Theory of Elasticity
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Lecture – 44
Complex Variable Method (Contd.)

Hello everyone, this is the second lecture of this week and what we have been doing is we are; we are been reviewing the some of the concepts of Complex Variable. And those concept we will be using to formulate the elasticity problems and then subsequently solve them. So, today's topic is the integration on complex variable.

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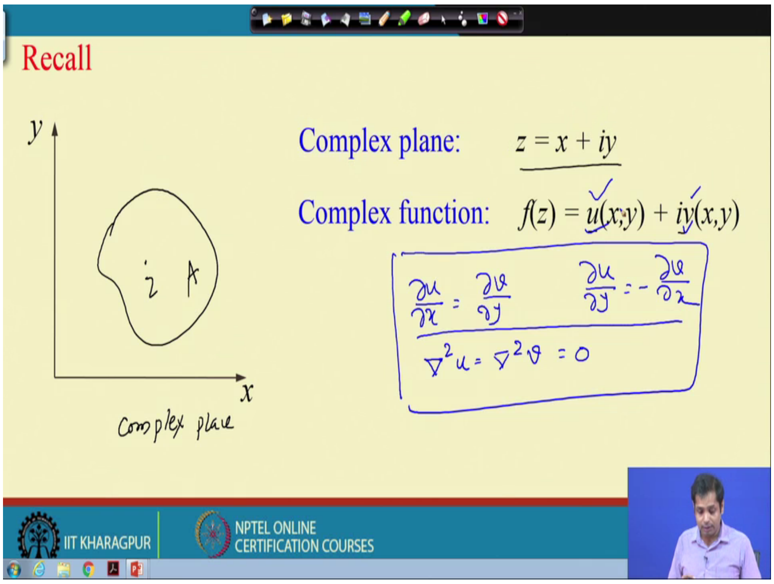

Now before that if you recall this is if we have a complex plane this is the complex plane. Then that Z is any point here Z, and that Z is written as that is a any complex any point on this complex plane is written as. Now similarly we have a complex function; that complex function can be written as a real part and the imaginary part u and v. And we also discussed if this function is a analytic function over a domain in a complex plane. Suppose this is the domain in a complex plane and over this domain this function is over region A this function is analytic function. Then u and v they satisfy Cauchy Riemann equation, and what is that equation? If you recall that equation is del u del x del u del x is equal to del v del y, and then del u del y is equal to minus del v del x that is the Cauchy Riemann equation.

And we also have seen just the as a consequence of this equation; if this equation further extended. Then we can show that that del 2 u is equal to del 2 v that is equal to 0. Means u and v both are harmonic function what if they satisfy the Laplace equation. And u and v are complex conjugate with each other right. So, this equation and these properties, this equation in is very important in a sense; just we will demonstrate that through an example before we actually start discussing integration on complex plane. See if you if a function is analytic function over a given region A and then that is represented by real and imaginary part like this.

And if you know any one of these suppose, if you know u then using these relations Cauchy Riemann equation you can determine v. Or if you know v, then using this relation you can determine u right, because there that is why they are called harmonic conjugate. They are harmonic functions and conjugate with each other they are called harmonic conjugate. Let us demonstrate this through one example now take one example.

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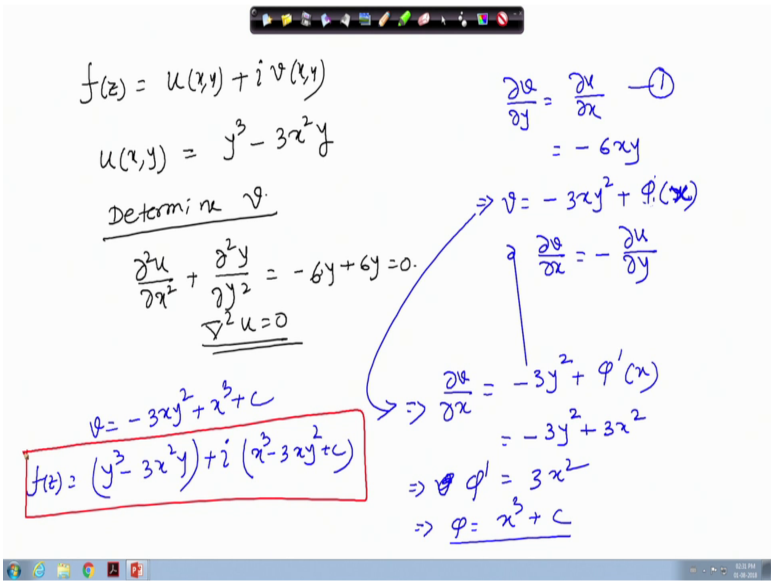

Suppose any function take any function any complex function f z say, that is equal to we know that u of x y plus i into v of x y, v is the imaginary part.

Suppose this function is a analytic in a given region ; and what is known as u of x y is known as it is given as say y cube y cube minus 3 x square y. Now the question is then determine v ok, that is the problem. Now what information we have? Information we have is f z is a analytic in a given region that is why they satisfy Cauchy Riemann equation. The both these functions u and v are harmonic functions and u and v are harmonic conjugate and this u is given ok.

