**Theory of Elasticity**
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**Lecture – 45**
**Complex Variable Method (Contd.)**

Hello everyone, this is the third lecture of this week. And today's topic is Conformal Mapping.

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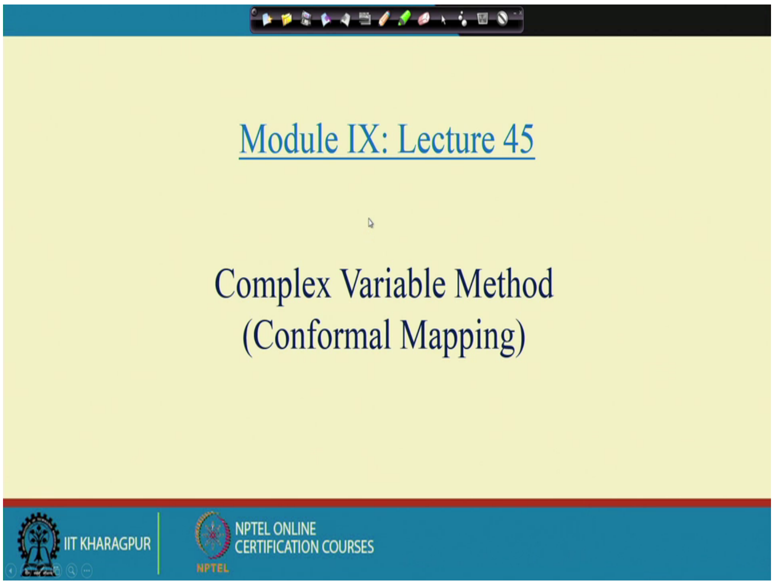


You see, the mapping is essentially a transformation or a function ok, with some time with special features.

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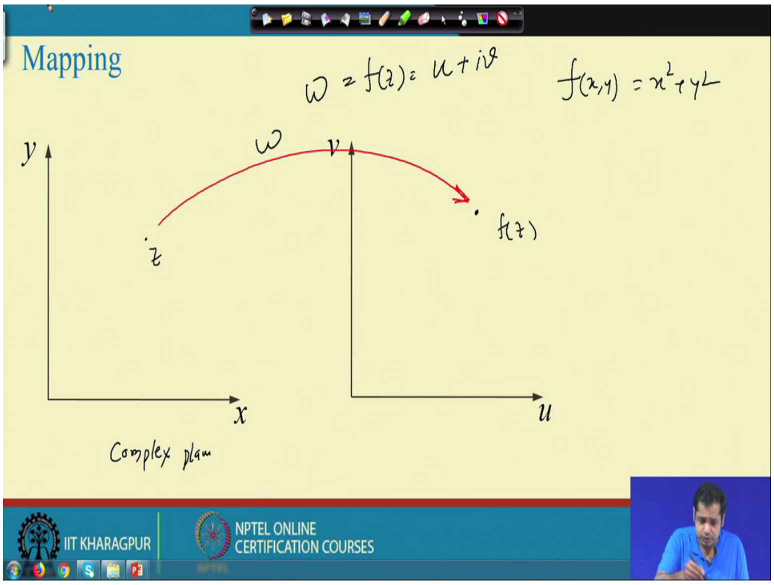


For instance, suppose, if we if we write a function like this, suppose if you write a function f x is equal to f xy is equal to x square plus y square.

So, this is essentially a mapping; mapping from one space to another space mapping transformation from one space to another space. For instance, in complex variable approach, suppose if this is a complex plane complex plane. And if we have any point z and if you have a function f z and if you know the function is equal to u plus iv that is how we can write a complex function and suppose this space is u and v.

So, corresponding if this is s and corresponding say this is say f z. So, this is a map this is a transformation, this the this is called a Transformation. Sometime which is also written as or we will be suppose w is a transformation this is the w. So, this is a mapping and then it is just not the mapping it is conformal mapping, means there is certain properties need to be satisfied during this mapping and that is what we are going to discuss today ok.

