Theory of Elasticity
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Lecture - 39
Boundary Value Problems in Elasticity (Contd.)

Hello everyone, this is the second class of this week. In the last class we will derive the governing equation in terms of stresses and the another part of the boundary value problem is the this is the boundary condition and today we will see how the boundary conditions to be written and then what is the form of that boundary condition and what information we can extract from that form ok.

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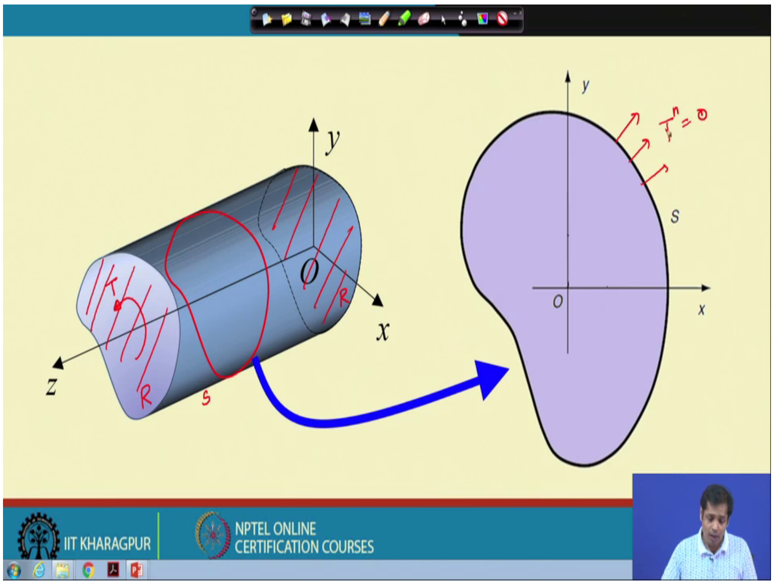

So, if you recall, if this is a problem then what we are interested in? What the problem description is this is subjected to a torsion; suppose a torsion like this at this boundary. Now, if this has when we say boundary conditions means all the boundary we have to specify the conditions. What are how many boundaries we have here?

One boundary we have is the both ends of this means the on the cross section where boundary where the torsion is being applied this is one boundary and another boundary is at z is equal to 0, which is at this point. So, these are two boundaries two ends we have.

Suppose, these ends are denoted as R and then between these two ends we have the surface of this of the solid and suppose the surface is S. So, when we write boundary condition, we have to specify boundary condition both on S and R. Let us start with R, S and then we see what is the boundary condition on R. So, if we take a cross section like this cross section and this is the cross section like and then these surfaces S.

Now, look at this problem or by just looking at the problem we can say the boundary conditions are what is the boundary condition in S? The S is a stress free boundary we do not apply any external load or external pressure on this surface. So, essentially it is a traction free boundary. So, if we have a traction on this boundary on this boundary these traction will be 0 this traction on this boundary will be 0.

Now, what we have to we are going to do now is we are going to we are going to see how this information to be written in terms of mathematical equation. Let us do that.

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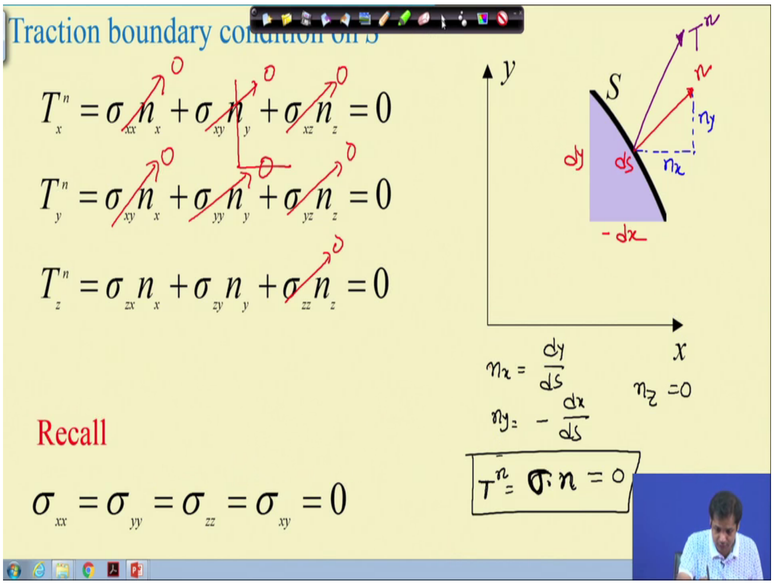

So, take any section for instance if we take a section here take any section. So, this is the corresponding section and suppose this section this distance is d x, this distance is d x and this distance is d y to be precise this distance is minus dx because x is in a other direction to x ok.

