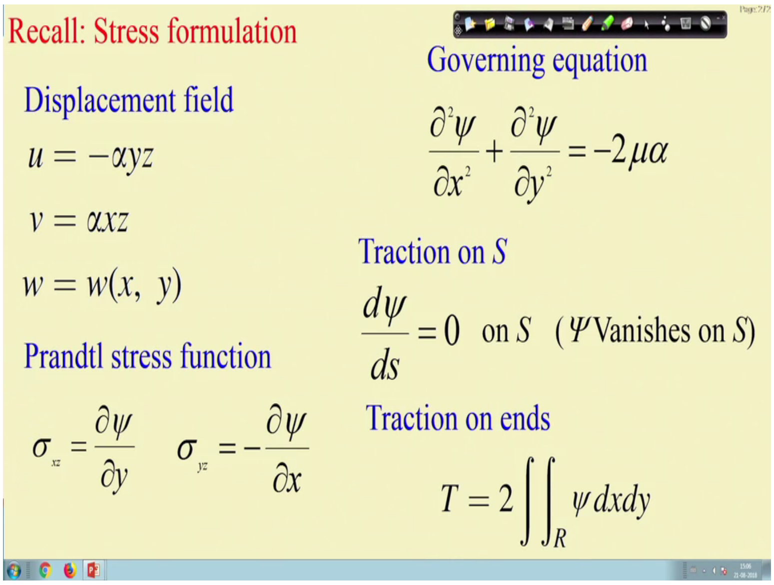
Theory of Elasticity
Prof. Amit Shaw
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 40
Boundary Value Problems in Elasticity (Contd.)

Hello everyone, this is the third lecture of this week. In the first 2 lecture we derived the for governing equations and the boundary conditions, stress based governing equations for torsion problem in elasticity. Today we will just demonstrate those equations and though that formulation through 1 example. Now, the example that will be will be solving today.

(Refer Slide Time: 00:37)

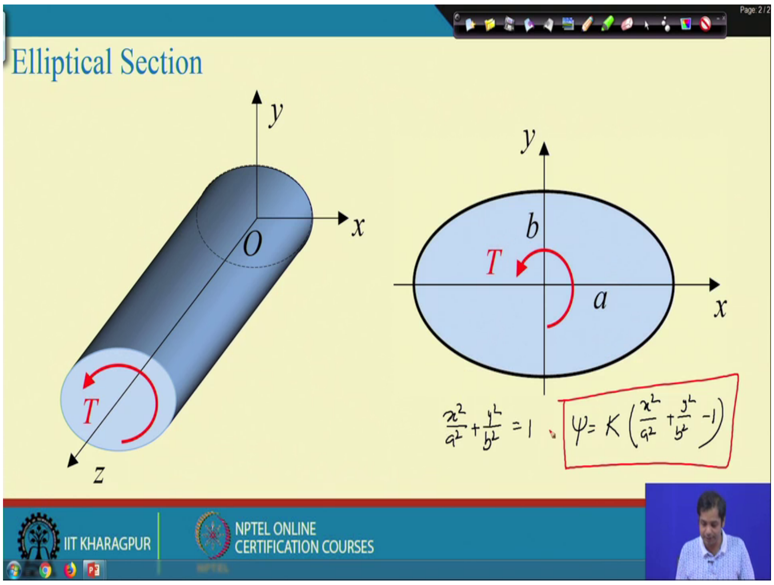


Just before that example, if you recall these are the these are the summary of this is the summary of the entire thing that we discussed in first 2 classes. This is the assumed displacement field, it comes with the though we in previous class we discussed this summary just to keep the continuation. This displacement field we its starting from the assumption that your angle of twist is linearly proportional to the length of the or the length direction.

And this is the stress function, we derive we define stress function as this and then finally, the governing equations for this is a Poisson's equation which is this 1 and this is important the traction boundary condition on S. The direct interpretation of the traction boundary condition on S is del psi del is ds is equal to 0. It tells you that psi has to be constant on the surface and we can take that constant value is 0. So, that psi is vanishing on this surface and this is the traction boundary condition on the ends.

Now, if you take psi vanishes of the surface then other traction boundary conditions the stress free boundary condition that automatically vanishes on these surfaces. Now, let us start with this example.

(Refer Slide Time: 01:53)



This is the example that will find out expression what are the torsion, what is the angle of twist, what is the displacement out of plane displacement all this expression that we find out for this problem.