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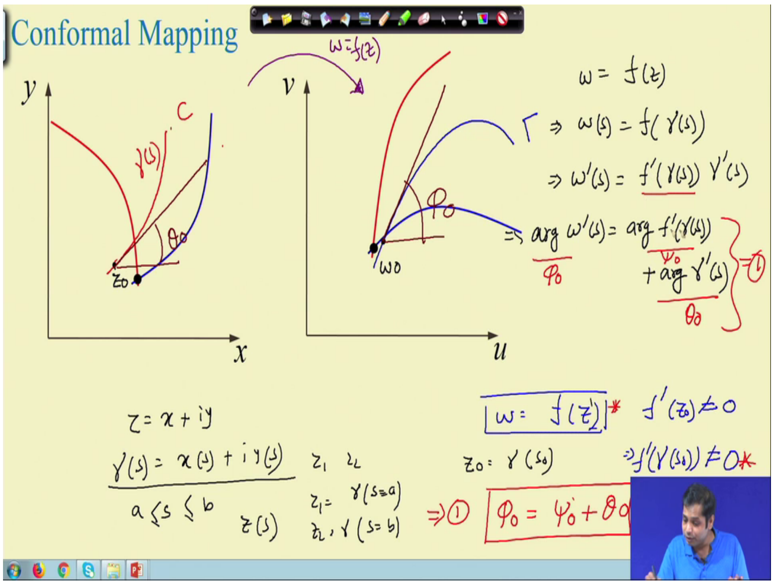


Now, you see suppose consider two plane, one is a complex plane x y plane or the z plane and another one is the function plane uv plane. Now, you take any curve for instance if we take any curve say this is any curve say c in z plane in the complex plane ok.

Now, so if we recall we can write we know that z is equal to x plus i y x plus i y. And then any curve can be written as a gamma s which is x of s plus i of y of s, that is how we can. So, s is essentially greater than a to b. So, s is the. So, at any particular value of s, we can have we can find out we have the z is a function of function of s.

So, at if this is this curve is defined between z 1 to z 2. So, z 1 is essentially your z 1 is gamma at x s is equal to a and z 2 is equal to gamma at s is equal to that is how we discussed in the previous class right.

So, suppose any complex plane c is a curve and then which can be parameterize which can be parametric definition of that curve is this ok. Now and then suppose on the transformation on this and suppose this the you have a function defined as defined as this. You have a function w defined as f of z.

And suppose this f is analytic function on this curve and also f dash z f dash, we will come to this point. So, suppose f z is a transformation from x y plane to y u v plane.

Then if we have a curve c then corresponding transformation on uv plane is suppose that transformation is suppose this is the this is the curve we have suppose this is gamma.

So, essentially gamma is the transformation of curve c, through this transformation and this transformation is this transformation is w ok. Now suppose now consider a point here point say at point on c suppose this point is z 0, where z 0 is equal to z 0 is equal to essentially gamma at s 0 ok. So, suppose at z 0 gamma z this function is analytic function we have already defined what is the definition of analytic function and also suppose if that z 0 z 0 not equal to 0. This means f dash gamma of s 0 that is not equal to 0.

So, we have two properties one is this function is z this function is analytic at z 0 and also that the derivative of this function is not equal to 0 at this point at z 0 ok. Now if it is then what we have, then can we can we write can we write like this suppose; w is equal to w is equal to f of z and then with a using the parametric definition of z. we can write that w s is equal to f of then z can be written as gamma of s ok.

Now, if we differentiate both side then what we have? We have dash s that is equal to f dash which is of gamma s and then gamma dash s ok. So, this is the derivative that we also use in the previous class ok. Now this is the derivative, now you see since we have we have we have two condition, one is this derivative exists because when we see it is an analytic function then this derivative exists and another thing is this derivative is not zero this derivative is non zero. See these two condition is important ok.

If these two conditions are what we are going to exactly do is if two conditions are satisfied then what happens if we have a curve c in x y plane. Then if it is transformed through these through this mapping w is equal to f z. Then what happens between these two map this between these two these two curve in this transformation what happened that is we are going to we are going to find out ok.