So, and suppose this is the unique normal, this is n, which define this plane and then this has two components; one is the horizontal component and one is the vertical component.

So, and this is n y and this is n x. So, n is essentially an unit vector. In 3 dimension, it is it has nx, ny nz, but now since now we are just taking a section, so, it is essentially on a plane. So, you have nx and ny.

Now, so, how this nx and ny is related to each other? We can we can write the relation if this distance is S, if suppose if we have this is S, the entire surface is S and then we can write n x as we can write n x n x is equal to dy by d S and similarly we can write n y is equal to minus dx by ds; minus because this is this distance is minus x it is in other direction.

And then what is n z? N z in this case will be nz in this case will be 0 because we are taking just a plane. So, there is no nz for this. So, this is the in components of components of n right. Now, once we have that if you recall how we define traction traction on a plane? Traction is defined as suppose on this surface if the traction is traction is this is traction suppose T n then T n is defined as T n if you recall that is equal to sigma dot n. So, sigma is a second order tensor, it is a Cauchy strain tensor and then T and n T T traction is a vector ok; this is the this is the traction this is the stress on this or the traction at this point.

Now, since this is a stress free boundary, there is no traction on the boundary. So, there is no external load on this boundary. So, naturally this has to be 0. So, this is the boundary condition we have ok. Now, let us expand this equation let us write traction in terms of their components and then see what this information how we can simplify this expression.

Now, if we do that so, let us you recall this is the expression for expression for traction in terms of stresses and we also recall; now what we have is from this we have n z is equal to 0. So, n z is equal to 0. So, all these terms straightaway we can make them 0 ok.

And then in addition to that what information we have? Recall, in the last class we discussed for the for the chosen displacement field we have these stress components are 0. If these stress components are 0, then straight away we can make them we can make some of the quantity from this expression we can straight away make them 0. Then what; so, sigma xx will be 0 and then sigma yy will be 0, sigma zz will be sigma zz will be 0 and then sigma xy will be 0. So, this will be 0 and then this will be 0.

So, essentially what we are left with? We have then what we are left with? Let us write that. So, what we have is now then? We have traction as only z.

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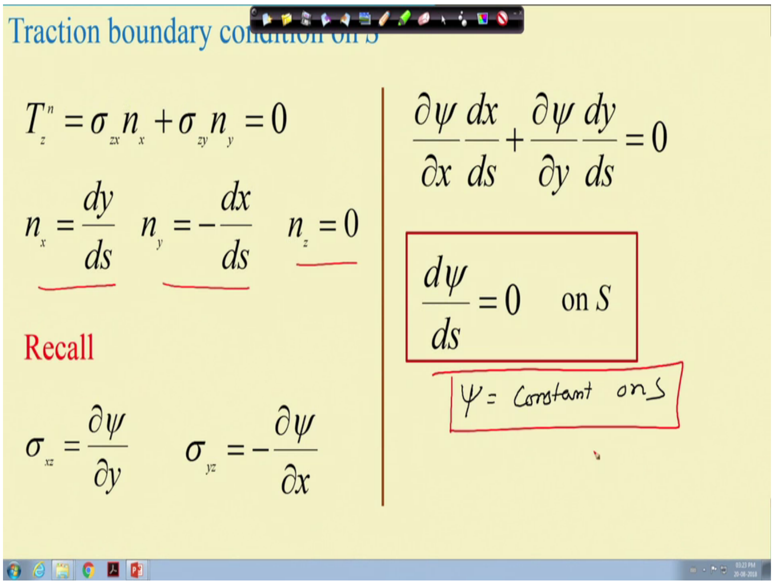

Because all the other all the among three equations three components of traction in x direction, traction in y direction, they become 0 because the corresponding stress components and the and the and the components of an n z is equal to 0 eventually we are left with the thought this is the component of traction.

