical coordinate system, cylindrical coordinate system and how its tensor quantities, how looks like in these coordinate system, we will learn these things in the next lecture.

Thank you.