Theory of Elasticity
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Lecture – 04
Introduction to Tensor (Contd.)

Welcome. This is lecture 4 for module 1 where we are discussing Tensors. So, in the last class we basically discussed tensor algebra. So, one of the important part of which is we learned in the last class that tensor inner product of tensor which is basically the;

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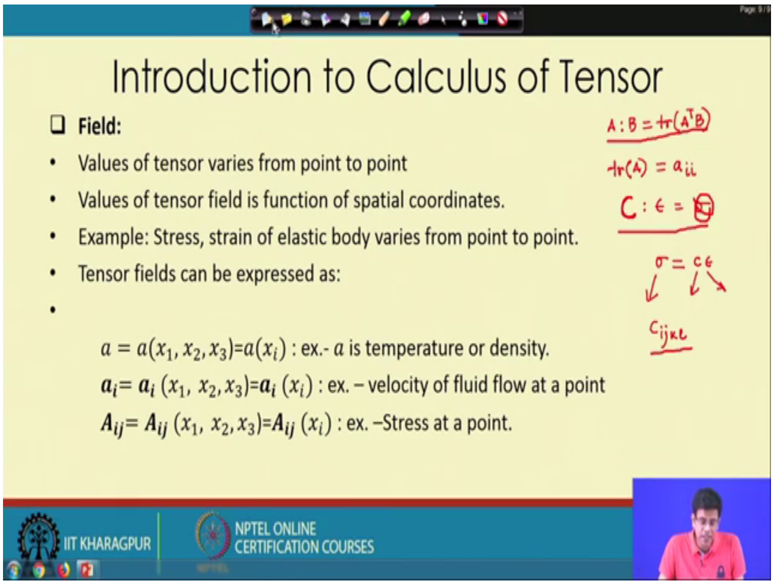

So, just to remember things I just wanted to write it once again. So, A B contradiction B or A inner product B, where A and B both are tensors. So, it is represented as trace of A transpose B. So, trace we know it is the sum of the diagonals of this product. So, A transpose B. So, both of these are second order tensors. So, now this is you see this is contraction. What does this contraction means? This is sometimes we called as contraction. What does this contraction means? Now, see these both of them are second order tensor. So, this trace is a scalar function that we know because if I write trace of A matrix which is or trace of a second order tensor.

So, what is this? This is trace of A a ii. So, ii is a summation implied in it. So a 11 a 22 and a 33. So, you see that inner product of two tensors are essentially a contraction in a sense that tensor is a, this has two directions, but trace does not has any direction. So, stress is a scalarso from a second order tensor, it comes to the scalar. So, similarly if I take the inner product of a fourth order tensor with the second order tensor, so it will give me another second order tensor.

So, this is important and this why I am discussing here. So, if you have seen the Hooke's law, right the Hooke's law all of us have seen in this form that sigma is C epsilon right, but C is a constitutive matrix right and epsilon is a strain vector and sigma is a stress vector, right. So, these are in the void notation, right which we have discussed in the last class. So, these are the vectors, but in here these are the tensor. So, sigma is a second order tensor, epsilon is a second order tensor, but what is C? C is essentially the fourth order tensor. So, this fourth order tensor essentially has four components; for instance if I write it in component form, it has to be C ijkl. So, this we will discuss when we will discuss it in material behavior. So, this is elasticity tensor or sometimes known as the Hooblian Tensor. So, these fourth order tensor is essentially if I take the inner product with a second order tensor which is strain, then it will reduce two times. So, its dimension from fourth order tensor to second order, it will operate on a second order tensor and it will convert this to a another second order tensor.

So, this has to, this probably we have not discussed in this last class. So, this we have to keep in mind. So, again we will see it in detail in when we will do it in the material behavior. Now so in this lecture we will basically introduce the calculus of tensors. So, calculus is basically probably all of you have learned what derivative is, what gradient of a vector field is, what divergence of a vector field is. So, now we will concentrate on what these quantities in a tensor field is. So, I am using the word field, right so let us define what field is.

So, values of tensor varies from point to point. That means, a field can be displacement, a field can be stress, right. So, if it is a displacement as we know displacement is a vector, so we will call it a vector field, right and if it is a tensor for instance stress, stress is a tensor field, second order tensor field. So, the values of these vector or tensors or it could be a scalar field for instance temperature. Temperature could be different at different points so it is a scalar field. So, values of tensor field or values of this field varies from point to point. So, naturally if it is a value and if it varies, it may vary or it may be constant also.

So if it varies from point to point, it will be a function of x y and z, right. So, example stress strain of an elastic body varies from point to point. So, naturally it is a function of x y z. So, tensor field or zeroth order tensor which is a scalar field can be a function of a of x 1 x 3 x 3, right. So, we write it in short form a of x i. So, if it is a vector field, then for instance velocity displacement or any other quantity which is also a function of x 1 x 2 and x 3. Remember here that a i; a as three components here when it is a vector and all three components that is a 1 a 2 and a 3 are all dependent on x 1 x 2 x 3.