Now, again we can recall the expression for n x and n y, just now it is just now we derive n x is equal to this, n y is equal to this and nz is equal to this and if we substitute that in these expression then what we get? And before that we recall the stresses are written in terms of these in terms of a stress function chi and this is the relation between the stress function and the and the and this and the corresponding stress components. What we do is now we substitute this information and this information into the first equation and then what we have? We have this expression ok.

Now, this expression is essentially we can simplify this as this. So, this is important. So, what you see, what it says that. So, what we can write if this is equal to 0 then if I have to if I have to integrate it integrate it. So, then this integration will be psi is equal to psi equal to constant on S.

Recall remember psi is not constant everywhere since at the surface in this case it is S is the traction free boundary and when we substitute when we when we impose that boundary condition and when in that boundary condition is written in terms of psi what we get is psi has to be constant on the surface.

Now, constant means, so, it could it we take you can take any values, but for general practice at least for a simply connected domain, we can assume psi is 0; just to give you an example.

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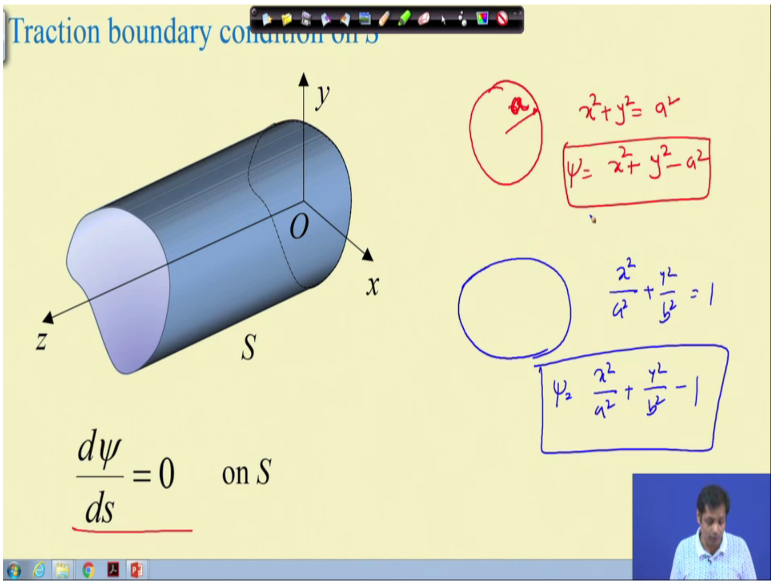

Suppose, we have a domain; for instance, we have a domain suppose a circular domain ok. Take a circular domain of say radius is a radius is r or take a radius is a. Then what is the equation of the circle? Equation of the circle is x square plus y square is equal to a square. Let us assume psi is equal to x square plus y square minus a square. We have to start with an assumption of psi right by very similar to this Airy stress function. We assume some expression of phi, which has certain constant and then those constant study determined through equilibrium equations through the governing equations and also the boundary condition.

Similarly, here also you have to assume some expression for psi. Now if we start if we take this assumption psi, you see that psi is 0 on this boundary. So, psi vanishes on this boundary, which is consistent with this definition of psi.

Now, similarly if we take another if you take an ellipse, suppose the cross section is an ellipse which is defined as x square by a square plus y square by b square is equal to 1. Now, if I take psi is equal to x square by a square plus y square by b square minus 1 and then what happens? Psi vanishes at the boundary and then it is also consistent with this expression. So, when we in the next class we will have we will solve this two problem; one is shaft with circular cross section and the shaft with elliptical cross section there will this point will see what is the advantage of taking psi like this as the outer surface of the cross sectional profile of the of the section ok.

So, this is the condition that to be satisfied on the surface. Now, once we have that let us move on and then next is let us then boundary apply the boundary condition on the two ends. So, if we apply boundary condition on S that boundary condition essentially reduce to a condition and that condition says that del psi del S d psi d S has to be equal to 0 or the psi which can be interpreted in this way, psi is a vanishing function on the surface. So, this is the first boundary condition on the surface. Now, let us write the boundary condition on the two ends ok.

