Theory of Elasticity
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Lecture – 38
Boundary Value Problems in Elasticity

Hello everyone we are going to start the 8th week of this course. What we have been doing from last few weeks is we have derived the formulation for boundary value problems in elasticity. Then we applied the boundary value problems those formulation two different problems like we applied to plane stress problem, plane strain problem. In the previous we will applied two different bend beam bending problems and in this week we will apply those equations, but in a slightly different form to torsion problem.

So, topic of this week is torsion. Now the first 2 lecture of this week we will derive the formulation of torsion we write the equations boundary conditions in a nutshell we derive the boundary value problem for torsion and then subsequent weeks we apply them for different problems in torsion. You see whenever we start any, whenever we learn any theory or when we whenever we formulate any many theory one of the most important part of the theory is the assumption.

Every theory is based on certain assumptions; assumption is important because assumptions are essentially limitation of this theory. If you recall the torsion is it is not the first time that you are hearing torsion or you are we are going to solve some problem in torsion, we study torsion in our strength of material course and if I recall what we studied in that course. That we if we and if you recall the mostly at that time you applied all these formulation to or you derive the formulation for torsion of a circular shaft.

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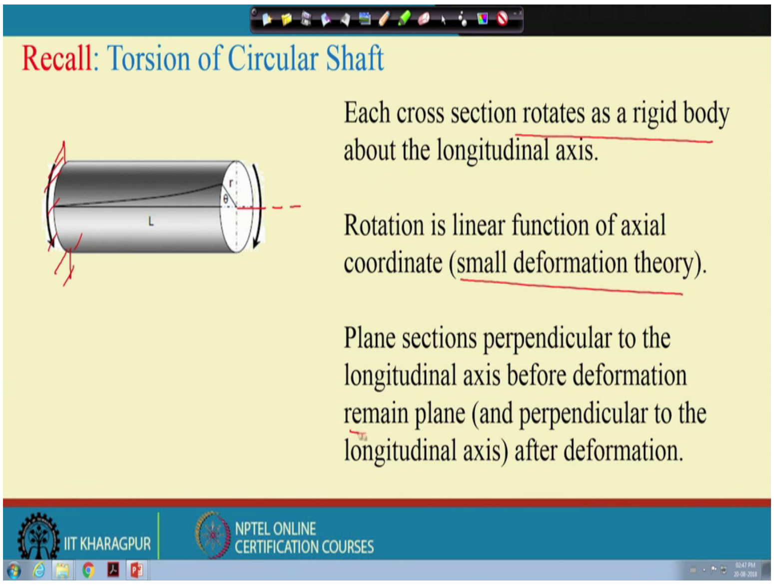

Now, the assumptions that we used those while deriving that formulation other than the it is a elastic obey Hooke's law of all the geometric assumptions that we had is. First condition was if you recall each cross section rotates as a rigid body about the longitudinal axis if, this is a shaft and this is the longitudinal axis of the shaft this is the longitudinal axis of the shaft. Then every cross section these cross section rotates about this shaft as a rigid body rotation of this of this longitudinal acts about the longitudinal axis.

And if this end is fixed, if we take this end is if we take a one end is fixed then rotation at this at this end is 0 and the rotation at this end is maximum and also we assume that rotation of that as I said the rigid body rotates about the rigid body that rotation is a linear function of the length. Which essentially is the 2nd assumption the rotation is linear function of x axial coordinate and this is true as long as our deformation is small.

Then another assumption was plane section perpendicular to the longitudinal axis, if we take any section which is perpendicular to the longitudinal axis any plane section. And then after the deformation that plane which was plane that section which was a plane before deformation even after deformation they remain as plane. So, these are the very important assumptions that we had while solving torsion is about solving a torsion problem; a torsion is circular shaft using strength of material. Now, as I just now I said assumptions are limitations there are many places these assumptions are not valid.