So, now if we take once we have this expression, now if we take argument both side then what we have argument of now both all are essentially the complex variables right. Argument of w dash s that is, is equal to argument of f dash of gamma s plus argument of gamma dash s ok.

Now, let us see what these three arguments these this and these and these they physically signify. Now if we draw suppose draw a tangent at this point draw a tangent. Suppose if we draw a tangent which is at then this suppose this angle is theta 0. Similarly suppose the transformation of corresponding transformation of this point z 0 is equal to this, which is say w 0 and similarly draw a tangent at this point. If we draw a tangent at this is the tangent at this point and then corresponding angle say equal to phi 0 phi 0 ok.

Now, this is your this is gamma s this is let us use this is gamma s and then these tangent is actually your gamma dashes this one. So, argument of this will be what this will be theta 0, right. So, this will be theta 0 this will be theta 0. Now similarly now you see this is this is w transformation of this curve and its derivative is slow is w dash is essentially this one is the w dash and its argument is equal to this angle phi 0. So, essentially this will be phi 0.

Suppose this is equal to psi 0, now then from this from this equation what we have? From this equation we have phi 0 is equal to psi 0 plus theta 0 this is important ok. So, what we have now on this slide is this now if we have a curve define on a complex plane say curve c and its projection or the transformation is gamma and if we take a point on this curve say z 0.In this case and at z 0 if this transformation this f z that is analytic function and also the derivative of that transformation is not zero these two conditions satisfied. Then this is the relation we have and this relation tells you that.

Now, essentially if what is if we have to find out, what is the physical interpretation of psi 0. See theta 0 is the theta 0 is this angle of if we have a slope here and this is the corresponding angle and phi 0 is the slope at the project on this plane. And this is the corresponding angle and essentially psi 0 which is equal to psi 0 is equal to phi 0 minus theta 0, it is the change in the rotation of this slope. The slope we have and how these slope changes is orientation how these rotates phi 0 is a psi 0 is essentially the measure of that.

But so this is the things this is the equation that we have from this figure right. Now consider now we have one more curve define on x y plane of this [cum/complex] complex plane, but passing through the same point z 0 and then look and we considered the transformation of that curve through the same transformation and if we do that.

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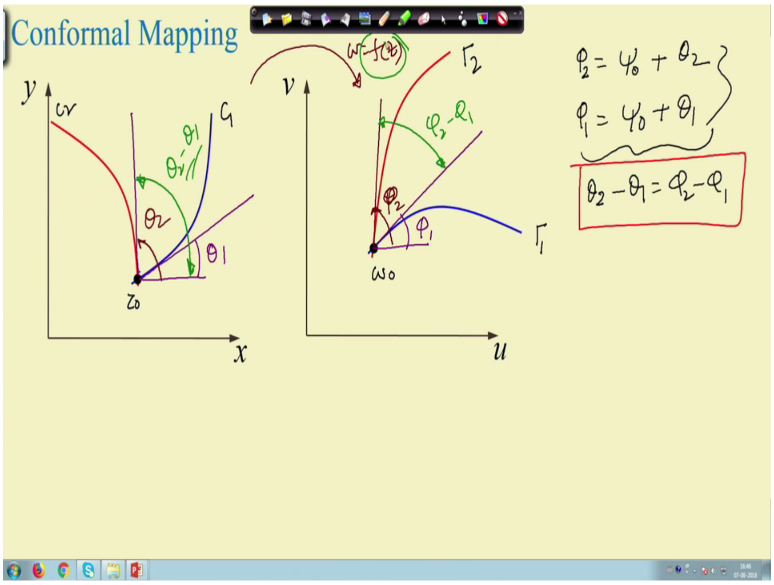


Suppose what we have is we have two curve in this case it is say this is c 1, this is this curve is say c 1, and this curve is say c 2 and corresponding this is gamma 1 and this is gamma 2. And we have a transformation this transformation from this to this is equal to w is equal to f of z ok. And these point is z 0 this point is z 0 and corresponding point is w 0. So, w 0 is essentially the map of z 0 ok.