Now, how do we define two ends are essentially your plane and how do we define those planes? This is the normal vector and suppose these normal vector nz is equal to 1.

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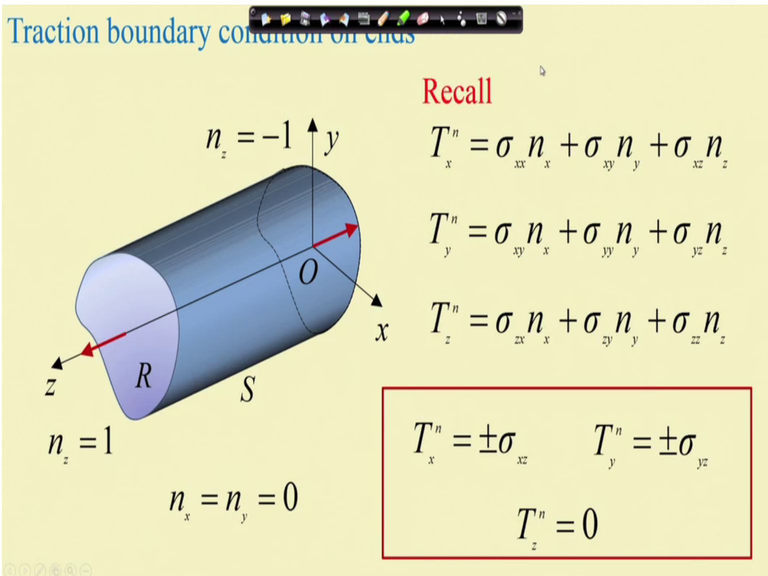

Now, on this plane and on this plane nz is equal to 1 then on this plane it will be n z is equal to minus 1 and nx and ny will be 0 because this is normal to this plane, this is normal to this plane and then we have nx and ny is equal to 0; very just the look at the difference between when we when we apply boundary condition the surfaces nx and ny where nonzero and nz was 0, but now in this case when we apply the boundary condition at both ends we have nx and ny 0 and nz is plus minus 1 ok.

Now, if it is now let us recall now this is the expression we have on the traction. Now, before that before that say before that let us find out what are the boundary we have on this surface.

Now, let us on the surface what we have? We can have a force in this direction.

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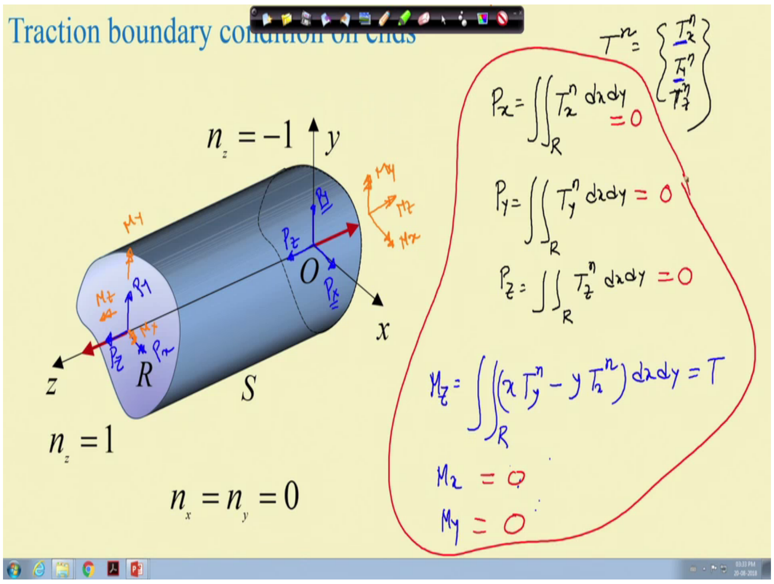

Suppose that is force P z, we can have force in x direction, we have P x and we can have force in y direction which is P y. Similarly, here also we can have force in x direction which is P x and then we have force in y direction we have P y and force in z direction we have P z. So, we can have three forces. I am just writing a most general case.

For a torsion, we do not have actually P x, P y, P z they will be 0, but whether these condition whether it is automatically being satisfied or not on what information we need to impose to satisfy this condition we have to see that. So, these are the very general case. Now, in addition to that what are the things we have? We can have moments also.