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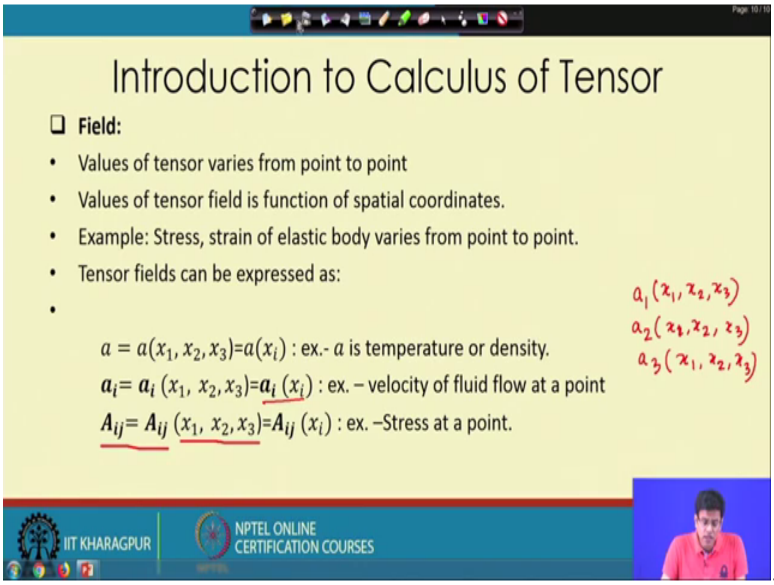

So, if I write it little in a expanded form, so a 1 is or in a component form a 1 is also a function of x 1 x 2 x 3, a 1 is a function of x 2 x 1 x 2 and x 3 and similarly a 3 is also a function of x 1 x 2 x 3, right. So, this is the short form of this and we write it a i x i. That means, a i's are all function of x i's, right. Similar to this as a vector field, we can define tensor field also. The tensor field A ij is essentially all components of the tensor is dependent on special coordinate system or the x 1 x 2 x 3 system. For instance, stress varies from point to point. So, x 1 x 2 x 3 are all a special coordinate system, right. So, once we know the concept of field, a field could be scalar field, a field could be a vector field, a field could be tensor field also.

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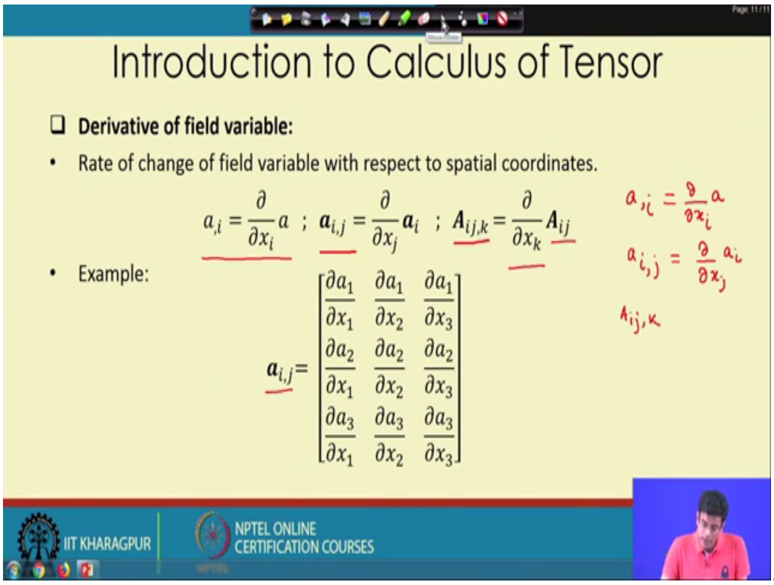

So, now the derivative of the field variable. So, once we know the field, so the field variable comes, right. So, the field variable for instance if I want to take the derivative of a scalar field, so it is dou dou x i of a, right which I write it in short form a, ix. So, this is important because we will be using this. So, when I write comma, that means a, i. That means, I am implying that del del of x i of a, right. So, similarly a i,j; that means I am implying a i,j. That means, I am implying that del of del x j of a ith component. Remember in the previous slide we have seen that a i are all functions of x 1 x 2 x 3. So, ith component of vector a is differentiated with the jth component of the special coordinate system, right. So, these are the special variable, right.

So, similarly if I write it in this, if I expand it, it will form like this kind of 6. So, each component will be differentiated with respect to special system. Now, similarly we can define a tensor. In case of a tensor, we can define since tensor has two directional component. So, when I write a ij, k, this represent my del del of x k a ij. So, aj a ij th component which is also dependent on x 1 x 2 x 3 is differentiated with x k. So, this is important to remember here because we will be using more this shorthand notations, right.