Now, let us apply let us first see whether they are harmonic function u is harmonic function or not. Harmonic function is del 2 u del x 2 plus del 2 y del y 2. It will be if we differentiate it with respect to this then it will be minus 3 minus 6 y and then this is 6, this is 6 y and then from this also we have plus 6 y so this is equal to 0.

So, del 2 u is equal to 0. Therefore, u is harmonic that property is being satisfied here now next we have to find out what is v. Now you know that if we apply the Cauchy Riemann equation, we know that del u del x is equal to. So, del v del y that is equal to del u del x. Now if we if we differentiate u with respect to x then what we have is; you have to go back yes. Now if we differentiate with respect to x then what we have is minus 6 minus 6 x y.

So, del v del y is equal to minus 6 y from there; we have v is equal to then it becomes minus 3 x y square plus some function, which is let us not write f here, because f we are using for different purpose. Let us write a different notation say, phi of phi of y right phi of; this is with respect to x this is integration with respect to y. So, the constant will be it is a function of x, so this is the expression of v. So, this is we obtained from the first expression of Cauchy Riemann equations.

Now, the second we know that del v del x is equal to, del v del v del x is equal to minus del u del y. So, let us find out that from this expression if we get del v del x. So, this gives us del v del x is equal to minus 3 y square plus phi dash x. And that is equal to del u del y or minus of del u del y del u del y you can obtain from this expression. And del u del y will be minus will be 3 y square plus 3 3 y square plus 3. It is again it is going in a different way say yes, minus 3 y square and then we have plus this one is plus 3 x square.

Now, if we differentiate this v, then we have v is equal to; then not v if we from this expression what we get is that phi dash is equal to 3 x square right. Now if we phi dash is equal to 3 x square then from this we get phi is equal to; if we integrate it we get x cube x cube plus c. Now this C will not be a function of y because phi we know the phi is a function of x only so C will be constant.

So, if phi is equal to this then if we substitute that phi here then we have finally, v is equal to minus 3 x y square plus phi, phi is plus x cube plus c. So, finally, function f z is equal to u is this; y cube minus 3 x square y. This is your u plus I into v and v will be your x cube minus 3 x y square plus c, so this is the final complex function. So, this through this what we try to demonstrate is for; if you know the function is a comp analytic function. And if you know one part of the function at the real part or the imaginary part, apply the Cauchy Riemann equation you can obtain the other.

But now this is very important because when we apply complex very complex variable approach for solving elasticity problem, we will be using these equations extensively ok. Now this was once this is done, let us move on and then the start the actual topic of today's lectures the integration on complex plane.

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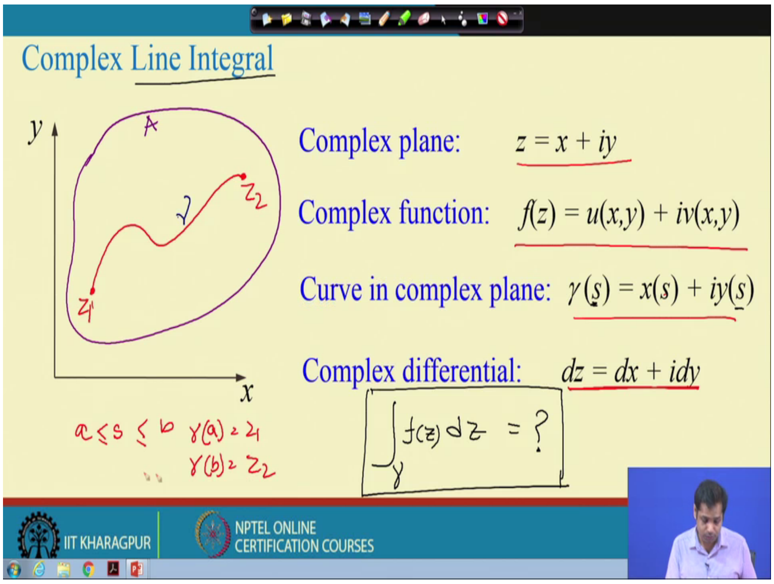

So, let us start that. So, next is before we do that you see there are 2 important things. Suppose we have we have 2 points one point is here, we have a curve like this and this is one point and this is another point.

Suppose this point is a z 1 and this point is z 2. Now this entire curve is within a domain say, draw the domain within a domain like this. And say region A or region this is a domain A and the domain within this domain A the function f z. Say write the function f z the function f z here, it is written here function f z is analytic function. Now what we are interested in; we are interested into find out integration of f z say integration of f z between 2 points z 1 and z 2.

But before that we want the integration of f z over this curve. If suppose this curve is gamma this curve is; say this curve is gamma. Then what we want is we are interested in it is integration of f z dz over this gamma, what is this value? This is that is why this is called line integral this integration is over this line ok. We want to determine this that is our fine final objective. And after determining this we then we will see through that derivation through these through this process, we will establish some of the very important theorem in integration on complex plane.

Now before doing so let us now this is the curve which has 2 point z 1 and z 2. Now let us let us parameterize the curve through a parameter s to a parameter s.