So, we can have say moment we can a moment about x axis about x axis means we can have moment this is this is M x; similarly we can have moment about y axis which is M y and we can a moment about z axis which is M z on this surface.

Similarly, on this surface we can have moment about z axis, which is M z and then moment about x axis, which is M moment about y axis x axis which is M x and then moment about y axis which is M y. So, at every boundary at both ends we have three forces and three moments ok.

Now, how do we represent? Say force let us find out if I have to write the force in terms of stresses, if I write to what is the expression for P x? Expression for P x will be that now on the surface suppose T n is the traction T n is the traction and T n has three component; one is T n x then T n y y and T n z. So, essentially T x, T y and z are the components of T n in xyz direction.

Now, if I have to write what is P x then P x will be the, this traction is defined at a particular point and this traction has to be integrated over the entire area we get the total force in x direction. So, P x will be integration over the entire area R and then T n x x component and this will be dx dy; this will be the component to total force in this will be the force in x direction.

Similarly, if I have to write out force in y direction P y, P y will be integration over the entire area I component of traction in y direction integration of that. So, this will be T n y dx dy and then similarly P z will be integration over R over R means the entire area and T n z dx dy right and this is true for this and this.

Now, both the ends similarly if I have to write say moment in a particular direction, then the moment will be say lets write the expression for a moment z. Moment z will be what? Now what are the components, what are the component of forces which will contribute to the moment? P y will contribute and then P x will contribute; it means that T x will contribute and T x will contribute. So, moment will be T x into multiplied by y integration and then T y multiplied by x integration and this will be integration over y then x into T n y, the y component of the traction minus y T x x component of the traction that is integrated dx over this will be Mz.

Now, if I have an external force say externally applied torque at that end at the considered end is T then this has to be equal to T ok. Similar way we can write expression for M x and expression for M y ok. Now what are the boundary conditions we have here? M Y; look at we only have the boundary this is a general case. Now, if we reduce this if we now consider only the case we have in our hand where you have a shaft and on that shaft at one end we are applying only torsion ok.

Now, in that case this will be equal to 0, there is no external force being applied in x direction because there is no extension or stretching we are considering in this case; there is no force in y direction, there is no force in z direction; all these forces will be 0.

Now, let us see the moment; only moment being applied at this section is about z axis which is the torque, but no other moments being applied at this end. So, M x will be 0 and M y will be 0. So, these are the boundary conditions a traction boundary condition the entire thing is the traction boundary condition we have at the both ends; total six conditions. All the forces are 0; M x and M y are 0 and the M z in T M z will be the externally applied torque ok.

So, now let us see what is. So, if I write it then what we what is the expression we have? Now, this is just recall this is the expression for torsion, this is the expression for traction and if we substitute if we substitute these conditions nz and nx is equal to 0, essentially we have these components are nonzero this is 0 component ok. In the previous case when we apply boundary condition nos on S only T z was non 0 and T x and T y were 0 ok.

Now, then what would be the conditions? Conditions will be you recall these are the condition just now we wrote ok.

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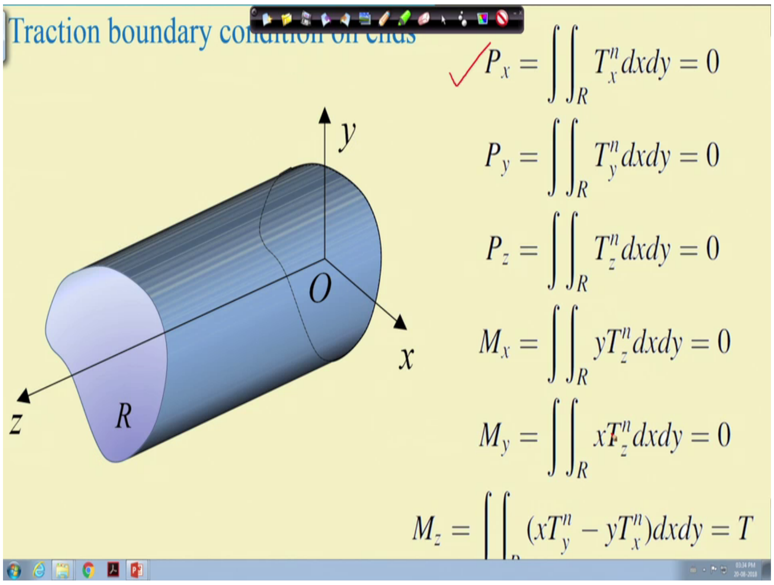

These are the force and these all the all these force and moments will be 0 only is this is nonzero.