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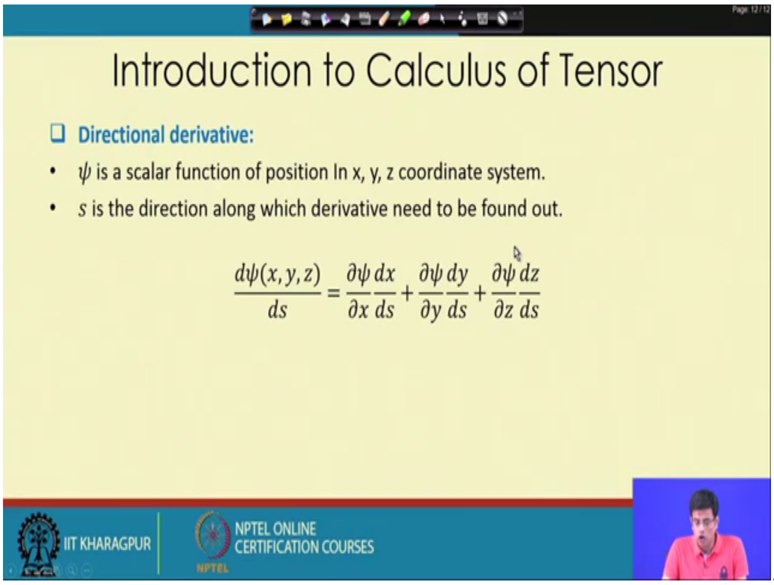

Now, I think all of you know what derivative is, but if I say it say directional derivative, then probably you do not know.

So, let us know what directional derivative is and why we need to do you know more in this course we are introducing tensor. So, today a tensor directional quantities for instance stress, strain, these are the directional quantities. So, when I take derivative, the derivative could be along its direction, right. So, these motivators to learn what directional derivative. Now, before going to a tensor directional derivative, let us understand what if there is a scalar. So, psi is a scalar function of xyz and s is the direction along which derivative mean to find. So, it can be also done, right which is simply del psi del x del x del s and so on. So, parameterization of a curve if you remember, any function which is xyz dependent can be taken derivative in this form, right.

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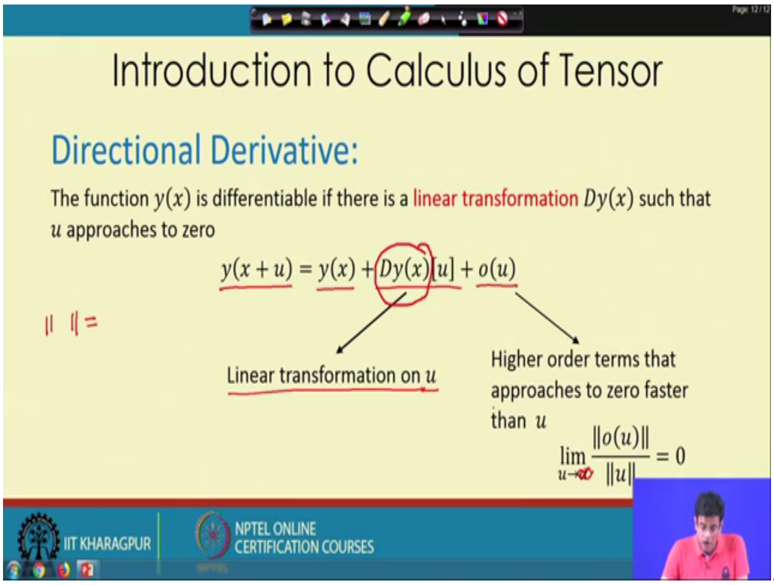

So, it is I think all of you know right. Now, let us learn what directional derivative. So, a function y x here whether I am not saying whether it is a scalar function, whether it is tensor function, whether it is a vector function, I am not defining anything. So, let us write it in a simple form what the directional derivative. The function y x is differentiable if there is a linear transformation D of y of x, such that u approaches to 0. Now, where this incomes from? So, let us expand it y with a x plus u. So, if I write it in this form if you have seen the basic differentiation formula, this is nothing but the basic differentiation formula or you can obtain it from Taylor series formula also. So, this thing this quantity I am talking about is that linear transformation which this D of y of x.

So, it is a linear transformation on u. Now, this is the higher order trans which approaches to 0, right. Now, what does this means which approaches to 0 means that limit so limit of u tends to infinite. So, this term if I divide with the norm of u, so this is essentially the norm this quantity is known as the norm for instance Euclidean norm from a vector. So, it could be a distance, for a tensor it could be a maximum norm, it could be l 2 norm or the distance norm or the Frobenius norm in case of a second order tensor. So, this kind of quantity should go to 0, but let us not bothered about these things right now. So, let us see what this linear transformation means actually so these let us.

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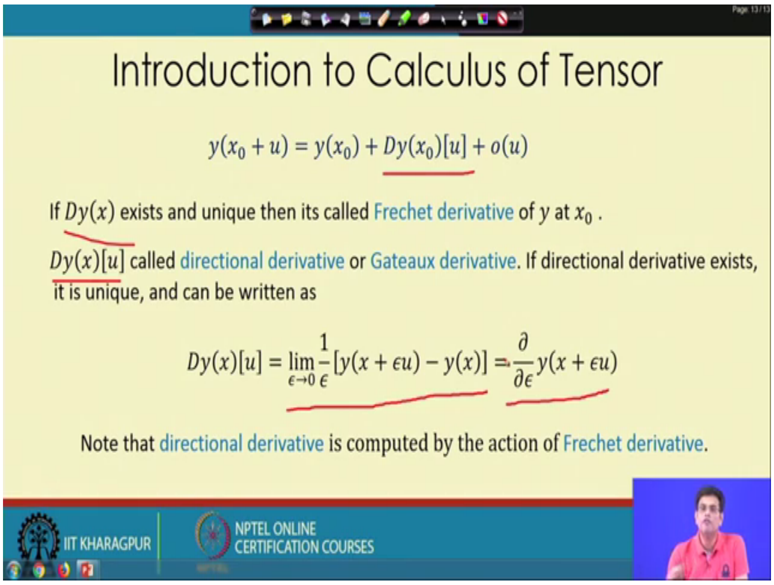

So now if this linear transformation exist and unique, then it is called simply derivative or fresher derivative probably you have heard this. So, a derivative of u y at x 0, right and then, this total quantity this total quantity is known as the directional derivative or the get out derivative where there is a imply direction u is there. So, if you look these two expressions carefully, this is a frechet derivative and this is a directional derivative. So, directional derivative actually operates on the frechet derivative. So, that means the directional derivative is computed by the action of the frechet derivative. So, we will see what this means.