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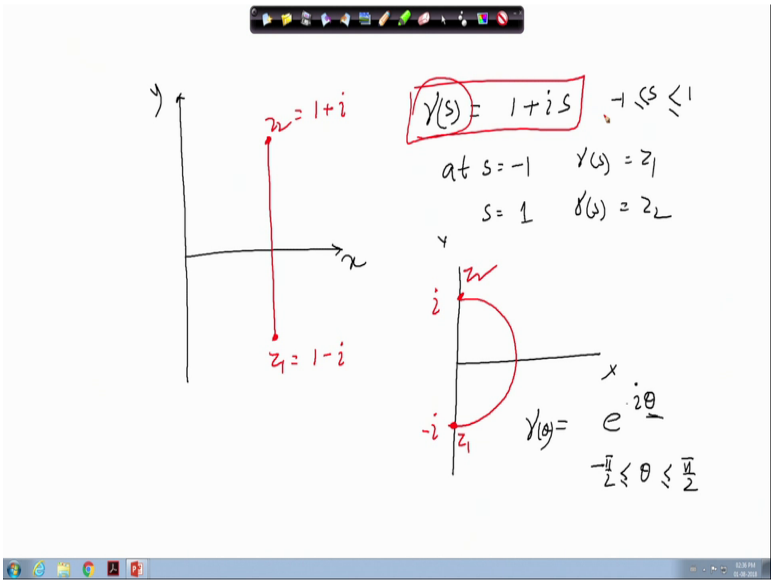

Now, it means suppose we have suppose, for instance if we have a complex plane like this complex plane like this. This is x and this is y and of course, this is minus y. And now suppose we have a curve like this is z 1 and this is z 2. And z 1 is suppose 1 minus i and this is z 2 is equal to 1 plus i.

Now, this is the line in this case the straight line. So, what is gamma s now is in this case gamma s gamma s we can write that 1 plus i into s, and s varies from minus 1 to 1. So, at s is equal to at s is equal to minus 1. So, gamma s actually gives you z 1 and s is equal to

1 gamma s gives you z 2. Now the entire curve is now written as this as this. Similarly, for instance if we have if we have another, if we have at if we take another example like this, it is a complex plane again and then you have a curve like this.

Suppose this is minus i and this is i now then gamma s can be written as in this case gamma s, gamma can be written as e to the power i theta e to the power yes, e to the power i theta, where theta varies from minus pi by 2 to pi by 2. So, this becomes gamma theta. So, in this case instead of s we use theta because it is an in polar coordinate ok. But now the entire curve is parameterize by a parameter theta here.

Now, you see here at theta is equal to minus pi by 2, when theta is minus pi by 2 you get this point this point which is your z, which is your z 1 and at theta is equal to pi by 2 we get this point which is your z 2 ok. So, similarly in any curve if we have that curve suppose can be written as gamma s ok. Now depending on the curve what is this expression and what is the bounds on, what is the limit of s that is different ok.

Now, let us let us go back to our so what we were doing is this right. Now this is this is a curve within a domain A which is an any function f z it is analytical. And to start with this domain is a simply connected domain, if you recall in one of the one of the earlier classes we discussed what is simply connected domain, what is not simply connected domain or multiply connected domain. And there we discussed that there are some situations where we where we encounter multiply connected domain and we have to deal with them.

Now, to before with that let us assume it is a simply connected domain. Now f z is analytic on this so gamma s is the gamma s is this curve, we want to find out this integration. And then dz is like x z is equal to x plus iy dz can be is written as dx plus i into d y. Now then first you see and another important thing is your here s is equal to in this case say, s is s varies from a to b. Like the previous example s varies from minus 1 to 1 or s varies from minus pi by 2 to pi by 2.

In this case suppose s varies from a to b with gamma s gamma a gives you z 1 and gamma b gives you z 2. And in between z 1 and z 2 everything is obtained by this expression right. So, suppose let us find out that integration now.

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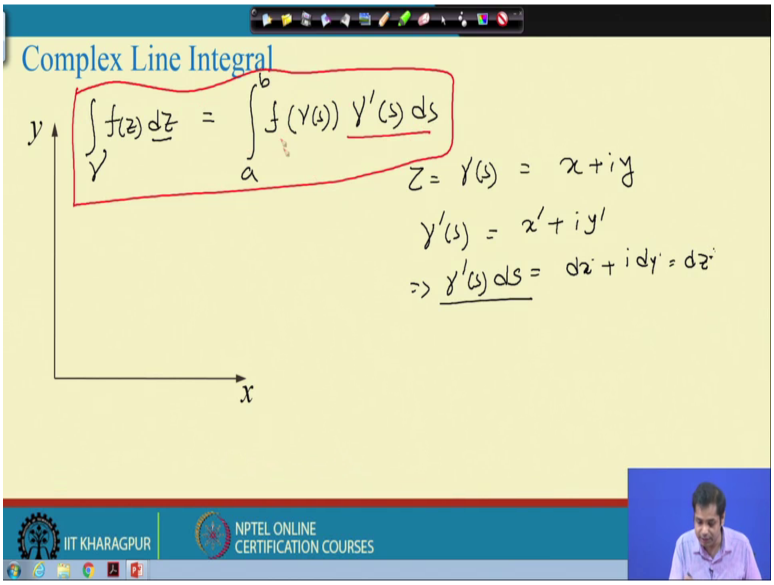

So, we are interested in integration of f z dz over this line gamma. Now this can be written as say gamma is a line whose limit is a to b. Now just the change of variable here it is written in terms of z. If we write everything in terms of in terms of s; following our previous this definition of gamma and z what we have is; then this expression will be a to b, then f of gamma s into gamma dash s ds.