Now, this problem the reason why we are starting with in the subsequent classes we will be will be will be solving few other examples as well, but the purpose of particularly taking this example as our first example is you see we already know the solution of circular shaft under torsion, we already have the solution from strength of material right.

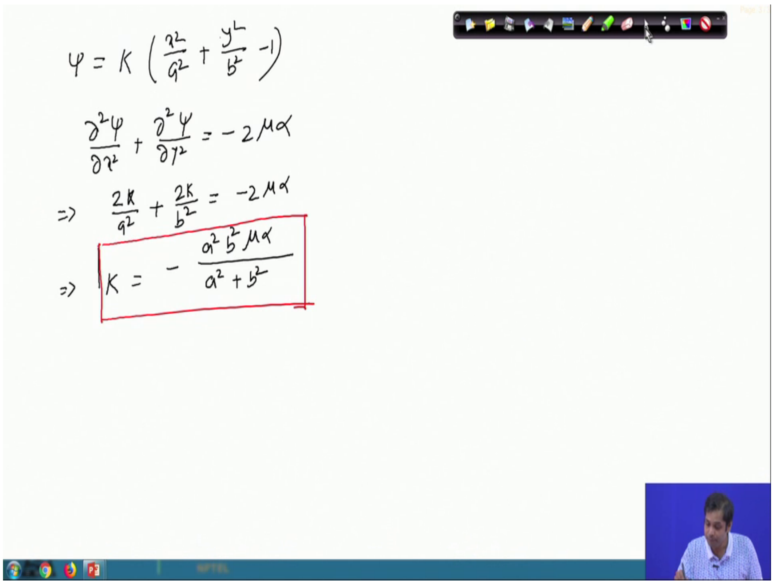
Now, whenever we derive that solution we had certain assumptions and our if you recall our starting point of entire discussion that we have been doing the starting point was some of the assumptions that we had in the, in case of circular shaft some of the assumptions are not valid.

And the plane section remain plane the out there is no out of plane deformation those assumptions and the every section rotates as a rigid body those assumptions are not valid if your section is non circular section. What we do is we will try to find out the, we will find out the solution of these examples and then in the limiting case when a becomes a is equal to b it becomes a circle, we will see that whether we get the solution that we obtained from strength of material.

So, let us start if you recall the first step of any; first step of this is we have to start with an assumption of psi and we take the assumption, we take the expression of psi such that it vanishes at the boundary. Now what is the equation of this equation of this boundary? Equation of this boundary is x square by a square plus y square by b square is equal to 1 right, this is the expression.

Now, takes assume psi as some factor K into x square by a square plus y square by b square minus 1. If you take psi as this then automatically irrespective of the value of K, automatically it vanishes at the boundary. So, is will take psi as this. So, let us do this exercise.

(Refer Slide Time: 04:29)

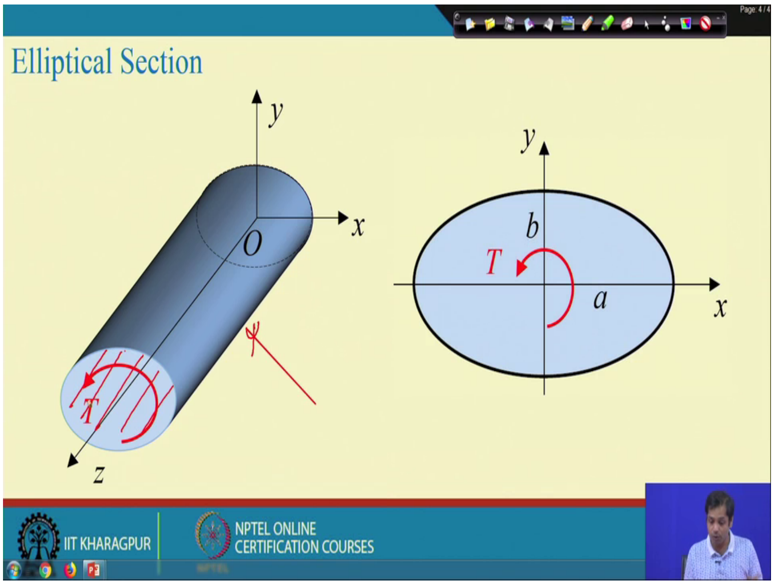


Now, so, we have psi is equal to K, K is a constant that we need to find out is x square by a square plus y square by b square plus y square by b square minus 1. So, this is psi. Now psi has to satisfy the traction stress free boundary condition is automatically satisfied by this psi has to satisfy the governing equation which is the Poison’s equation and what is that poisons equation? If you record del 2 psi del x 2 plus del 2 psi del y 2 that should equal to minus 2 mu alpha when mu is the shear modulus and alpha is the angle of twist per unit length.