Now, the directional derivative exist and if it is unique, then can be written as the usual derivative formula where epsilon tends to 0, right. So, this is the formal definition of the directional derivative.

Now, let us see some example of how we use this. So, what we have learned? We have learnt two things. One is frechet derivative and one is directional derivative. Now, let us see some example.

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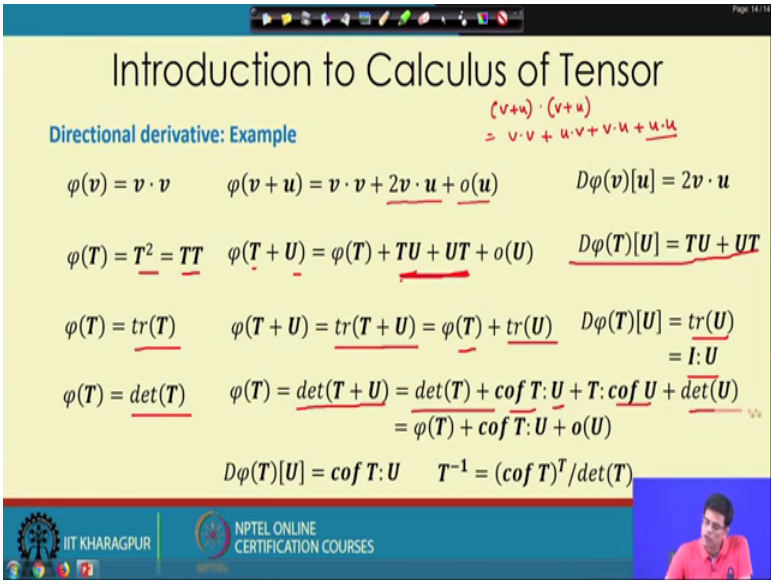

Suppose we want to take the derivative of a function which is v dot v, v is a vector. So, it is a scalar function dot product will lead to a scalar function. So, v dot v. So, what is the directional derivative if we want to compute the directional derivative? So, phi of v plus u is essentially v dot v 2 v dot u plus o u. How it comes? You just write, you just substitute in case of a v, you write v plus u and then, multiply it. So, dot product we can just simply write it like this v plus u dot v plus u which will give me v dot v plus u dot v plus v dot u plus u dot u, right.

Now, this v dot u and u dot v, I can club together because all of these are scalar quantities which I can club together and put like this to read out u an order of u is essentially u dot u. So, now if you compare the directional derivative formula essentially directional derivative is this, right and frechlet derivative is 2 v, right. So, we can write it in component form also, right. Now, if there is instead of a vector if it is a tensor, right tensor argument right phi of T. That means, T is a second order tensor. So, if it is T 2, that means I can write it T into T, right. So, this if I want to find out the directional derivative T plus U if I write and the same way if I write, so T U and U T remember here. So, this is the second order tensor. So, this I cannot write T U equals to U T because as we know we can think of a second order tensor as a matrix. So, matrix multiplication is not commutative in general. So, that means a b not necessarily equals to b a. So, I have to write it in this form T U plus U T. So, if I see in this if I want to find out the directional derivative which is essentially T U plus U T. Now, suppose one of the popular function we are using or we know today is trace of T, now trace is a linear function if you look carefully. So, I think I have also discussed this trace is a linear function. So, trace of T plus U which is essentially trace of T plus trace of U, right.

So now if we see the directional derivative which will be essentially trace of U. So, trace of U I can write that I contraction U or I inner product U because I inner product U is essentially trace of I transpose U. So, I transpose is essentially I and I multiplied by U is essentially U, right. So, I can write it in this form. So, now determinant of T determinant is the third invariant of stress tensor or strain tensor we know this. So, determinant if we want to take the, find out the directional derivative along the direction of U, so this is contrary to the trace determinant is a non-linear function which cannot be just write det of T plus det of U. So, if I expand this determinant of T plus U, then det of T cofactor of T inner product with U and T cofactor a inner product with cofactor of U and then, det of U, right.

So, these two quantities goes into the order of U and then, this could be the; this is the my directional derivative. So, D of phi of T along the direction U see whenever we use tensor arguments, our direction has to be the tensor direction. For instance, in case of a vector we always use a direction as a vector, but here in case of a tensor, it always has to be in the direction of a tensor. Now, those probably who do not know what cofactor of T, it is the adjoint transpose which we have from our matrix algebra knowledge, we can find it. So, the inverse of a tensor I can define it in like this cofactor transpose cofactor of that tensor transpose divided det of T determinant of T. So, in a matrix algebra we have seen this and we will just note it little.

