**Theory of Elasticity**
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**Lecture - 43**
**Complex Variable Method**

Hello everyone, we are going to start today the ninth week of this course and the topic that we will cover in this week is Complex Variable Method.

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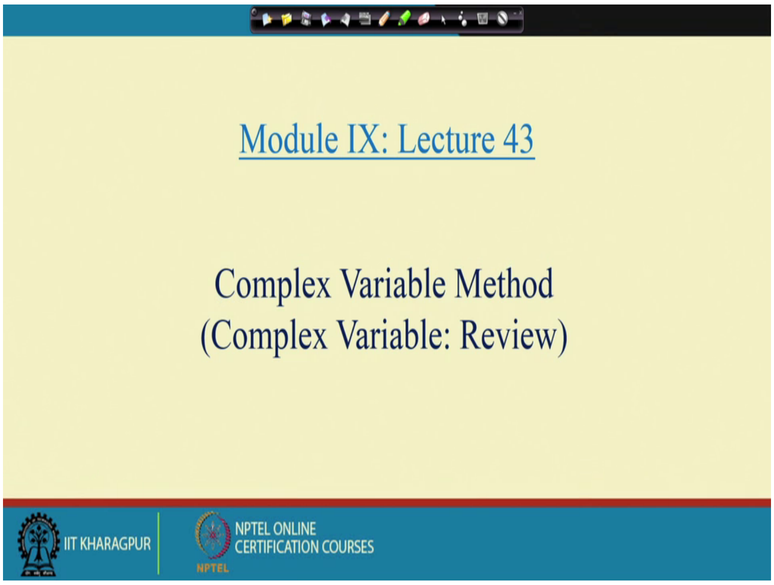


You see, we have derived the governing equations; then we have we have we have seen over the last three weeks that how that govern equations can be solved for different problems in elasticity. And the methods that we used mostly the power series method either in Cartesian coordinate system or polar coordinate systems ok. Now, there are some methods, there are some problems in elasticity, we will see their application of those methods that we have studied so, far is slightly tedious.

And therefore, you need to look beyond the approach the governing equations remain same, the equilibrium equations and then compatibility equation, also the using the potential how that equations can be converted into a single equation? That concept will remain same.

But the methods through which we solve those equations, we have to look beyond some alternative method. So, that there are large class of problems can be solved and here we have a method which is based on complex variable. Now, in this week we will learn how to formulate those equations, we have done in we have we have derived this equation in Cartesian coordinate system, polar coordinate system.

Now here we learn how to derive those equations or rewrite those equations in terms of complex variable and then what are the how to solve those derived equations? And we will we will we will realize that, the many complicated problems with complicated geometry will be easier to solve using complex variable method, which was otherwise difficult using the approach that we have studied so, far.

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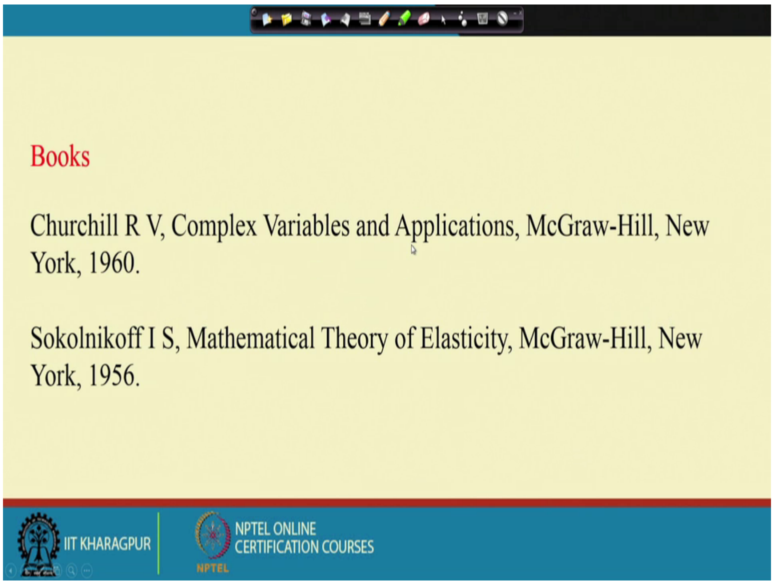


Now, being the first lecture of this week, today before that, there are two books is very important; one is the complex variable is details of the complex variable is given in book by Churchill which is a very good book on complex variable. But the applications of those applications of those concept in the different problems in elasticity is given in some of the chapters in this book. The second one; the mathematical theory of elasticity ok.