Now, then similar to the previous slide let us draw two slopes here. First slope is this one this is drawn on c 1 and draw another slope on suppose then these angle is this angle is theta 1 ok. And similarly you draw a corresponding slope here and then this angle is phi 1. This is how we define we call theta 0 and phi 0. Now then we have two curves, now draw another slope on these curve suppose these angle is use different color just. So, that we can understand this. Now, these angle is suppose theta 2 and then draw another tangent here and corresponding this angle is theta sorry phi 2 ok.

Now, then what is the relation between theta 1 and phi 1 and theta 2 and phi 2 recall the relation between relation between these are that phi 2 is equal to phi 2 is equal to psi 0, 0 means psi was the argument of if you recall psi was the argument of argument of f dash gamma s 0 at this point since this point is same for both the curve. So, here also it will remain psi 0 psi 0 plus theta 2.

Similarly, we have phi 1 is equal to psi 0 plus theta 1. So, this is the relation how this is the relation through is theta 1 phi 1 and theta 2 phi 2 are related. Please note that here this point z 0 is common both the cases both the curve point z 0 is common and at point z 0, the function is analytic and the derivative is not zero. and psi zero is equal to the argument of this at s is equal to s zero or z zero ok. And since this point is same your both the cases psi zero will remain same.

Now, from these two from these two what expression then we can write? We can write that theta 1 minus theta 1 you theta 2 minus theta 1 is equal to phi 2 minus phi 1 this is important. Now what it tells you? What is theta 2 minus theta 1? Theta 2 minus theta 1 is essentially the angle between this two slope. This is theta 2 minus theta 1 right theta 2 minus theta 1.

Similarly, phi 2 minus phi 1 is this phi 2 minus phi 1; this is phi 2 minus phi 1 ok. So, what it tells you that if you take a point if you take a point or if you define first a transformation say w is equal to f z from x y plane to u v plane. And then suppose that transformation is that function is analytic function and if and at that particular point it is not zero and then if you take two curves passing at the same point z 0. And then this the slope between the angle between these 2 slopes which is theta 2 minus theta 1 and then if you transform this curve you project this curve on u v plane and their angle between two slope is phi 2 minus phi 1 then these two angle remain same.

So, as I at the beginning I said the mapping is essentially a function sometime with some special features. Here it is a function it is an analytic function, but with the special features that the angle between slopes at a given point at this point for the function is analytic. The angle between the slopes remain same they preserve. That is the special features we have here and that is the reason why this mapping is called Conformal Mapping. The angle between these two points are preserve.

Now, probably this thing will be clearer if we, if we if we if we give if we demonstrate this through an example. , But before that please note the two conditions that the function needs to be analytic at that particular point and then the derivative of that function should not be zero. Now if we say if we find out if we take if we suppose if you have a domain D here in this case.

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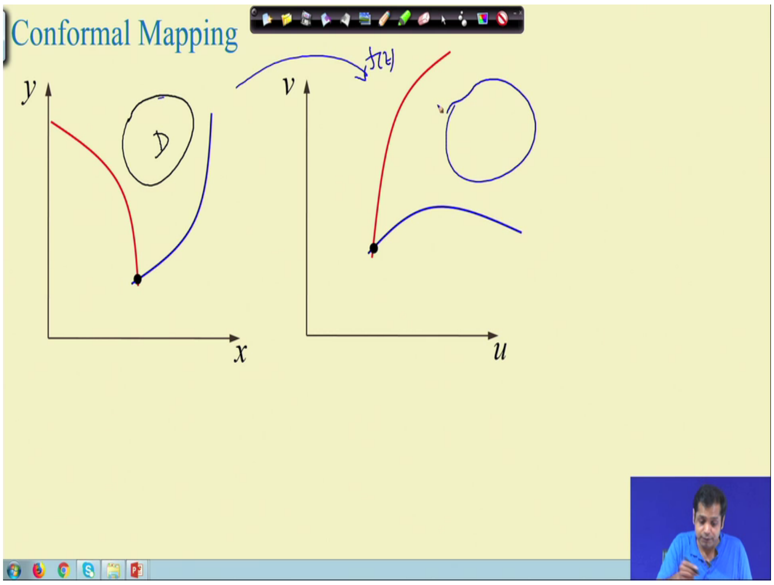


Suppose if we define a domain if we define a domain D and then its corresponding transformation is corresponding transformation is this.