Let us find out that see what is gamma s, gamma s is equal to, what is gamma z is equal to, what z is equal to gamma s right which was equal to x plus i y. Then what we have is we have gamma s dash is equal to x dash d x dash plus i into y dash. Or from this we can have that gamma dash s d s that is equal to d x dash plus i into d y dash. Which is dz which is equal to dz right no not this there is been no dash that is equal to dz right.

So, if we substitute dz by this then we have this part right. And then the first part is very straightforward because f of z is equal to gamma s so this is the thing now. So, this is the line integral on so if we have if we have to find out any line any integration of any complex variable over a line then this is the expression for this.

Now, we will just give we will demonstrate what is the how this expression can be used through one example.

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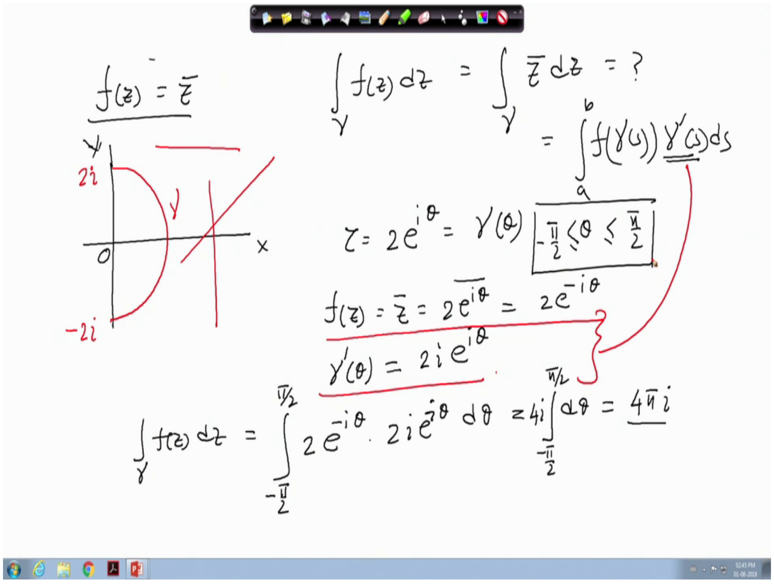

Now taken one example, take the example of the same thing that we consider a plane like this; consider f z f z is equal to say z bar ok. You can take any functions for demonstration let us say zx is equal to z bar. And then suppose your this line is this is the complex plane. And suppose the f z is analytic over the entire plane or some domain which within which, we take one curve suppose this is the curve this is the curve.

And these suppose this is minus 2 i and this is 2 i and this is gamma. And what we have to do is this is o we have to find out this integration. We have to find out integration of f z dz over gamma or integration of in this case z bar dz over gamma ok. This is we have to find out. Now we have seen the expression in the previous if you recall the expression is this; that expression is this will be integration of this is integration of a to b f of gamma s, and then and then we have gamma dash s right gamma dash s d s.

Now, the first step would be to write this entire curve through as a function of s. And if we do that just now we have seen the same example we can write z is equal to 2 e to the power i theta. In this case because earlier case it was minus i 2 i that is why it was e to the power I theta, it is minus 2 i to 2 i 2 e to the power i theta. And that is in our case is gamma theta right. So, instead of s we have theta l because we are using polar coordinate system. And what are the values of theta? Theta is minus pi by 2 to pi by 2 pi by 2.

So, a is equal to minus pi by 2 b is equal to pi by 2. Now we have to find out this integration. Then what is f z? F z is equal to z bar which is equal to 2 e to the power i theta bar. And that we know that that function of the count this your, if it is the is the function of the conjugate can be obtained by just function of the conjugate it will be minus i theta.

Now so f z is equal to this so f z is written in terms of this parameter theta. And then gamma is also written in terms of this parameter theta right. Now let us we have to find out what is gamma dash theta. So, gamma dash theta we have to find out that will be equal to 2 i, 2 i from this expression 2 i e to the power I theta. So, we have this and then we have this. If we substitute this into this equation the entire thing into this equation, then what we have is this. We have that integration is equal to integration, f of z dz over gamma that is equal to where integration over a to b means; minus pi by 2 to pi by 2. Then we have a gamma s gamma s is equal to 2 e to the power minus i theta 2 e to the power minus I theta and then gamma dash theta this will be 2 i e to the power i theta d theta right.

Now, it will be there will be d theta as well fine. So, once we have this then essentially it will be integration of the 4 I, we can take out and this is d theta then minus pi by 2 to pi by 2. And the finally, what expression we have is this is equal to 4 pi i. So, this is the integration of f z over this line. Now you can try same example same expression of f z you can take some different lines suppose a line like, this a line like, this a line like say this, line like this any line you can take, but that line should belongs to belong to that domain over which that function is analytic function.