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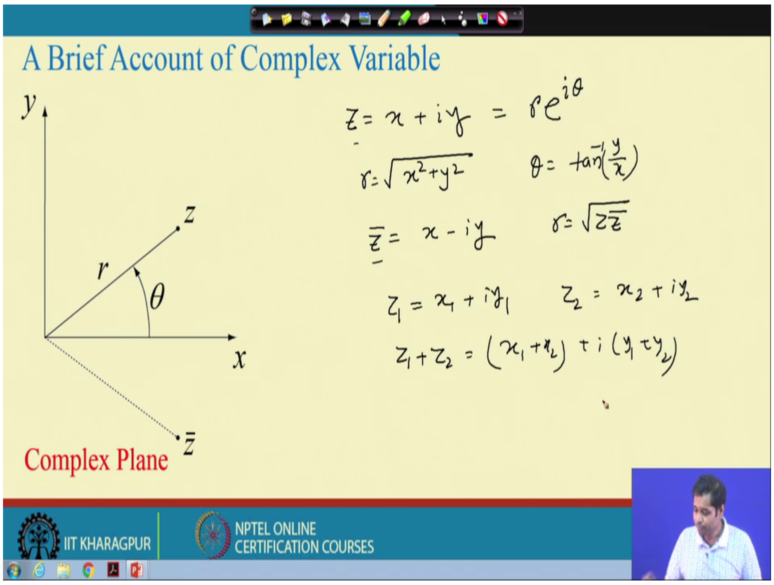


So, being the first lecture of this week we will spend some time today to just revisit some of the important concept in complex variable. You have already studied complex variable in your engineering mathematics also in school also in school maths. We will just revisit, we will just briefly review those concepts today ok.

So, so, if we have so, we define a complex variable a complex plane, which is if you have an x and y, then this is the complex plane. And we know that the complex variable z is written as is equal to x plus iy right, where i is the x is the real part and the i; i is the imaginary part.

Now, it can be also written in terms of polar coordinate system, using r and theta coordinate and this becomes r r e to the power i theta right. Where r is essentially, if you can write x square plus y square which is called modulus of z and then theta, is equal to tan inverse y by x y by x which is called argument of z we know that right.

Then we have the complex conjugate z bar can be written as x minus iy, which is z and z bar are the conjugate each other. And then r is related to r can also be written at z, z bar so, these are the things that we know ok.

Now, then some operations in complex variable, some algebraic operations between different complex variable, for instance, if you have a one complex variable, z1 x1 plus i1 i1 plus z 2 is equal to x 2 plus iy 2. Then z1 plus z 2 can be written as the addition can be written as the x1 plus x 2 and then plus i into y1 plus y 2 the real part, they can add be added together separately and the imaginary part becomes separate. So, then so, this becomes essentially, if we have all these algebraic operations, you see, different algebraic operations we already know this ok.