Now, if I substitute that the entire expression of psi in these equation what we get is the first one will be will be 2K by a square and the second one will be plus 2 K by b square and then we have minus 2 mu alpha. From this if we find out the expression of K, the expression of K will be minus a square b square mu alpha divided by a square plus b square. This is the expression of K.

So, this expression in the psi when K is equal to this, it satisfy traction free boundary condition is also satisfy the governing equation. Now, another boundary condition is what? Another boundary condition is the he condition at the end where you have the torsion; means, if we take this; if we take this, so, we have already specified, we have already specified boundary condition here, we have already specified boundary condition here by psi by taking psi by taking expression of psi as this.

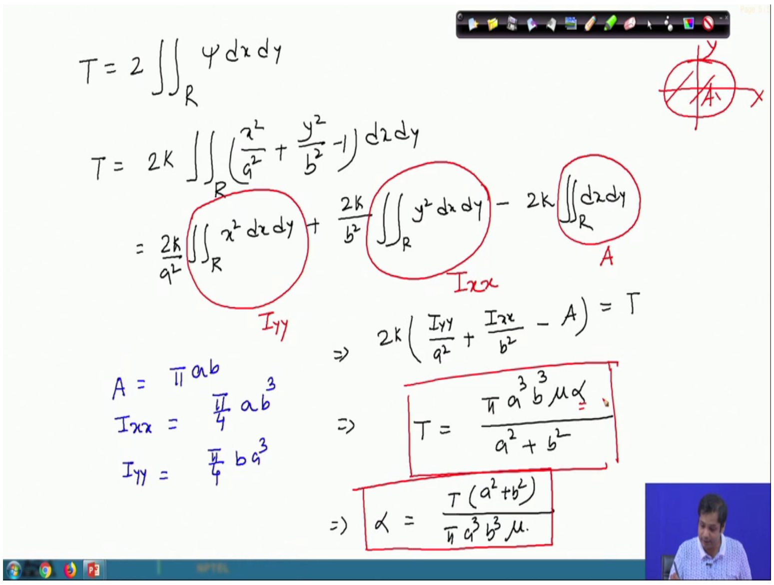
(Refer Slide Time: 06:47)



We already we already had this psi already satisfied the governing equation.

Now, another thing is we have to the psi has to satisfy the governing equation this boundary condition here. What is that condition? This condition tells you the torque at this point is equal to T. So, if you recall the condition was another condition was the torque.

(Refer Slide Time: 07:21)



If we write on that boundary T was equal to 2 into integration over R over the entire cross section into psi dx dy.

Now, psi was this if we substitute this psi in the expression what we have is then T is equal to T is equal to 2K which is which is then integration of R integration over R; then x square by a square plus y square by b square minus 1 then dx dy.

Now, let us write these all these expression; we had three terms in the expression, let us write them separately. What is that? This gives you x square 1 by take a square out. So, this is essentially x square dx dy plus if we take 2 K by b square out then this gives you y square dx dy now then finally, we have 2 K integration of dx dy over R.

Now, look at this term you are already familiar with this term. What is this term? This integration of these term is essentially if this is the ellipse this term is essentially area of that ellipse right. So, this is the area of this ellipse cross sectional area of this ellipse. Now, look at this term. This is if this is the x axis and this is the y axis then these term is essentially second moment of area about x axis right. So, this term is second moment of area about ax Ixx or I double x.

And what is this term? This term is essentially second moment of area about y axis. So, this is Iyy right. Now, for an elliptical section like this we already know these values this area is area is equal to. So, we already know the integration of this value area is equal to pi into ab and then first the second moment of area about x axis that is equal to pi by 4 into ab cube and then Iyy essence gives you pi by 4 into b a cube.