And then following the same approach we can find out integration over these lines. Only thing is the first step or the important step is that line needs to be written in terms of gamma through a parameter s. Now if you recall at the beginning of the complex variable I gave you 2, references one is for elasticity problem and another one phase another one is for complex variable. The book by Churchill; there are lots of many examples of similar kind of examples in fact, this example or both the examples. The earlier one and this one is taken from by the same book by Churchill. You can go through that book there are many examples please do these examples and the exercise problem; so, that these steps will be clear to you ok.

So, this is how we obtain the line integral. Now let us move on so now, but before that ok. Now that we discussed already; now another important thing; so, what we now know is how to determine if you have a complex variable, a complex function, analytic function, then how to determine the integration of that function over a line now.

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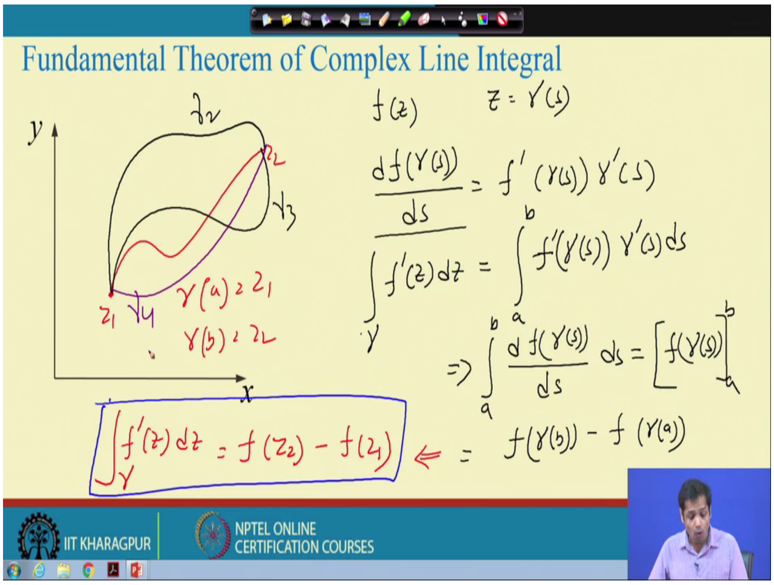

Suppose now let us before there is important thing is that; we are that we will be introducing shortly is, fundamental theorem of complex line integral.

But before that before we state this and then subsequently the Cauchy’s theorem. Before we state that let us first prove it prove something do some exercise and through that; we find out some property of this line integral. Now so suppose f z is a function we know, now f z is a function and z is written as gamma into gamma of s ok. Then d of f of gamma s with respect to s ds we can write as; f dash of gamma s and then gamma dash s right this is just simple chain rule um.

Now, if we have an integration like this; integration of f dash z dz not the function the derivative of the function over line integral gamma ok. Then that can be written as say a to b we have f dash gamma s that is the just now we have derived this expression into gamma dash s d s. So, if f is a analytic function then that f the f dash z the derivative exist right. So, this is the expression of this that we just derive an short badge in the in the previous slide.

Now, this you look at this can be replaced the entire thing, the entire thing can be can be replaced by this. So, this can be written as integration of a to b. So, that thing becomes d of d f of gamma s and then ds into ds. Which is essentially becomes f of if you integrate it f of gamma s into of boundary values that a to b ok. This eventually becomes f of f of your b f of gamma b minus f of gamma a.

Now or f of gamma b and gamma b is what; if we have if we recall these expression. This was is z 1 and this was a z 2, f of gamma of a was z 1 and gamma of b was z 2. So, essentially this is equal to this we get that is equal to f of z 1 f of z 2 minus f of z 1 ok. And that is equal to your this expression integration of f dash z d z over gamma, this is important this expression is very important.

Now what this expression says this expression says that; if we want to integrate any functions f dash z over this line. Then that is equal to that function evaluated at z 2 minus the function evaluated at z 1. Now suppose if we have if we have say another curve like this; this is say gamma 2 gamma 2, or we can have say another curve like this which is say gamma 3, or if we have say another curve like this which is as gamma 4.

So, if we have several such curves, but the common thing is that the starting point and the end point are same z 1 and z 2; so, between z 1 and z 2 if we have if we can have infinite number of curves right.

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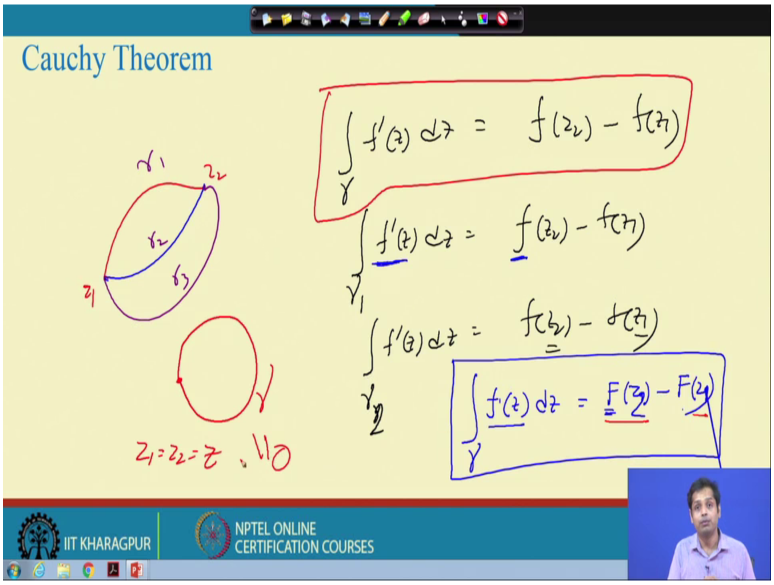

Now, if I have to integrate the same function, the same thing over different paths, but their end points are same z 1 and z 2 then what does it mean? It means that just now we wrote that integration of f dash z dz over gamma that is equal to f of z 2 minus f of z 1.