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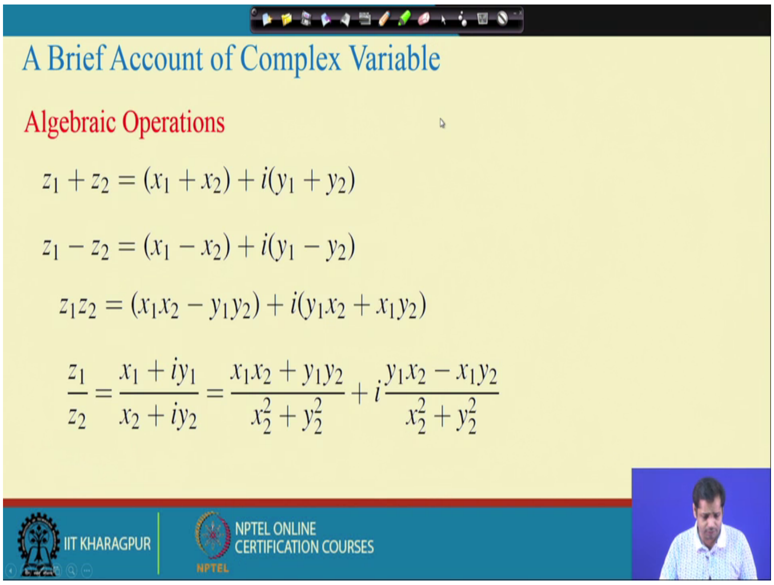


Now, once we know that let us now define how to write a function represent a function in terms of complex variable.

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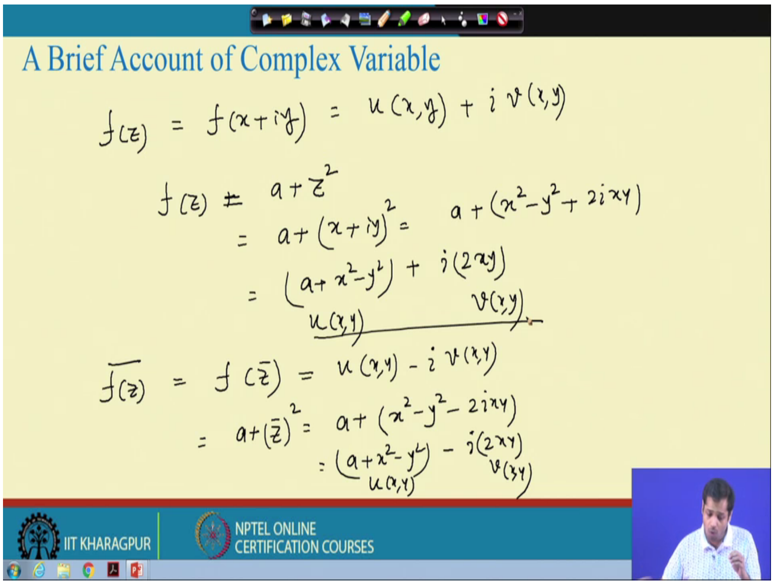


So, suppose z is a function take a function any function suppose f f z is a function of complex variable fz and that can be this is essentially is f of x plus iy and this can be written as u of xy plus i into v of x y. So, u and v are the functions, u is called a real part and the vi corresponding to the imaginary part ok.

So, for instance,; if we take a function say fz is equal to take any function say is equal to say a plus z square, a plus z square then it becomes a plus x plus iy square. And then which is essentially becomes a plus x square minus y square plus 2 i x y so, this becomes a plus x square minus y square and then plus i into 2 xy.

So, this is u x y and this is v x y so, any function if we have that function can be written in terms of two separate functions u and v. We will shortly see, what is the relation of u and v, what are the conditions that u and v satisfy? For a function to be differentiable we will see shortly those things.

Now, similar to the conjugate similar to z bar we can also define the conjugate of complex function and that is equal to f of say the conjugate of complex function. And that is essentially, we can write is function of z bar and if we in terms of the same variable we can write it u of x and y minus i into v of x and y.

For instance for instance, if we have to if we take the same function az, then this becomes what this becomes a plus z bar square which is equal to a plus x square minus y square, then minus 2 ixy. So, this is a plus x square minus y square and then minus i into 2 xy so, this is u of xy and this is v of xy. So, conjugate can also be written like the z z itself, the variable itself ok.

Now, this is now once so, we have now next is let us see what is the now you see, all these equations that we derived that is derived in terms of x and y right or in polar coordinate in terms of r and theta. Now here, we have to derive the equations we have to rewrite the equation in terms of z and z bar.