Now, let us see. So, we said that this is the condition right this is the boundary condition needs to be satisfied. Now the way we define stresses and the interaction let us see whether that definition is consistent with this boundary condition or not. We will prove it all do that exercise only for the first case and the other case you can apply the similar way ok.

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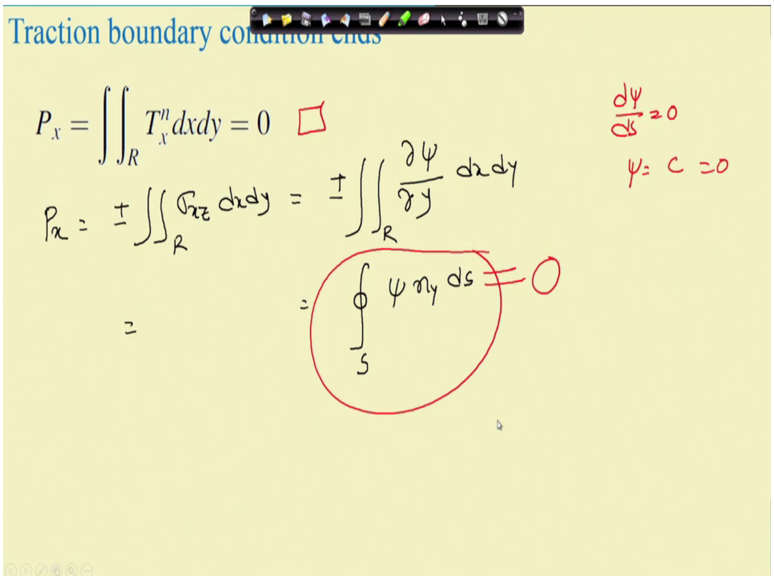

So, first case is this is the expression for P x ok. Now, if I substitute the expression for torsion if expression for T x and if you if you look at T x was how much? Just in the previous slide, if your T x was sigma plus minus sigma xz; it plus and minus depends on which end we are considering, at a z is equal to 0 or z is equal to or n nz is equal to minus 1 or nz is equal to plus 1.

So, but the main thing is that tau x T the component in x direction is sigma xz. So, let us let us write that. So, this expression becomes this is equal to this is equal to. So, if I write this as plus minus. So, this become plus minus integration over R this is sigma x z just now we have check sigma xz and then this integration is over dx and dy. So, this has to be equal to 0, we do not know whether this is equal to 0 or not this is P x this condition is to be proof the first 1. So, this is the P x ok.

Now, let us substitute sigma xz sigma xz by by the definition of the stress function. So, this become plus minus integration of R over the entire cross section and this becomes del psi del y and then dx dy right.

Now, here if we apply Greens theorem, Greens theorem already discussed in when the concept of tensor we are discussed in the second week second and third week Greens theorem was discuss at that time and if you recall that Greens theorem, if you apply Greens theorem then this becomes this becomes this integration over it was integration over the area and this becomes integration over the boundary this integration over the boundary this is a closed integration and this becomes psi n y n y d S ok

So, essentially it project integration from area to blind or from solid to surface in this case it is on the area and it is on the boundary of the area. So, integration is being projected on the boundary. So, these gives.

Now, look at just now we discussed what was the condition when we applied condition that we obtained when we apply boundary condition or NESS, the condition was d if you recall the condition was d psi d S is equal to 0 which give psi is equal to some constant and we said that this constant for simply connected domain at least this constant you take 0; means psi is chosen such that it vanishes at the boundary and if it vanishes at the boundary and this integration is over the boundary straight away it become 0 ok. So, this is proved. So, by choosing psi which vanishes at the boundary automatically all the boundary conditions that you have the traction boundary condition you have at the both the ends are satisfied ok.