Now, between f and f f z 1 and z 2 we have many curves right, we have many curve. So, gamma 1 gamma 2 gamma 3 and so on; it means that if we integrate want to integrate over gamma one f dash z dz that is also f of z 2 minus f of z 1. And we if you want to integrate on say gamma 2 gamma 2 of this that is also becomes f of z 2 minus f of z 1 and so on. Any curve you take as long as your z 1 and z 2 are same then that integration is same. Means that integration if that function is a analytic function on a simply connected domain. Then this integration does not depend on their line integration, but this line integration does not depend on which line that integration is performed.

As long as the 2 points endpoints between 2 points that 2 points are same for different curves that this integration is same. And that is called path independence of this line integral right. So, these integrations are path independent. This can also be stated in the same expression; this same expression can also be written or stated in a slightly different manner. And this is suppose take different color if I instead of instead of integrating f dash, if I integrate say integrates a function f of z dz over any gamma and that will be F of z 1 minus F of z f of z 2 minus F of z 1, F of z 2 minus F of z 1, where F is the anti-derivative of small f.

In this case here f this small f is the anti-derivative of f dash, f dash is the derivative of f f is the anti-derivative of f dash. So, if we integrate f then capital F here is the anti-derivative of f. So, this is another way we can express the same thing the path independence. So now, with this then we introduce the important theorem is Cauchy's theorem.

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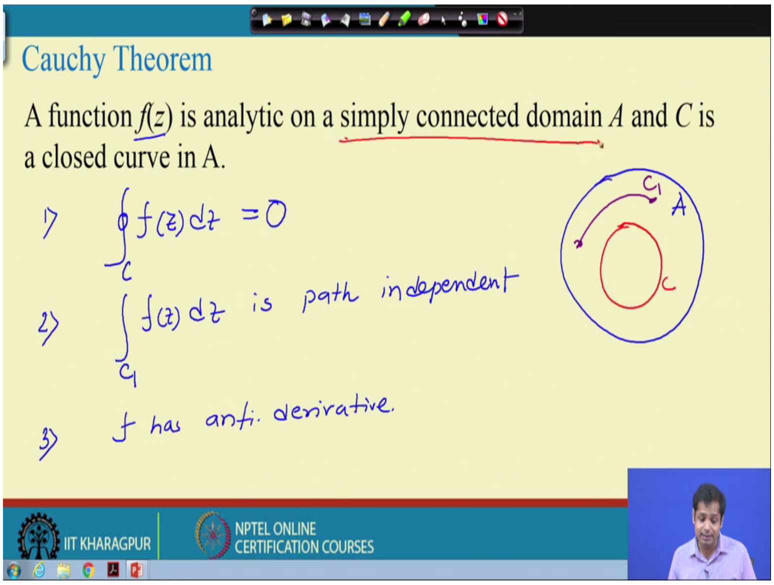

Now what it says; that if a function f z is analytic on a simply connected domain A. Then and C is a closed curve in A, for instance if we take if we take a complex plane like this or let us not draw the complex plane let us just draw the domain.

Suppose you have a domain like this which is a simply connected domain. Simply connected domain supposes this region is A ok. Now over this region the function f z is analytic ok. Means it satisfy all these Cauchy Riemann equation that their derivative exist ok. And then consequently satisfy the Cauchy Riemann equation and so on. And suppose you take any closed curve C, C is a closed curve ok. Then what happens? You recall just now we stated in the previous slide that if, if we integrate any function over any line over any line this is z 1 and z 2 and then their derivative is this.

Now, suppose this curve is integration your starting point is this, and this curve is n ends here means; in this case z 1 is equal to z 2 is equal to z ok. I mean the starting point of the end point then same. Now this is now gamma suppose, now if I integrate any function f over this line integral then what will happen? This will be f of z minus f of z means this integration will be 0 right.

Now, that is what the first thing says, that if we have if a function f is it is a analytic on a simply connected domain A. And C is a closed curve then what we have; we have the first is, we have the integration of f z dz a closed integration. Because it is then oversee that is equal to 0 that we that is a direct consequence of this path independent that just now we proved ok. Now the integration the line is a closed curve therefore, starting and end points same that is why it is automatically 0.

Now this is one point and the second point is what you obtain that it is in the previous 1 that is now in this domain if we have a curve any arbitrary curve like this, any arbitrary curve suppose this is say C 1, we have a starting point and endpoint different. But this inter curve belongs to the region A over which that function is analytic function. And then what we have is we have integration of integration of f z dz over C 1 that is path independent. It is just now we prove path independent.