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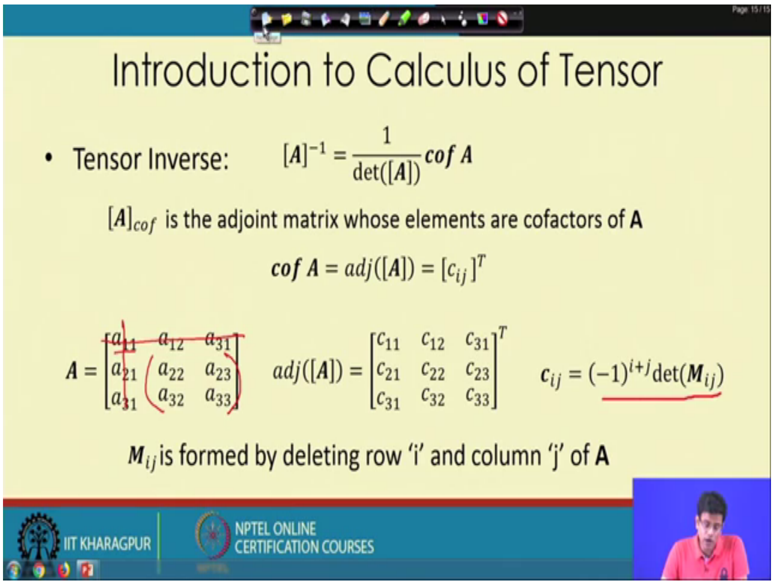

So, this is tensor inverse actually if a is A if i had to take inverse which from the matrix algebra knowledge, we can write cofactor of a divided by det of A. Now A cofactor or the cofactor of A which is essentially cofactor matrix or the adjoint matrix, right so, let us see how this cofactor matrix A is computed. So, if A is a 11 a 12 and so on, then adjoint matrix is C ij. Now what is C ij? C ij is this quantity which is essentially the determinant of the minors. So, M ij is from deleting ith row and jth column for instance M 11. So, M 11 is essentially this, right.

So, this portion determinant of M 11 is a 22 a 33 minus a 23 into a 32. So, c 11 is minus 11 plus 1 that is minus 1 to the power 2 that is 1 which is a 22 minus a 33 minus a 23 into a 32. So, c 11 like this we can find out c 11 and then, other quantities. So, this is a very well known matrix algebra. Now, this is actually is that adjoint matrix. So, this forms the transpose of these actually is the adjoint matrix which is essentially known as the cofactor matrix. So, in order previous case where we actually found out the directional derivative of determinant of tensor T which I have expressed, you can actually verify these things whether the determinant of two summation of two matrix can be written in this form.

So, this is very helpful to find out the directional derivative. Actually one can prove this is the determinant of the total determinant if you sum this matrix and then take the determinant and then, instead of that has to be equal to determinant of the individual matrix cofactor of T inner product of U and then, so on. So, cofactor of T inner product of U essentially trace of cofactor T transpose into U. So, this is essentially a scalar so this is also a scalar, so, this becomes a scalar. So, finally the determinant will be a scalar even though it is a scalar function. These quantities makes them the non-linear function so it is not just simply det of T plus det of u, this one can remember.

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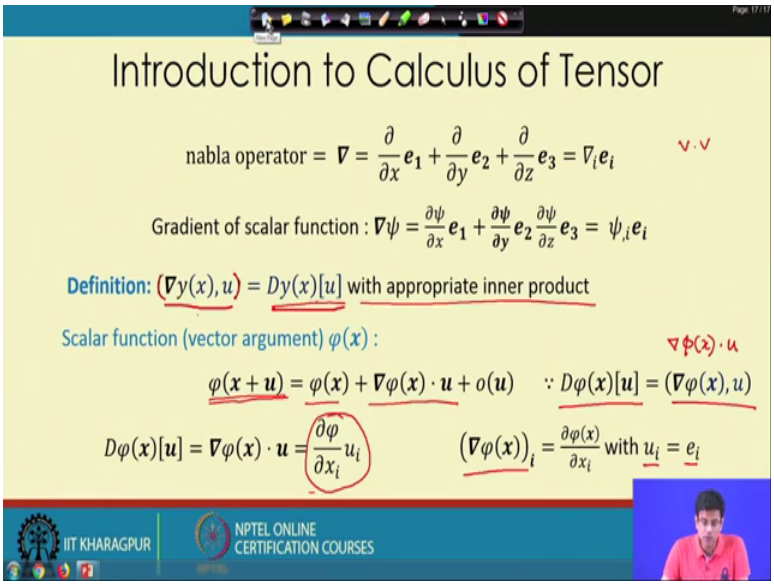

Now another interesting thing probably all of you have seen is nabla operator or del operator.

Del operator is a vector actually so del del x e 1 del del y e 2 del del z e 3 and so on. So, what we write is essentially del i e i. So, in a initial notation gradient of a scalar function, when I write radiant of a scalar function, it is just del of psi. So, del of psi means each component has to be taken, the derivative has to be taken with respect to each component. So, del of psi x del of e 1, these e 1 e 2 and e 3 are the unit vectors along the cartesian component direction.