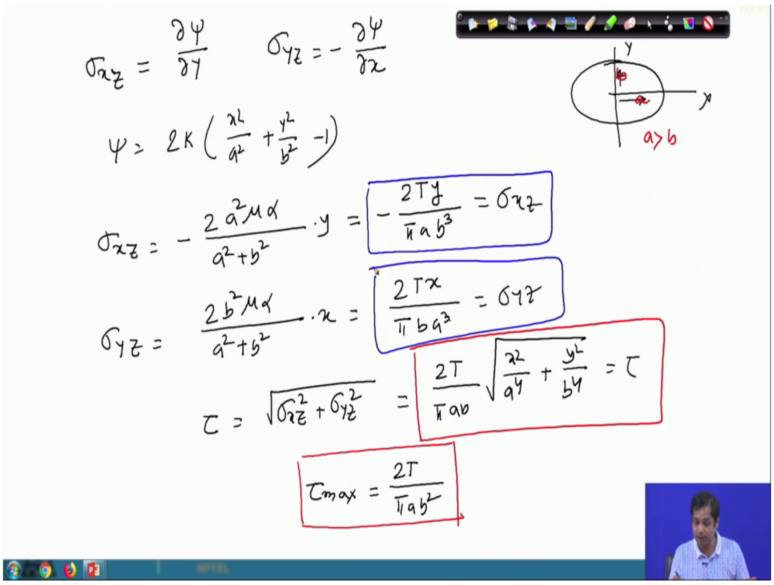
So, essentially these integrations are this and if we substitute this a Ixx and Iyy in this expression these expression becomes finally, these expression becomes 2 K by 2 K into this is Ix this is Ixx by Iyy by a square Iyy by a square and then please this become Ixx by b square and then these become minus this 1 minus area. So, this is equal to T. So, T is equal to this.

Now, if you substitute Ixx Iyy area from this expression and then final expression of T from this becomes T is equal to pi. You can verify this; a cube b cube and then mu into alpha divided by a square plus b square. So, this is the expression of this.

So, from this expression we can also write what is the expression for alpha. Then alpha will be T into a square plus b square divided by the same thing pi a cube b cube into mu this is these expression is also important. So, we have so, we already determine what is the amount of torque we have; what is the relation between torque and the angle of twist, great.

Now, you see the whether it is circular cross section or non circular cross section. One assumption we kept same and that assumption is the angle of twist is angle of twist is linearly proportional to the to the coordinate axis longitudinal axis. Now, once we have the angle of twist let us find out what are the stresses we have. Now, if you recall the expression for stress expression for stress expression for stress is sigma xz as a function of that is how we define the stress function that is del psi del y and sigma yz that is equal to minus del psi del x right.

(Refer Slide Time: 12:27)



Now, psi is equal to again same 2 K x square by a square plus y square by b square minus 1 and if I substitute we already have an expression for K we determine. If we substitute that expression in this substitute expression of psi in this expression in this stress stresses, so, finally, we get xz is equal to I leave it to you this becomes minus 2 a square mu alpha divided by a square plus b square into y.

And then this becomes the other stress component become yz that become 2 into b square mu alpha divided by a square plus b square into x. Now, already we have the expression for expression for torque, this torque and if you substitute these expression in the expression of stresses then these expression can be again further rewritten as minus 2 Ty by pi a b cube and then this 2 T x by pi b a cube. So, this is the expression for stresses.

Now, let us find out what is the these are the stress components; if we have the cross section is this and this stress component this is x direction, this is y direction x and y, the direction of xz it is it is acting on a it is in this direction it is in this direction this and yz will be in this direction. So, it is acting on a plane perpendicular to the normal to the y axis, but the direction of the stress is a direction of the perpendicular to the z axis and the direction of the stress is either in x direction or in y direction.

Then the total stress since, the for both these stresses they are acting on the same area, the resultant stress will be if you say the resultant stress the total resultant stress le let us write tau because that is how we that is the symbol we used in strength of material while calculating shear stresses for a circular shaft.

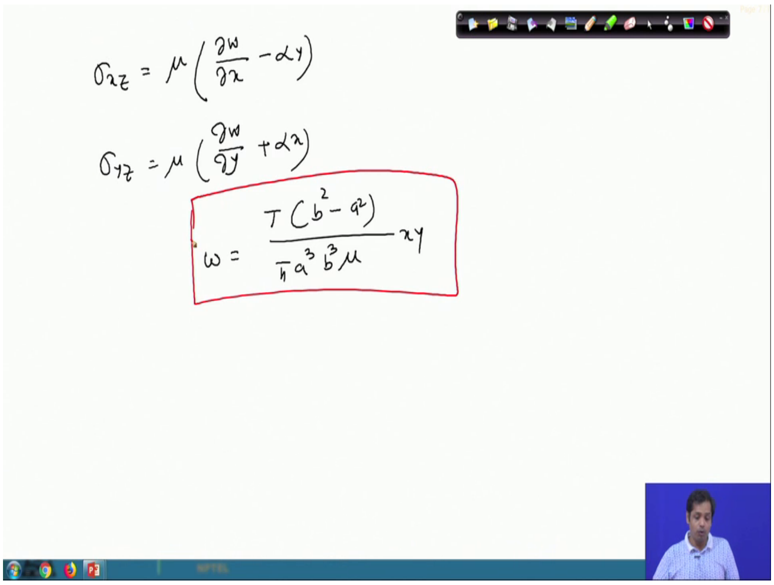
This will be sigma xz square plus sigma yz. Please recall we do not take the component of stresses like this the we can take the component of forces, not the stresses like this, but since both these stresses acting in the same area. If we convert them into force and take the component it will be same right.