So, what we need is? We need a relation we would not derive the irrespective of the coordinate system you choose, irrespective of the reference frame you choose the governing equations remain same. Equilibrium equations, compatibility equation, the essence of governing equations remain same, but only difference is the difference is the how you represent those equations right?

Now, we represented those equations in last few weeks we have seen in terms of xy and r theta. Now we need to represent those equation in terms of z and z bar so, what we need is? We need we needs relation between xy and z and z bar, let us find out that relation. So, what is that relation? You see if you take any function f any function f any function f is essentially so, take any function which is f of z ok, which can also be written as f of x plus iy right.

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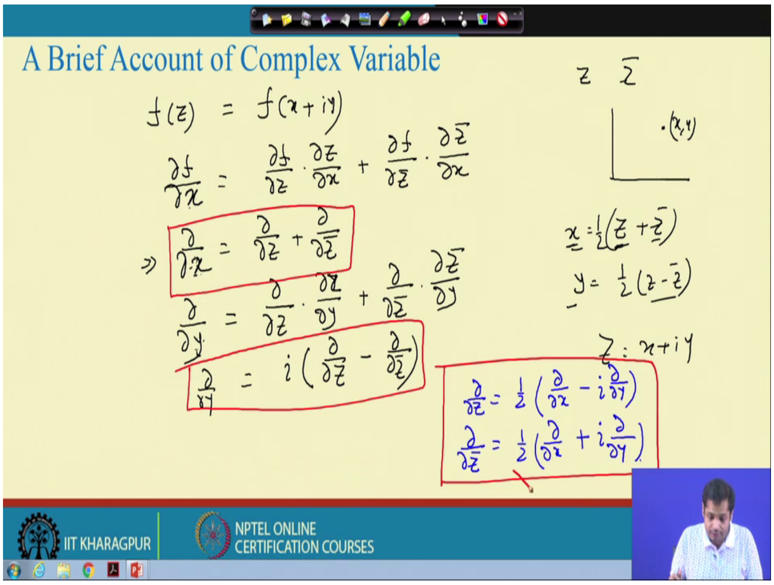


Now, if we have to differentiate this function with respect to z, then what we have? Del f del z now the same thing if we write here, then this becomes del f del z into del z del x just the chain rule. And then the del f del z bar into del z bar del x, please remember one thing here you may say that z and z bar z and z bar is if we know z then we know z bar.

But here, when we write these equations like this; we are treating z and z bar for instance, if we if we take a plane right. Now, in a plane we have two independent vector, two independent basis vector right, if we take a point in a plane. Then that poin[t]- in order to represent that point we need two basis right in a plane in three dimensional space we need three basis vector.

Now in Cartesian coordinate system those basis vectors are along the x direction and then y direction x and y. And if you take say polar coordinate system the two basis are two variables through which we can represent a point location of a given point that is r n theta right.

Now, in rn theta the independent variable, independent the basis need to be independent linearly independent, then only it can be used as a basis right. Now similarly, x and y are independent, now here we are using z and z bar as basis to represent a given point, location of a given point or represent something.

Now, you may say so, zz and z bar has to be they have to be independent, you may say that z and z bar this is not independent, because if we know z you know z bar, but the properties required two basis vectors to be independent, we can prove that z and z bar satisfy those two those properties.

And in a sense we can take z and z bar independent two basis. And therefore, if we take a any point say if we want to take if we if we know z and z bar. So, x coordinate of any point say x and y x and y so, x can be represented at z plus z bar half of this and similarly y can be represented at an half of z minus z bar.

So, essentially, it is it we are just transforming the basis, instead of x and y we are treating z and z bar has our basis ok. And that is the reason we took z and z bar separately ok. Now so, therefore, so, del z becomes what del z z becomes del z del x if you if we see that, del z del x become z is equal to x plus iy del z del x becomes one and similarly del z bar del x becomes one. So, this become del, del z plus del z bar del del z bar del del z bar.