And third was since it is a analytic on these domain that the anti-derivative f has anti derivative this is these 3 is very important. So, this is a direct consequence of f z being an analytic function and then the path independence, but then all these are put together and then finally, we have this outcome from all these exercise. Now let us now let us extend this for a for a for a non simply connected domain, the here in this case the everything is all the domain the assume is it is simply connected domain.

Now, very, very often we encounter non simply connect a multiply connected domain. Let us see can we extend this theory to multiple connected theory multi multiple connected domain and get some information which will be useful for the problems that will encounter in elasticity ok.

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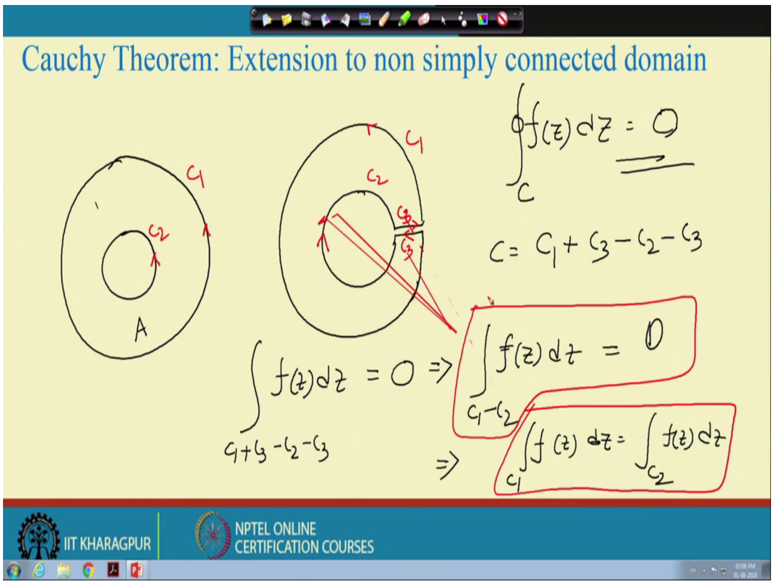

Now, suppose so let us do that exercise they can take a multiply connected domain suppose this is this is this is a multiply connected domain right ok. So, your domain is essentially A is he this one this one is the domain. Now suppose what you do is suppose this one is so you have in this your domain is this is the domain here right. Suppose this is C 1 and this is C 2. This outer boundary is C 1 and the inner boundary is C 2. So, anything in between this C 1 and C 2 is our domain now the domain A.

So, it is a it is not a simply connected domain right. Now what we have is first we write this expression the extension and then we try to prove it ok. Or let us let us first let us first do some exercise and then state the state the final outcome. Now suppose this domain assume you have the annular shape domain and you with a scissor you just cut it ok. For instance, if I have to give you just an example I have, I take a piece of paper a just so that you can understand this.

Suppose if I take now this is this is now a multiply connected domain right. Now if you look at this is a multiple connected domain. So, in this case this is the C 1 and this is the inner boundary is C 2 ok. Now what we do first is let us with a sissor, let us cut it like this ok. Now this domain if you see this domain; this domain is now multiple connect this domain is now simply connected domain. If we if we assume that this and these 2 are separate if this is another this is one boundary and this is another boundary means; in this case if we have something like this. For clarity I am showing this space but it may not be that this is C 1 and this is C 2 with a scissor we cut it.

Now, this domain is now your a is now simply connected domain and suppose this boundary is C 3, I do not know this is some time ok. So, suppose this boundary is C 3. So, we use different color this boundary is C 3 and this boundary is also C 3 ok. Now for this domain we have 4 boundaries one is C 1 and C 2 and all these boundaries when we all these boundaries are continuous boundary of course. And another important thing is that the line integral the line one property we missed, that the line should be piecewise continuous line so that the integration exists over that line.

So, though it has 2 it has 4 boundaries now C 1, C 2, C 3 and then C 3 and suppose you are going in this direction in an anticlockwise direction. Here also you are going in an anticlockwise direction so this direction if you consider positive. So, the direction is required because when you integrate it you have to you have to you are integrating from now from which point to which point. That is z 1 to z 2 or z 2 to z 1 a to b that because of that you are moving in which direction over that line that direction is important.

So, suppose this is the direction we are moving so on C 1 it is a anticlockwise direction C 2 also it is anticlockwise direction. So, C 1 it is anticlockwise direction and then, but now let us find let us get a function f z, let us get a function f z which is analytic over this region right. Now what happens then we know that integration of this function closed now if we integrate over the entire curve.

Now, if you see without removing your pen on this board without removing in your pen you can you can cover the entire boundary whereas, in this case without removing your pen you cannot cover the entire all the boundaries. And in this case you can do that it is simply connected domain ok. Now if since these boundaries are continuous you can integrate over the entire boundary. Suppose you integrate over the entire boundary. Then we know dz that is equal to 0. So, that is on entire boundary if I integrate that is equal to 0; that is the Cauchy's theorem that is now we discussed like.

Now, now the boundary C has now 4 part one is one is C 1, C 2, C 3 and another one is C 3. Now when you integrate it your direction is important right, suppose we are we are going you are moving in this direction and integrating like this. So, if we move in this direction so this is C 1 and then we move to C 3 in this direction, we move to C 3 in this direction, and then we move to C 2 in this direction and then when you come here we move to C 3 in this direction ok.