Now, this quantity is a vector quantity, right. So, scalar even though psi is a scalar function, the gradient of this scalar function is a vector quantity so this has to be remember. Now, another general formula for finding out gradient is essentially the inner product with this inner product with a vector u which is the direction here. So, it is essential if you take the inner product and with the del operator of this function with the u and then, these two quantities is equal actually this is how gradient is defined. So, with the appropriate product, inner product, so why they appropriate product inner product the word appropriate inner product because u could be a vector, u could be a tensor or x could be a vector x could be a tensor.

So, that is why in case of inner product defined on a vector field which is essentially dot product in case of a tensor product which is essentially the inner product of two tensor which is again the scalar. So, when defining the gradient, the proper inner product has to be defined. Now, this could be; those who are interested this could be elaborated much more and study further on these aspects. So, now our objective is to find out the scalar product, scalar function here. Now, if I write from our previous knowledge of directional derivative if phi of x is a scalar function with a vector argument that means for instance the function we have seen v dot v both of these are vectors, but their dot product is scalar. So, such kind of function if I want to write in the sum with another vector u so which is essentially phi of x plus u phi of x and del phi of x and u. So, from our definition, this del phi of x dot u which is essentially I have represented with a braces here essentially to define the inner product, so this del phi of x u is essentially del D phi of u so it is equal to the directional derivative.

Now, this quantity is essentially del phi of x dot of u right. So, if I write this thing and then, if I write it in component form, my directional derivative my gradient I can find out gradient of a vector field very easily which is essentially the component form del of phi x i which is del phi of x i del phi of x with x i.

Now, how can I find out here? This is essentially if I take u i as e i because e i these are unit vectors along the x direction y direction and z direction.

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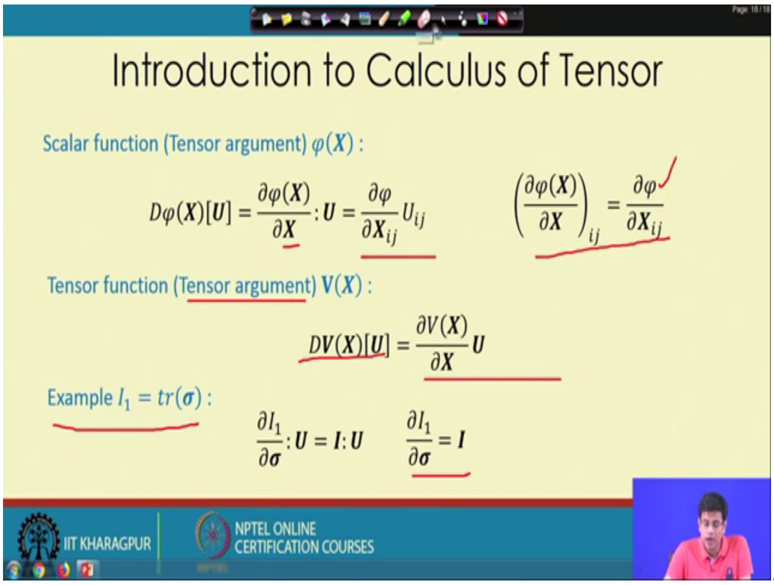

Now, similar to this if we find out scalar function with a tensor argument, the same way we can find out the gradient, but the function is tensor function, but arguments are the scalar arguments are tensor. For instance, trace; trace is a scalar function which is the, but the argument is tensor trace of a second order tensor.

So, if I now want to write the gradient, so I can write it del phi of X ij with del phi of del X see this is the tensor. So, derivative with respect to tensor function, so contraction u. So, if I write it in this form so in the initial form it will look like this. So, gradient of a tensor for scalar function with tensor argument in a component form is this. Essentially see phi is here is a scalar function and this has to be remembered. Now, similarly I can expand it for a tensor function where I can just simply write it in this form with the proper inner product defined. Now, for instance if I see I was talking about the trace. Trace is the first invariant if I want to find out the gradient of i 1 with respect to sigma, then this formula tells me that which is high.

So, this sometimes we will use these kind of treatments for finding out the required derived quantities in this course.

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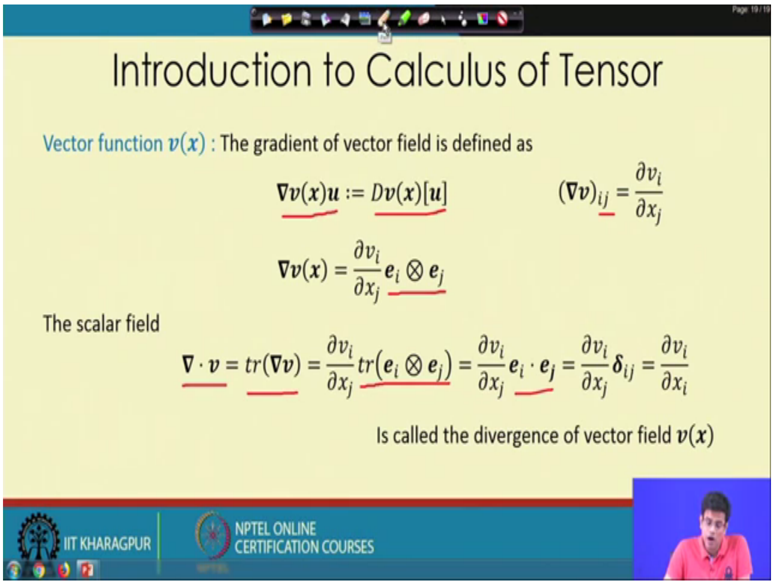