So, similarly, what you can do is? You can you can you can write del del y may del del z bar you can write. So, this become you can also write del del z bar and if you do that then, this becomes the same way this becomes ok, there is there is there is this is this is a small thing it is not z, we are differentiating with respect to x here.

Ah Therefore, it has to be x ok, it has to be x because that is it has to be x right, it has to be x, that is why it is x it is x similarly it has to be x. Similarly, if we take del del y, del del y become del del del z into del y del z del y and then plus del del z bar into del z bar del y ok.

Now, if we do that and then what we have is finally, we have this becomes, this becomes i and this become del del z and then minus del del z bar. So, this is the relation that we have right now so, del del x is this and del del y is this ok, del del y is equal to this.

Now, if we solve this two then, we have a relation we can have we can we can show that del del z is equal to you can try this, del del z is equal to half of del del x we will be using this expression i into del del y. And then del del z bar , but this equation is important because, this equation we will use to prove one or two introduced rather one very important theory in complex variable, now this is the expression that we have.

So, this expression what this expression tells us this expression tells us how z; z bar is related to x and y. So, using this relation, we have a really we can actually we will be using these equations later to transfer this expression form to find out first the relation between u and v the functions for real part and imaginary part.

And then subsequently we also will be using this equation to prove some of the important concepts, some of the introduce some of the important concept ok.

So, once we have this; the relation between del and del z bar, now let us see what is the derivative of any function so, what we have done in so far.

We have we know what is z bar, complex variable; you know what is the complex functions, what is the conjugate of complex functions? Then we also know what is the relation between this differential operator with respect to z z bar and the operator with respect to x and y? Now let us let us let us find out what is the derivative of any function in complex plane?

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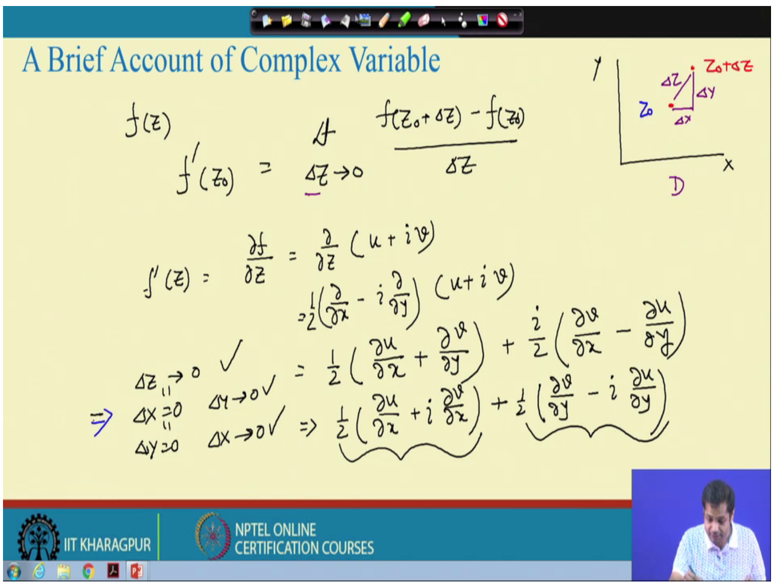

Now, suppose fz is a function ok, fz is a differentiable a single valued continuous function ok. Now then, the derivative of that function at z 0, say f dash z. We know that you use the very first definition of derivative that we learnt, that is f z 0 plus delta z, minus f of z 0 divided by del 0, this is the derivative right.

So, any function if we take, if any function in complex plane, suppose this is the complex plane x and y and this is any point say that point is z 0, this point is z 0 z 0. So, we take another point which is z 0 plus, z 0 plus delta z and then this delta z needs to be delta z tends to 0 ok.

Now, then what we have is this is delta y and this is delta x right delta z ok. Now so, if f z is a single valued and continuous function, then we say that this at point suppose this domain is represented by D ok.