Now, so, these exercise you can try for other component also. So, this condition you can similar way you can show the other boundary conditions also they satisfied. So, these are all boundary conditions you can do the same exercise and then check whether the definition of psi the chosen definition the way we are choosing psi if they satisfy these boundary conditions or not.

Now, in this case we are now interested only on the last part because that gives you the that gives you the relation between the externally applied torque and the internal stresses. Now let us see the last one.

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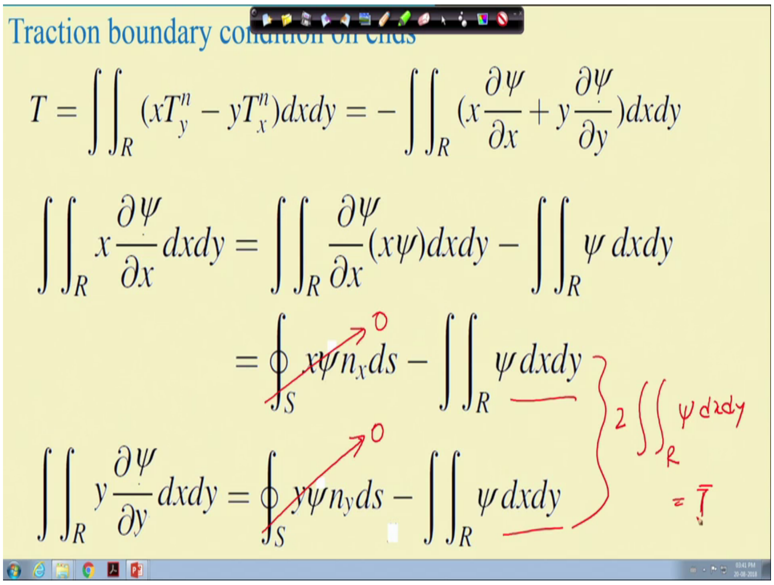

Now last one is this is the torque expression of torque that we have.

Similarly, we substitute the expression of T y and T x and then if we substitute that and then and then substitute those stresses in terms of the stress function then this is the expression we have. Now, this integration is let us this integration as two part let us see what are the first part first part comes. The first part is this. Now, if we integrate it by a parts this one then this integration gives you this ok.

Now, from this take this part if you apply the Greens theorem the way we apply Greens theorem just now. If you apply green theorem I mean projecting this integration on the boundary and this expression becomes this and this will remain as it is.

Now, similarly if you. So, this the first part the expression become this. Similarly, if you take if you integrate the second part in term in parts and then apply the greens theorem what final expression you have is this is your final expression.

Now, look at this expression. Again the same condition phi stored inside that such that if vanishes at the boundary. So, this straightaway this becomes 0, in this case also straightaway it this becomes 0. Then what this becomes? So, this is T is equal to we are left with only this part and this part. So, this part becomes then 2 into if we if we add them this part become 2 into integration over R psi dx dy that becomes your T and then we have a negative sign here.

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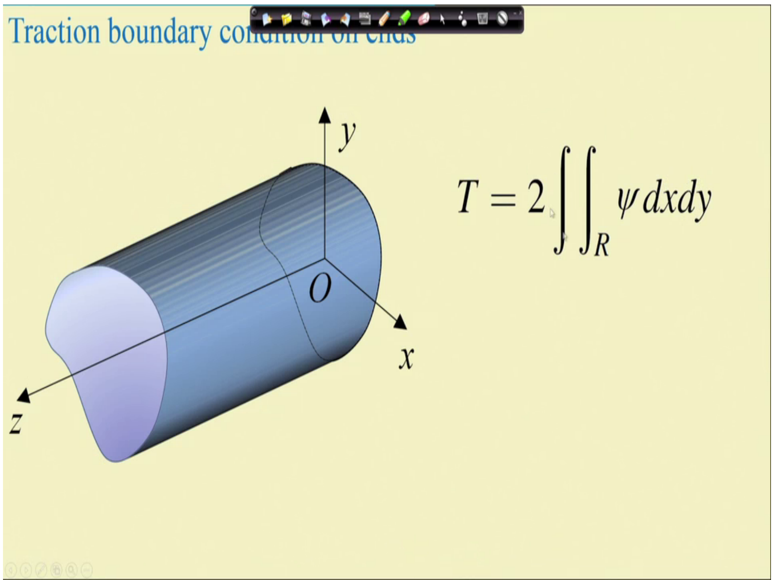

So, final your relation between the applied torque and the stress function is this. So, this is essentially the last boundary condition.