So, now another important thing is a vector function. So, gradient of a vector field even though inner product is defined, this quantity is not a scalar quantity so this is a vector quantity. So, if I write it in the gradient of a vector field in a component form, this will look like this. So, the definition of a gradient of a vector field is this. It is essentially equals to the directional derivative. So, del of v of x is a gradient of vector field which is essentially in a component from del v i del x j e i tensor product e j. Now, remember this is expansion that means this is a matrix.

So, gradient of a vector field is a matrix. So, this has to be remembered in mind. So, gradient actually increases the number of direction. So, one it was a vector field so now it becomes a gradient of that vector field becomes tensor field. So, it has two components ij. Now, another important quantity probably all of us know divergence. How the divergence is defined? So, we know that del dot v is the divergence of v, right which is essentially I can write trace of gradient. So, if I write it in this form or from this component form, so del v i del of x j is essentially trace of e i e j. So, trace of e i e j I can write it is as simply dot product. So, this proof is very nice actually a dot e i e j and then, it becomes a del i j and then, del v i del x j.

So, this is known as the divergence of a vector field. So, now another quantity probably all of us want to know is divergence of a tensor field.

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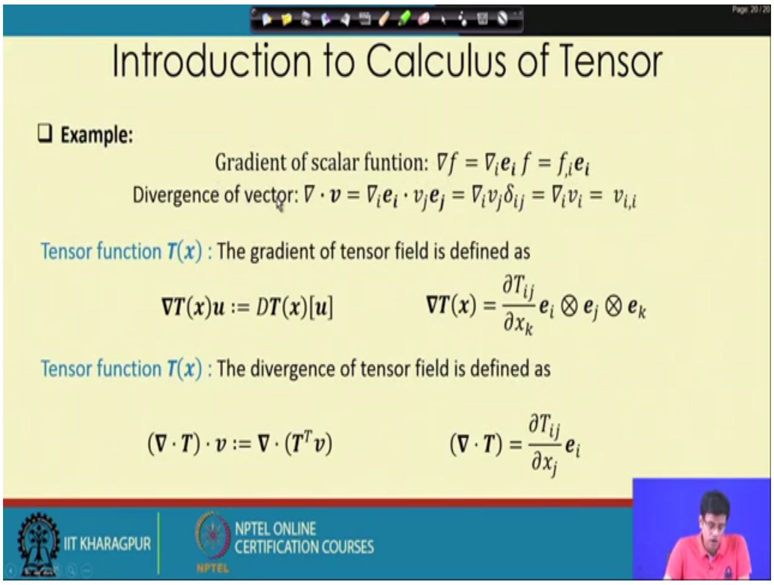

So, divergence of a tensor field is similarly increases another directional level which becomes a third order tensor. So, these are the general formula and actually we can define for any order of tensor so this becomes a third order tensor. Now, similarly divergence, sorry this is a gradient of a tensor field and the divergence of a tensor field also becomes a is defined like this del dot T dot v so del dot T transpose v.

You see if this is a second order tensor, this quantity becomes vector and this quantity becomes dot product. So, now this in a component form, I can write it in this form. Now, the next we will see some example on this.

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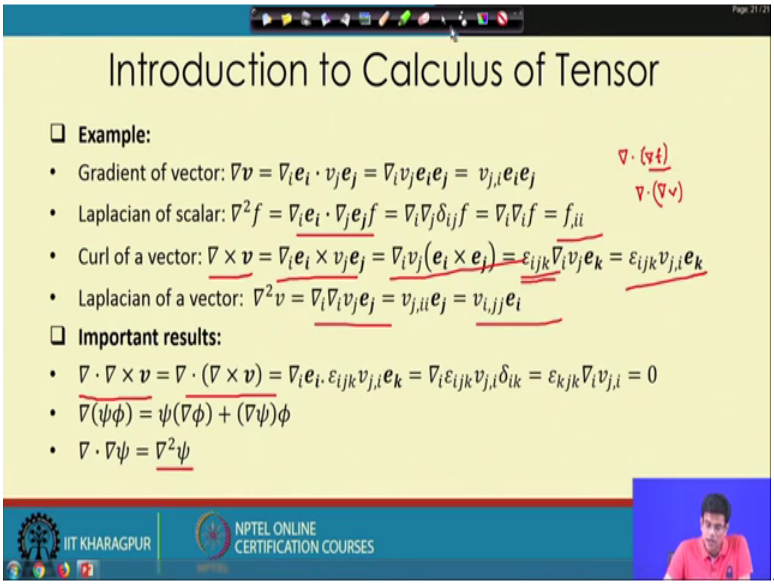

So, gradient of a vector which is simple we know that v i e i dot v j e j and then, v j, i is essentially we have seen this. So, another important thing is Laplacian. Laplacian is essentially del dot del. So, del of v for instance if I can write it in this form that del dot del f, right so in this form I can write the del f becomes the vector and then, again it is a dot product with there so it becomes a scalar, right.