Now, then this is the derivative of f at any point z 0 in D; it exists ok, then only we can we can say that it is differentiable at at point z 0. If it exists; not only that if it is independent of the fact that how delta z tends to 0? What it gives you? The delta z has to be this delta z tends to 0, but how these delta z approaches 0, tends to 0, independent of that fact this definition this f dash z 0 you should exist ok.

Now, what it means? Now let us let us let us write that expression for instance for instance, if you if you recall our del f f dash z f dash z is equal to what? It is del f del z right. Now f is equal to so, this is del f del z, f can be written as u plus iv that is what we have seen. So, this can be written as del del z of u u plus i v ok.

Now, what is what was the expression for del del z? If we if we recall the expression of del del z was del del x minus i into del del x minus i into del del y right. So, if we substitute that, what we have del del x minus i into del del y there is a half here and then we have u plus i v right.

Now if we just write this expression and take the real and imaginary part separately, then what we have is u del u del x plus del v del y and then plus i by 2 of del v del x minus del u del u del y right.

Now, what is now important here is? You see now what is you see here, your this is the expression of x, expression of fz ok. Now when I say that this exists irrespective of the fact that how delta z tends to 0, it means that, this equation should be valid. If f is differentiable and single valued then, this should be valid for delta x is equal to 0 and allow delta z to approach to 0 along delta y line.

And similarly, if we take delta y is equal to 0 and allow delta z to approach 0 along delta x. So, irrespective so, this equation is valid how the combination of delta x and delta y, delta x plus i delta y oppressed to 0, but it may it has to be valid if delta x is equal to 0. But delta y approaches to 0 delta x is equal to 0 and delta y approaches to 0 or delta y is equal to 0 and delta x approaches to 0, in both the cases this equation should valid.

Now, you consider the consider the fact that when delta x is equal to 0 and delta y approaches 0, since essentially this is derivative with respect to y. And when delta y is equal to 0 and delta x approaches to 0, this is essentially derivative with respect to x. It means that in both the cases then, if we take the in both the cases this derivative should be should a should be the same.

It means that, this is important it means that now let us first separate the x, x part and the x part and the y part. Let us first do that so, half of del u del x and then plus i into del v del v del x.

And then, we have we have then we have, this is equal to your plus half of del v del y, del v del y minus i into del u del y ok, yes, that is that is fine.

Now, it means what? When in the first case, you see what I say these delta z has tends to 0. In one case what happens that delta x is equal to 0 and allow delta y tends to 0 and delta y is equal to 0 and allow delta x tends to 0. So, this is derivative with respect to z, this is derivative with respect to y, this is derivative with respect to x. Then what it says that irrespective of the fact how delta z tends to 0, if that exists then this should be equal to this should be equal to this ok.

So, this is equal to, this is equal to this, if it is then what happens you look at this so, in one case, this portion will be the derivative and in the second case and in one first case this portion would be the derivative.

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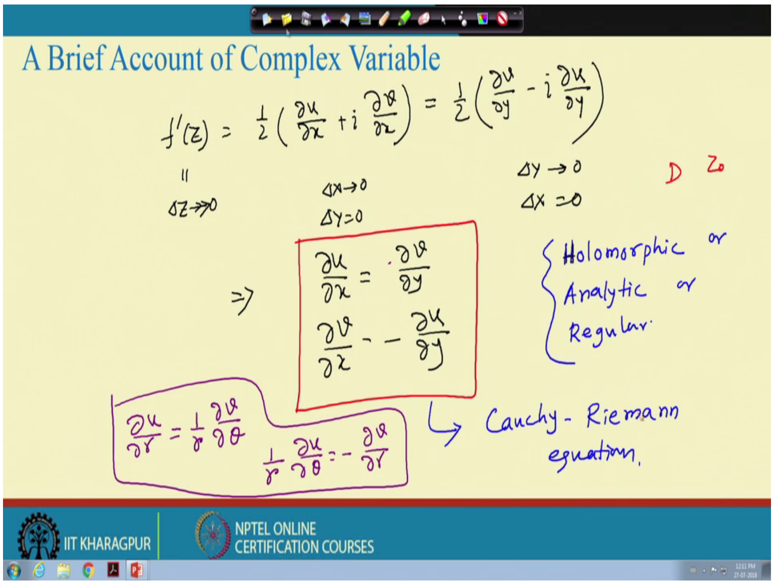