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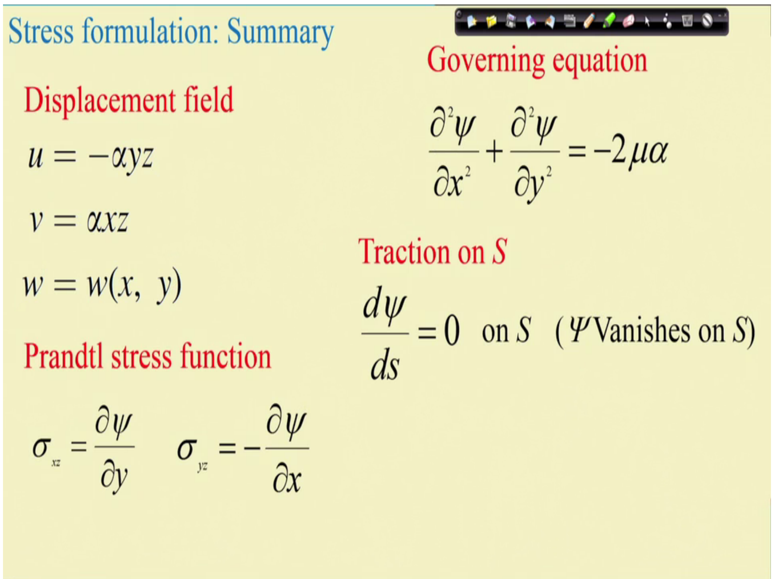

So, let us quickly revisit everything and summarize all the equations that summary is you see what we did is we started with we started with this definition of displacement field and the assumption was the small deformation, the displacement is small and that is why the rotation can be taken as linear function of the longitudinal axis. This is the assumption and this assumption gives you that rotation is a linear function of z and when you translate that rotation from that rotation, when you get the displacement corresponding in the in the different direction, we get u and v as this.

And then another assumption we had is the outer plane deformation, we have we assumed it is only a function of x and y it is not a function of z. So, this is the displacement field that we started with ok.

Now, that we discussed in the last class. Now, once we have this displacement field, from that displacement field what we did is we calculated the, we for this displacement field gives as some of the strain components 0 and some of the stress components 0. Then we define stress function; stress function is how the stress is the nonzero components of stresses. These are the two nonzero these are the two nonzero components of stresses and how these two nonzero components are related to stress function; this is the stress function ok.

So, this and then what we did is, we substituted this stress function in the equilibrium equation and also in the compatibility equation and combine the equilibrium compatibility equation and finally, we have a Poisson's equation and this is the equation right; this is our governing equation.

Now, once we have the governing equation next thing is we have to apply the boundary conditions. In this case, since it is a stress formulation we have to write or we have to apply the boundary condition on stresses; means, we have to write traction boundary conditions.

Now we have two boundaries; one is on the surface of this on the surface of the shaft and another is two ends of the shaft and if we apply the boundary condition on the surface then what we have is boundary condition on its it gives you d psi ds is equal to 0 and this give you psi is constant. We can assume this constant is 0 and which means psi vanishes on S and if psi if we take psi vanishes on S, we have just now seen that other traction boundary condition on at both ends they are automatically satisfied.

And then if we apply the boundary conditions on at the ends the final expression that we have is this. So, this is the summary of this; summary of the equations what we discuss in the previous class and what we have discussed today. So, this is the summary.

Next class what we do is we first we demonstrate these equations the application of this equation through two example and then and then we compare the results that we obtained from elasticity solution, we compare the results with the solution that we already had from strength of material and then we see whether they are is there any discrepancy and we then we try to understand what could be the possible reason for this discrepancy and then we also apply these equations to some other problems in torsion. I stop here today see you in the next class.

Thank you.