So, so this is the direction we are moving. So, C constitute of what? C constitute of your in this case the C becomes that, C 1 which is the anticlockwise direction that is the positive direction for us C 1. Then in C 3 also we are moving anticlockwise direction so this is C 3. Now C 2 we are moving. So, in C 2 the direction we assume here is this one is positive so C 2 becomes negative. And then if this is positive this direction is positive C 3 this direction becomes negative so this is C 3.

So, entire boundary has 4 parts and these 4 parts are C 1 then plus C 3 minus C 2 and then minus C 3. Now if we that if you write that integration here then what we have is; integration the same integration is now over C 1 plus C 3 minus C 2 minus C 3 integration of f z dz ok. And that is again that is equal to 0 that comes directly from this expression. Now C 1 plus C when you write C 1 plus C 3 minus C 2 minus C 3 means it is a closed curve ok.

Now, if this is then you see the see this C 3 this C 3 gets cancelled. And essentially what you have is integration of f of f of z dz over C 1 minus C 2 that is equal to 0. Or you can further write it f of z integration of f of z over C 1 dz that is equal to integration of f of z of dz over C 2. This is very important observation or this is also very important observation ok.

So, we will be using this observation when we actually derive these equations, but as I said we are we have tried to extend the Cauchy's theorem for multiply connected domain and get some information out of it some properties out of it. So, that that property can be used in somewhere in while solving while formulating the problems in elasticity. Now so we stop here today next class what we do is we will we will we will discuss a very important concept called. Ok before that another important another important extension.

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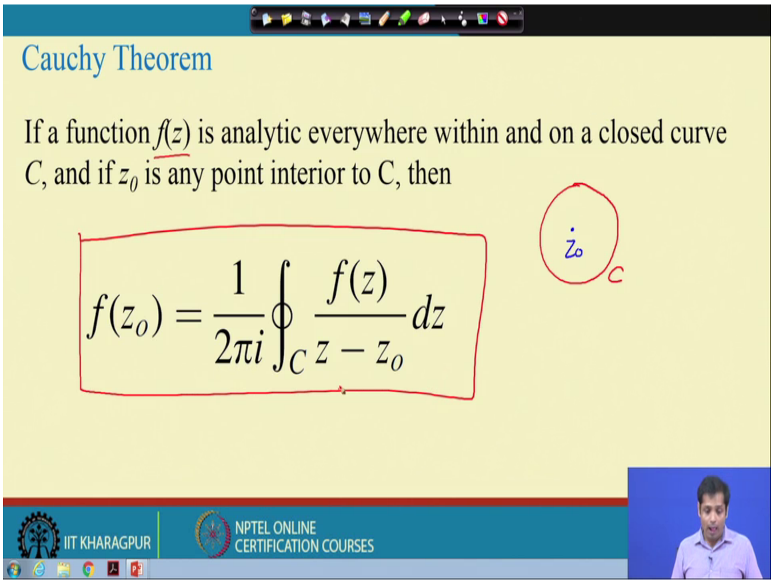

These extension is suppose if you have we are not going to prove this that you can get proved in any con any book on complex variable. If a function f z is a analytic everywhere within and on a closed curve. If you have a closed curve and within every point in this closed curve and on that curve with if any function f z is analytic. And suppose f z 0 is any point f and z 0 is any point in the interior of C this is C. This curve is C and interior of this C there is a point z 0 then we can obtain we can write f z 0 as this.

This also we will be using while x while during while we express any function in terms of their at a at a given point.

So, this we are not going to prove, but as I said prove is available in the book please go through them ok. So, we stop here today next class what we do is next class we discuss an important concept called conformal mapping. If you recall in the week when we discussed plane states and plane strain problem, there we solve some problems like we have an infinite plate with a circular hole and then plate is subjected to some in plane loading right.

And how we try to we solve that if we try to find out the solution using polar coordinate system. Now if it if the plate is circular if the hole is circular plate then yes it is very straightforward, you can see that at R is equal to something. And R is equal to R is equal to the radius of the hole your stresses are defined. So, all that inter curve can be defined very easily.

But then if it is not a circle, if it is a different shape, was an ellipse or a other shape then that applying that boundary conditions or the writing the equations may not be that straightforward. There we have a concept called mapping. Mapping is then what we do is we try to try to project these all these if you have a different arbitrary shape. For instance, an ellipse; then we try to project it on a circle and when you project it we define a map, a transformation between a circle and a ellipse. Then we know how to apply equations for a circle and then we use the same concept along with the projection, we will try to we will write the equations for an ellipse the governing equations for an ellipse ok. It was it has been too much verbose.

But next week when we start confirm the mapping, we will start with this example we will set the motivation and then see what is conformal mapping. With this thank you, but before I leave as I said because of the time constraint we cannot solve many problems numerical problems in this in the lecture, but you must refer to book on complex variable or book on elasticity. There are many exercise problems or even many example problems are given. All the detail proof of all this theorem that we discussed today they are given. And their extensions and their physical interpretation every discussion is given there. You please go through them and make yourself comfortable with all this concept ok. With this I stop today see you in the next class.

Thank you.