Now, if the function is this transformation say f z and if this function is analytic everywhere in this domain and everywhere the derivative of this is not zero, then this entire mapping is called the Conformal Mapping. So, if at every point that the condition that we satisfy can condition that we derived the angle preserving condition if that every point that condition satisfied then we say that the entire map is the conformal mapping. Let us find out let us let us give an example ok.

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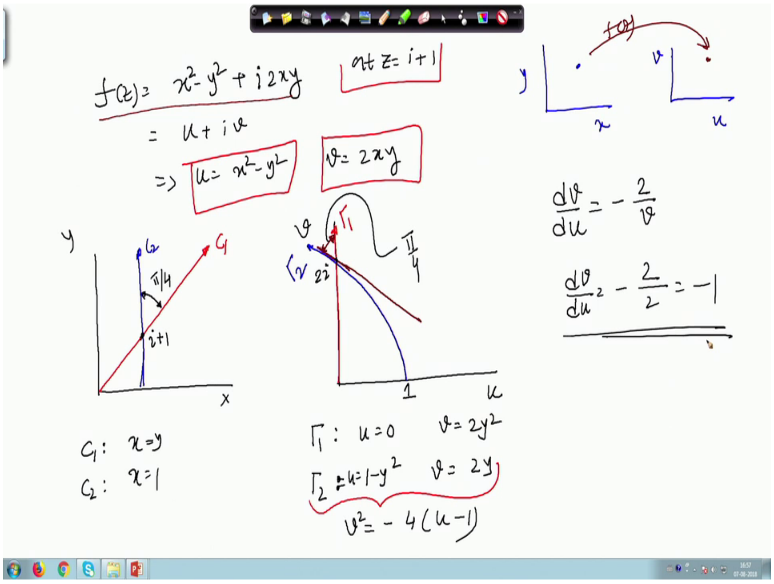

Suppose consider a function f z is equal to say f z f z is equal to say x square x square minus y square plus i 2 xy ok. Suppose this is the map this is the map between xy plane and uv plane. Now from this if you recall that we can write any function as u plus iv. So, this gives us u is equal to x square minus y square and v is equal to 2 x y right. So, for this transformation this is this.

So, it says that if you have if you take if you take x y plane and then u v plane u v plane. You know the x y you substitute x y here you get corresponding u v and then you get the corresponding point this is the transformation f z ok. Now we will check whether the transformation is analytic whether the transformation is they whether this mapping is conformal mapping and not just we take two slopes to two curves and then find out the slope between these two curve in the x y plane and then see what happens to this slope when it is transformed to project it into u v plane and then see whether these angles are preserved or not ok.

Now, let us draw this a let us draw an xy plane suppose this is the xy plane this is x and this is y plane ok. Now the condition is check whether the let us first complete the problem statement. The function is given now check whether the function is analytic or the or the it is a it is a it is a conformal map at z is equal to i plus 1 at z is equal to i plus one? This is the question ok.

So, we have to check in the previous in the previous example this z 0 is essentially now we have i plus 1 ok. Now so let us take this at i plus 1 suppose this point is i plus 1 ok, means x is equal to 1 and y is equal to 1 and then take two curves. Suppose these two curves are one curve is say c1; this is c 1 and say another curve is passing through the same point suppose this is a straight line this is c 2.

So, c 1 is essentially c1 is essentially your x is equal to y x is equal to y this curve and c 2 is essentially x is equal to one. Suppose we have two curves ok, now we know that this angle between these two curves is this angle is pi by pi by 4, right. Now, next what we do and this point is this point is i plus 1 now 1 plus i.

Now, what we do is now let us see when these two curves c 1 and c 2 transform through this transformation through this mapping. So, draw the corresponding space say u v space right this is u and this is v ok. Now, you see what happens to what happens where what happens to c 1 curve c 1 is c 1 is essentially x is equal to y. Now if we substitute x is equal to y then what will be gamma 1 gamma 1 will be gamma 1 will be if you substitute x is equal to y. So, for gamma 1 u will be 0 and v will be for v will be 2 y square 2 y square if we substitute x is equal to 1.