It means then, the f dash z is equal to f dash z 0, that is equal to or at any point. If you take f dash f dash z that is equal to your the first case, the first one is this del u del x plus i into del v del x half of del u del x plus i into del v del x. And that should be equal to half of half of let us see del v del y minus i into del u del y, del v del y minus i into del u del y that should be equal to this ok.

So, this was when delta z tends to 0, this was when delta x tends to 0, this was when delta y tends to 0, delta y is equal to 0, delta x is equal to 0, ok. And that is the condition if that is that exists then, how it approaches to z 0, how delta z 0 is just approaches to 0? Whether along the x line or along the y line irrespective of the fact they should exist and they should they should be same.

And therefore, these all are same, now, if they are same then, if you recall how we compare to imaginary to imaginary variable then, we compare the real part and the imaginary part. The real part becomes if we compare the real part, then del u del x is equal to del v del y and the imaginary part is del v del x is equal to minus del u del y, this is very important this is very important.

Now, so, you see; if a function if f is f if the D is the domain, if the D is the domain, D is the domain and if z 0 is the point in D. Then we have seen then what is the condition of derivative at the that existence of the derivative at z 0.

Now, if it is single valued and differentiable at every point in z, every point in D, then we say means this condition is valid not only at z 0 at every point, in every at every point in D, D is the domain. Then we say, this function is called in that case the function is called hollow I use different color, Holomorphic or analytic or regular or regular.

The point where this condition it is not differentiable then this does not exist, then at that that point is called singular point ok.

Now, if the so, if the therefore, if it is Holomorphic analytic function or regular function, they need this condition exists, this condition is true for every point in set every point in D. And the this we obtain this relation between u and v based on this condition. And therefore, if it is the function is Holomorphic analytic or regular, then this condition valid for every point ok.

Now this is called this condition this is called Cauchy ca equation ok. So, for an analytic function, all Holomorphic function, all regular function this Cauchy Riemann equations is this equations are valid at every point in the domain.

So, if we if a function is given to us and we have to check whether the function is Holomorphic or analytic or regular or not then, what we have to check is? We have to check whether we have to write the function first in terms of u plus i into i v and then check whether u v satisfy this relation or not.

Now, this relation tells you; tells you tells you another important story, let us find out that. So, this relation is so, let us ok, before that if we have to write this expression you can do this exercise in terms of polar coordinate means; the functions are written in terms of r and theta here we have expressed z in terms of x and y.

If you express z in terms of rn theta or it will be i theta, then also you will get some relation between their real part the function in the real part, on the function of the imaginary part. And that relation will be you can derive this that relation will be del u del r the this is the same Cauchy Riemann equations will be del u del r that is equal to 1 by r 1 by r del v del theta, that is the first equation.

And the second one will be 1 by r del u del theta, del u del theta is equal to minus del v del r del v del r. So, this corresponding to the first one, this corresponding to the second one so, this is the same it is Cauchy Riemann equation, but using r and theta expressing z in terms of r and theta. Now, so, let us as I said this equation the Cauchy Riemann equation tells another important story.