So, this is if we substitute this value the final expression will be 2 T by pi ab into x square by a square x square by a to the power 4 plus y square by b to the power 4; this is the expression for tau these expression is also important. So, these expression it tells you how tau is varying over the entire surface. If you take any value of x and y how tau is varying over x and y?

Now, what is the maximum tau? Maximum tau we can this is the expression of tau we can differentiate with coordinate axis and then find out for which value tau is maximum and if you do that for these case since a is equal to a is equal to b a is the major axis b is the minor axis, for these case the tau max the expression for tau max you can determine this expression for tau max will be 2 T by pi a b square. This you can determine by differentiating it with respect to the coordinate axis and then find out and make them 0 and then find out what is the value of x and y for which this tau is maximum. So, this is the expression for tau max.

So, already we have determined the torque, the angle of twist, the expression for tau max what is left is what is the displacement the out of plane displacement. Now the out of plane displacement, let us find out of plane displacement if you recall we had 2 this expression with your sigma z was sigma z is related to displacement.

(Refer Slide Time: 17:12)



Sigma xz that equal to mu into del w del x and then minus alpha into y right and then sigma yz is equal to mu into del w del y plus alpha into x that is the relation we already had right.

Now, then just now we had an expression for sigma x; this is sigma xz and this is sigma yz this expression just now we had these 2 expression right. Now, we can substitute these 2 expression in these equations and then from the first equation you differentiate this the entire thing with respect to x and then get a constant with a function of y and then from the second equation we can get the that constant.

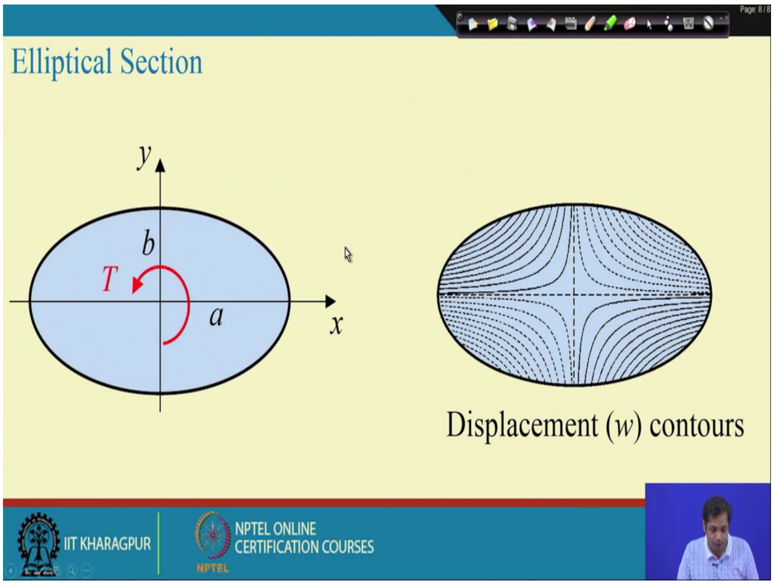
We have done it before whenever we have this kind of function this strain from strain we had to find out displacement we integrated this strain with respect to that coordinate axes and then we had a constant which the function of other coordinate axes.

And then another equation we use to get those constant, we have done it in the previous and if you do that exercise I leave it to you; if you do that exercise the expression for w will be will be T into b square minus a square divided by pi a square a cube b cube these expression is interesting x into y.

Now, this is interesting, this is the expression of w. So, we already have an expression for w. Now you could recall how to get an expression of u and vu it depends on u depends on u is u depends on alpha expression alpha already we know and therefore, if you recall, this is the expression we have. So, alpha for already we have an expression for alpha; if we substitute alpha here we get expression for u, if we substitute alpha you get an expression for v and just now we have obtained the expression for w.