Similarly, gamma 2 gamma 2 will be if you substitute x is equal to 1 it will be 1 minus y square u will be 1 minus y square and v will be if we substitute that this will be 2 y. Now if we if we if we combine these two this e 1 v then essentially this gives you an equation v square is equal to v square is equal to you can check it minus 4 into u minus 1.

So, what happens when c 1 this curve is mapped transformed through these transformation then it becomes gamma 1 and in this case gamma 1 is u is equal to 0 and v is equal to 2 y square means your gamma 1 becomes a line like this like this becomes gamma 1. And then c 2 which was a straight line with x is equal to1 and when it is transformed to the same map. Then this becomes and parabola with this u v the relation between the relation the equation of that parabola is this in terms of u and v.

And if we draw that parabola then this parabola would be something like this ok. Something like this so this is your gamma 2 ok. Now what will be this point this point will be we can check this point will be when v is equal to 0 then u is equal to 1. So, this point will be 1 this point will be 1 and if you check compute that u is equal to if we substitute u is equal to 1 here and this point will be 2 i 2 i ok.

So, essentially in this curve the at this point u will be 1 and v will be 0 at this point u will be 0 and v will be 2. So, this is the map of c 2 ok. And this is the angle we have this is the angle we have here now if we draw a tangent at this point this is the angle between these two curves ok. Now we have seen when in before the transformation in x y plane this angle is pi by angle was pi by 4. Now let us see whether this angle is pi by 4 or not ok.

Now, remember if this transformation this map this function is an analytic function at the given point in this in this case the given point is i plus 1 and this function is not zero. The derivative of this function is not zero where i plus i plus 1 then the angle preserved. What we do here is we will see that whether the angle preserve it not? Assuming not assuming you can you can prove it that I leave it to you, you can prove that whether this these function is analytic at i plus 1 or not, and then check whether the derivative of this function is not zero at i plus 1 or not. If that is done then the angle preserve and that is the part that we are going to show the angle preserves ok.

Now, let us find out how to find out the angle, now we have we have let us find out the slope of this curve this gamma 2 gamma 1 is very straight forward. This is just a vertical line then let us find out what is the slope of gamma 2 slope of gamma 2 in order to find out we have to find out del v du right.

Now, if you if you see that the du dv du will be in this is equal to minus 2 by v ok. Now look at this point we want to compute this slope at this point, now at this point just now we discussed v is equal to at v is equal to 2 and at this point u is equal to 1. So, if we substitute v is equal to 2 here then at this point your dv du dv du will be minus 2 by 2 is equal to minus 1 ok.

Then what would be the angle between these two this is a vertical line and this is the tan, this becomes minus 1 the tan theta become minus 1 ok. Then what happens this angle this angle is also this angle is also pi by 4. You can just with this figure.

So, you see. So, this is a conformal mapping where these two angles, if you take any arbitrary this is just for demonstration we have taken two angles two lines. So, that two very simple lines. So, that we know the angle, but if you can take any arbitrary slope any arbitrary points and if at that point the function is analytic and derivative is non zero.

Then you can show very similar way that the angles are preserved you cannot show we have already shown we have already proved it you can check the with any example for instance in this example if you take you can see the angle is preserved.

Now, that exercise I have done for you and let us see how it looks like.