So, now curl of a vector field. So, all these things you know because curl is a cross product. So, if I write it in component form, it will look like this. See we have used here the Christoffel symbols so this becomes this. So, curl of a vector field is a vector. So, Laplacian of a vector field so Laplacian is again I can write it in this form del dot del of u or v whatever it is. So, if I write it in this form so del i then in Laplacian of a vectors field will be a vector. So, these things we know already from our knowledge of vector algebra, vector calculus. Now, there are some important results these also we need to know. So, del dot del cross v which is essentially 0 and del of psi of phi is phi of del of phi and plus del of psi of phi and then, del dot del of psi is essentially del square psi so psi is a scalar function. So, this is essentially the Laplacian.

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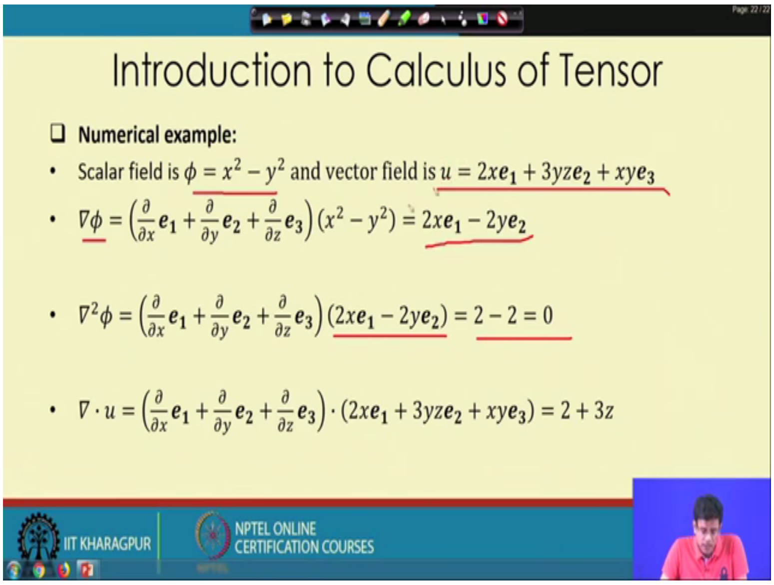

So, now we will go for an example today to see what these quantities looks like. Now, for instance if I define a scalar field of this and vector field of these, then what does this del phi means. So, del phi is essentially this del operator or the nabla operator, then x square minus y square if I take it, if I take inside, it is just the derivative with respect to x and just the derivative with respect to y since it is not a function of z, then this has this quantity is gone 0. So, now, del square phi, that means the Laplacian of that scalar function. So, essentially del phi dot del or del dot del phi so del phi actually we have computed from the previous part and then, if I just take again take it inside, so it becomes this.

So, it is a dot product, ok. So, now divergence of that u vector this is a vector. So, divergence of u again this is a dot product. So, this is essentially a scalar. So, now if you see this all those quantities here at scalar, but it is not necessary that all these quantities will be scalar, right.

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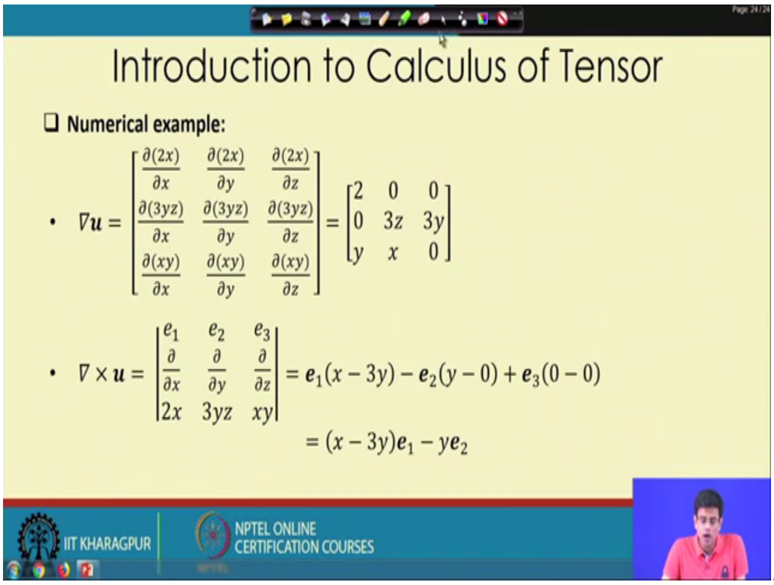

So, for instance the gradient; gradient as I have told that this vector is a, this will increase the dimension of the vector dimension of the field variable. That means, if I do take a gradient of a vector field, it will become a second order tensor. So, if I just take component wise take the derivative, it will look like this, right. So, curl is again is a vector. So, if I take the determinant of this, so the determinant in that cross product we have defined earlier cross product. So, it becomes this so it is a vector. So, this essentially completes our tensor algebra.

In the next class, we will review some of the integral transform integral theorems which will be useful when we will be doing the boundary value problem. So, in the next class we will go for the integral theorems.

Thank you