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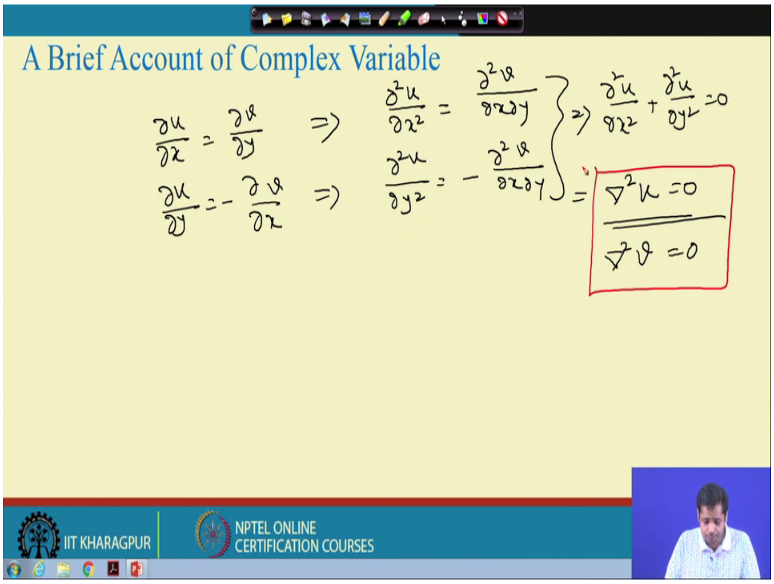


Let us find out that, let us see that that equation is Cauchy Riemann equation, if we write in terms of say Cartesian coordinate system. This equation is del u del x, del u del x is equal to del v del y and then del u del y del u del y is equal to minus del v del x.

Now, from that, if you differentiate this expression with respect to x, then we have del 2 u del x 2 is equal to del 2 v del x del y. Similarly, if you if you differentiate the second equation with respect to with respect to y, then we have del 2 u del y 2 is equal to minus del 2 v del x del y. And then you if we add them, this and this and from this what we get is del 2 u del x 2 plus del 2 u del y 2 that is equal to 0.

And recall and this is essentially del 2 u is equal to 0, this equation is called Laplace equation and similarly you can also show that del 2 v that is also 0, this is another important equation.

So, u and v if it is so, what we have suppose our function complex function, which is written in terms of in terms of u and v with real and v is an imaginary part, If that function is analytic or Holomorphic or regular function. Then even v they satisfy Cauchy

Riemann equations and another interpretation of that Cauchy Riemann equation is that even be both this both satisfy Laplace equation as well, this is the Laplace equation.

Now, if you recall the solution of the Laplace equation, the solution of the Laplace equation is called is harmonic. So, u and v from this since they are the solution of Laplace equation, this satisfies Laplace equation u and v are the harmonic function ok. But, if they are not analytic function, if the there are not Holomorphic at a given point, the singular point; these are not valid ok. For an analytical analytic function and Holomorphic function u and v are harmonic the satisfy Cauchy Riemann equation right.

Now, another important thing is; suppose you know u for a complex function, if you know u; v can be obtained by using the Cauchy Riemann equation. So, u and v are also conjugate of each other so, by knowing one function u or v you can obtain the another by using the Cauchy Riemann equation ok.

Now, so, what we have done so, far is we have just revisited some of the aspects of complex variable. We see what is we have seen what is complex variable, how it is written in terms of x and y and r and theta? Then we have seen some of the how the algebraic operations performed between two complex variable, two complex numbers.

And then also we have seen how to express the complex function. And then also we have we have seen how to transfer the derivatives operators between two definition one is complex definition, another one is Cartesian coordinates in terms of x and y. We have seen how these two operators are related to each other? Then, we have seen what is analytic function or Holomorphic function, what is the condition need to be satisfied a function to be analytic function.

And the condition is the Cauchy Riemann equation the u and v should satisfy Cauchy Riemann equation and the another interpretation of the Cauchy Riemann equation is u and v both are the solution of Laplace equations are both are harmony. And knowing one another can be obtained, that is why even if you are called conjugate function conjugate to each other.

Now, next class what we do next is we see very important part is the integration on integration in complex plane. There we introduce the path independent into integration, you recall in one of lectures we were discussing if the function is not simply connected domain is the domain is not simply connected.

Then what happens we will bring those kind of domain now and then see, how this how this complex variable approach help us to deal with those kind of those kind of domain. So, next class also we will continue with the with the review of complex variable integration in complex plane I stop here today see you in the next class.

Thank you.