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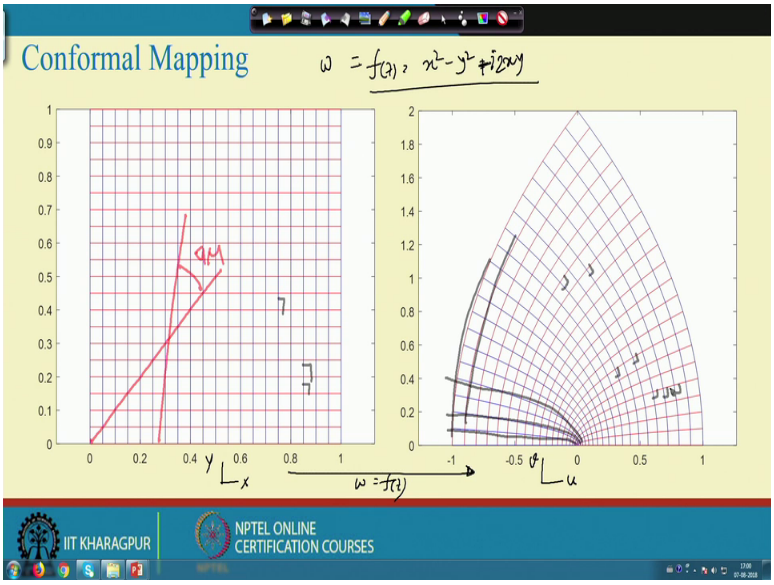


For instance yes, the same example same example means the same function in this case the function is your function is f z is equal to x square minus y square is plus 2 plus i into 2 x y this is the transformation this is the map w is the map. So, this is the xy plane and this is this is xy plane and this is uv u v and this is the map from this plane to this plane is equal to w is equal to fz ok.

Now in the previous example I took two curves one curve was one curve if you recall one curve was something like this and another one was one was this and another one was another one was this right another one is this and we saw that this angle was pi by four and we checked whether it is a pi by 4 or not.

Now, you see these in this figure there are randomly taken suppose we have if we it have a grid here. Now these all these angles are ninety degree all these angles are 90 degree ok. Now the blue one the transformation of blue one are this these are the transformation of blue one, and the red one which was the which is the horizontal axis here the transformation of horizontal axis are this

Now, you see the initially the angle was 90 degree, here also these angles are 90 degree this angles are 90 degree so , this is conformal mapping. So, if you take any arbitrary now similarly you can take you can you can further explore it or further do the similar exercise with different functions. As I say there is a book by chart Ceylon complex variable there are many other examples given exercise problems are given where map is given and you have to check or you have to you have to find out the angle. And then check the angles are preserver or not all different kinds of examples and exercise are given you do that exercise and then you can you can you convince yourself whether this how these angles are preserve.

Now, that is fine conformal mapping we discussed, but what we have to understand what is the use of this conformal mapping in the context of elasticity problem. At this point please note, this is the thing that we required for complex variable and next subsequent classes what we do is we will just use with apply this concept to formulate the elasticity problem and further solve them.

But at this point please note, we have discussed pure minimum thing that we require to formulate the elasticity problem a brief is a very brief account of complex variable method. In order to get the detail account please go through book and in order to get the get a comprehensive idea about the conformal mapping or different kinds of for instant any co different theorem with that we discussed in the previous class line integral, then a path independent in order to get the detail of all these concept in order to in order to have better comprehension of the all this concept you have to go through a book on complex variable ok.

Now, with this you see what is the use of this method, what how it will be used in your previous classes? Ok you see.

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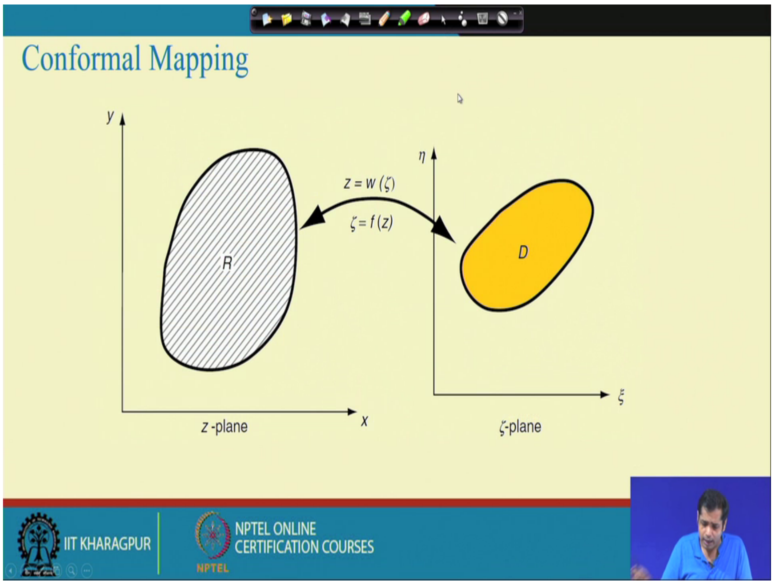