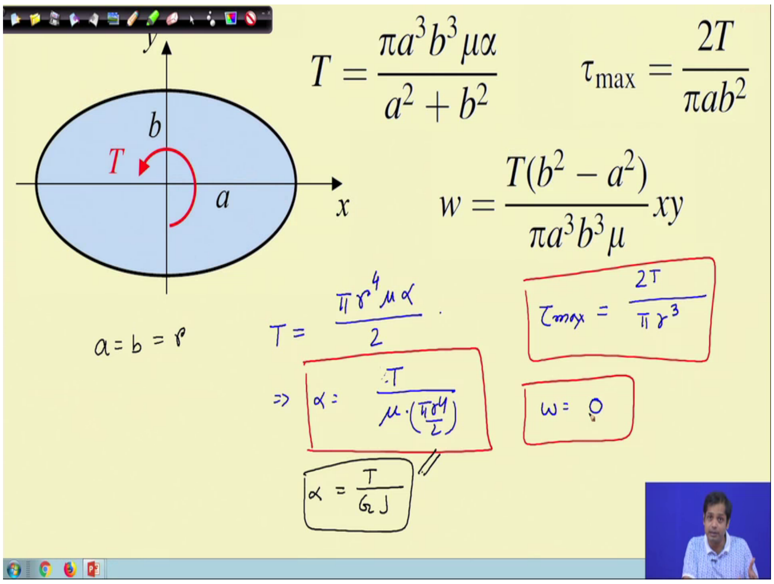
Now, so, let us summarize the solution that we have this is the problem that we discuss and this is the.

(Refer Slide Time: 19:35)



Now if you plot the expression for w that just now we had if you for the contour it looks like this.

(Refer Slide Time: 19:47)



Now, so this is the summary of this equation that we have just now, this is the torque and this is tau max and this is the w.

Now, let us find out from this solution what the solution is for a circular section. Now, if the solution if it is a circular section then what happens for a circular section your a is equal to b its that is equal to R.

Now, substitute a is equal to b in this expression, but before we do that; before we do that recall, let us first do that. If we do that then what would be the expression for T? Expression for T will be, now, then we have to after the substitution we have to compare the solution with the solution that we have in case of strength of material.

Now, T will be you if we do that then T final expression of T will be let us I have this expression with me this will be pi and then this will be R to the power 4 R to the power 4 and then mu into alpha divided by 2.

And then what you have? Then alpha will be this gives you this gives you alpha will be 2 T or let us let us just write T into T divided by mu into then pi R to the power 4 by 2; this is the expression for alpha. Then what will be expression for tau max? Then tau max tau max will be 2 T divided by pi R to the power R cube. And then what is the expression for w? w will be if you substitute a is equal to b here it will be 0.

Now, these three expression is important. Now, you recall what we had in the case of, what expression we had in the case of circular shaft that alpha by alpha which is your which angle of T square unit length that was T by G into J. What is G? G is the shear modulus, mu is the shear modulus. What is J? J is the polar moment of inertia and this is the polar moment of inertia for a circular section is pi R to the power 4. So, these expression and these expression is same.

Similarly, this is the expression that we had for tau max in case of a circular section. And for a circular section w is equal to 0 there is no warp interior takes place there is no out of plane displacement. So, what we have seen here? You see we already had the solution of circular section and when we had the solution that solution was based on certain assumptions and those assumptions are valid for a circular section and we just verified this those assumptions are valid there is w is equal to 0, there is no out of plane displacement if the section becomes a circular section which is very consistent with this with this solution.

Now, if the solution, if the section is not a circular section, here in this case w is not 0 and therefore, we have some out of plane displacement and therefore, that assumption cannot be cannot be applicable and in fact, when we solve when we derive this equation we never had that particular assumption.

So, next what we do is next we will again apply the same thing same formulation to other problems; other problem means, problem with other cross sections we will see if the cross section becomes a triangular cross section, if the cross section is a circular cross section is a rectangular cross section, then what happens to this, how to choose psi and then how to do this exercise for those for those cases?

Then will also see you recall all these examples this example or you they take the example of a triangular section rectangular section all these example your cross section is a simply connected domain. Now, consider a case that you are interested to interested to find out what happens to a pipe and you take a pipe these are all solids are solid shaft, but if you take a pipe and then you apply some apply some torque then what happens? And if you take the cross section that cross section is not simply connected domain.

So, we will also see what happens in those cross section, how to deal with these equations and how to assume psi for those for those domain. So, we will do that in the subsequent week. So, I stop here today. Next class will discuss few more problems with simply connected domain and then we go for the domain which are not simply connected. See you in the next class.

Thank you.