Now in the next in the subsequent classes what we see we for instance we will this is a conformal, this is how we define a conformal map we have a xy plane. Then we have will have another plane and then this is a domain c r is the domain in x y plane. And then we define a map like this z is equal to like this and we use this map to be conformal. We use a conformal map so that the angle between these, these lines are preserved angle between two lines are preserved.

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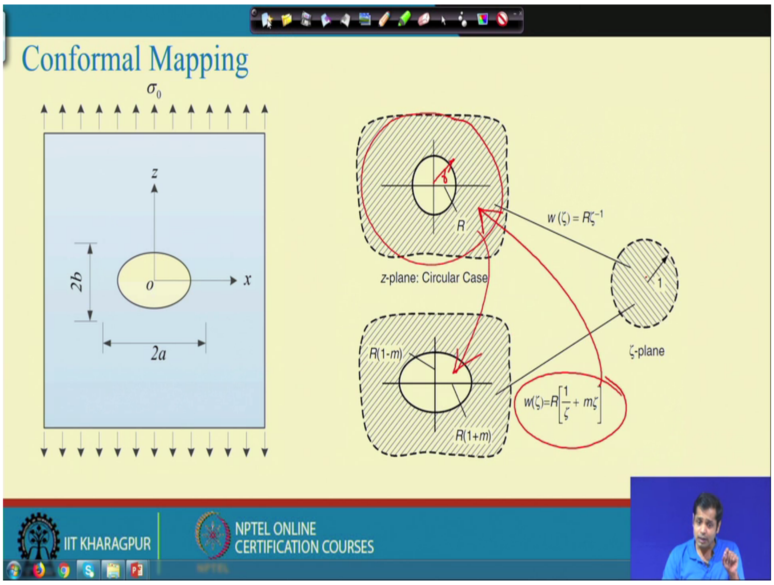

Now, the use of this you see this is a very important problem and if you recall in one of the classes with we i give a very brief idea about this problem. Now see we have to find out the solution of this problem and finding solution of this problem is just not an exercise um, when we do this we will discuss that in detail it will the solution of this problem will be the motivation of our next week lectures.

Now, we have to find out the in and in that context the solution finding the solution of this problem is very important. The problem is we have a plate, which is subjected to in plane loading and the plate has an elliptical hole, there. And then what we are interested? What happens to the distribution of stresses? What is the expression of? How do we get the way we have the closed form solution of difference, the stress distribution, and then displacement distribution?. Similarly we want the close from relation of the distribution of stresses on this plane.

Now, had it been a circular hole will be doing that one exercise of circular hole in the next class. I had not been circular hole it would have been very easier because you can the hole of this circle can be this for instance. This circle can be represented, the entire surface can be represented by one parameter r , but in this case it is not as trivial as circle. What we do now is, will define a map between this ellipse and a circle suppose. We have a circle and we define a map between this circle and ellipse. This is a map between this circle and ellipse and these map is a conformal map. That the conformal mapping, just now we discuss these map will be a conformal mapping.

And then once we once we have this mapping then what we do is we actually solve the problem on this place. But when we solve this we introduce this map into this into the solution into the formulation so, that we have to do the exercise on this but since as this map is introduced here final solution will be get of this problem.

This exercise will be this step will be clear when we actually implement this here on this slide what I want to tell you the point what I want to make is that the conformal mapping is important and this map will be using to solve this problem.

Next class first we derive the equations for elasticity plane elasticity problem. Using through complex variable method without any conformal mapping. And then see how the complex variable method can be used to solve some elasticity problem. We will demonstrate through some example. And then next to next class we will come to the conformal mapping. We will try to solve this example using conformal mapping concept.

With this I stop today, See you the next class.

Thank you.