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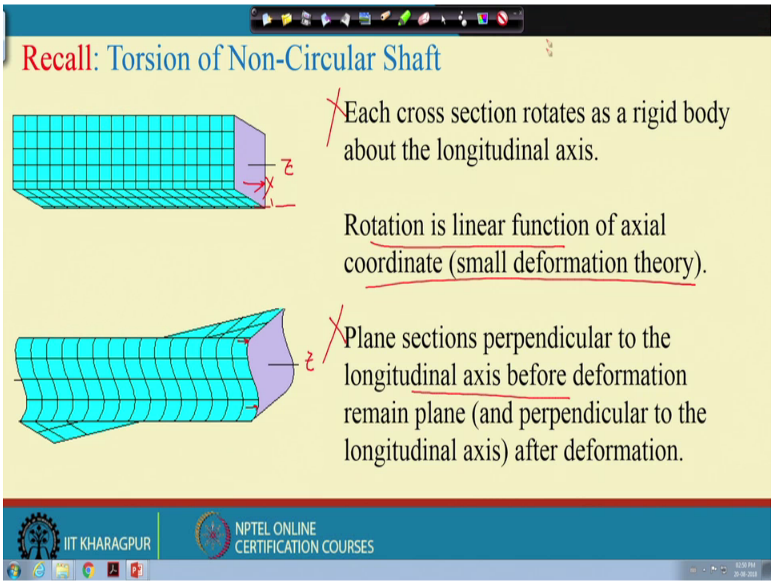

For instance, if we take a case of a non circular section for instance here the example is given as a square section. So, this is the un deformed configuration of the section and this is the deformed configuration of the section, you see the this first assumption the each cross section rotates is a rigid body about the longitudinal axis this assumption is not valid in this case.

This last assumption the plane section perpendicular to the longitudinal axis they remain plane that assumption is also not valid here this is the deformed shape and there is an outer plane deformation. So, this is the plane before the deformation and that section becomes like this so we can see there is some outer plane deformation, outer plane deformation means deformation if this is the z axis then deformation in z axis in z direction.

But in the previous case in the in the case of a circular shaft if this is this was the longitudinal axis there was no outer plane this there was no deformation in this direction this. That is why where is this that is why it remains plane right, but now in this case for non circular section they do not, these planes do not remain plane. So, first we cannot assume this, so these assumption is not valid for this class of problems this assumption is also not valid for this class of problems right.

So, now the that was the that was the limitation these assumptions were the limitation for the approach that we followed in strength of material course. And because of that limitation we cannot apply the same theory to problems with a non circular cross section because this non circular cross is that the deformation behavior is such they are not consistent with those assumptions.

So, before we start formulating the first thing we have to do is we have to remove these assumptions. So, these assumption this is no longer an assumption for our formulation, but we will stick to this because as long as deformation is very small, this can be this can be applicable. Now so once we discuss the assumption let us now start deriving these equations.

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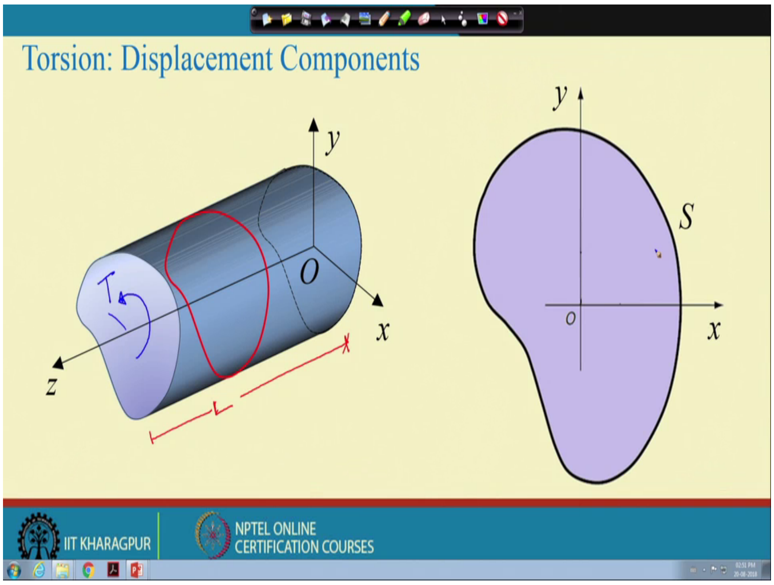

Take this is suppose a shaft which length is say this length is L and this is how we define the axis the longitudinal axis is the Z axis and X and Y axis define a plane in along the length of the shaft.

Now, take any cross section for instance, if we take a cross section like this any intermediate cross section and this cross section like this. Now what happens if we apply some torque here suppose if we apply some torque here. So, it T is the torque then what happens this cross section rotates and our assumption is if this end is fixed then at this your rotation will be 0 at this at this end your rotation is maximum. And if you take an any intermediate cross section this is an intermediate cross section So this intermediate cross section also rotate in this direction.

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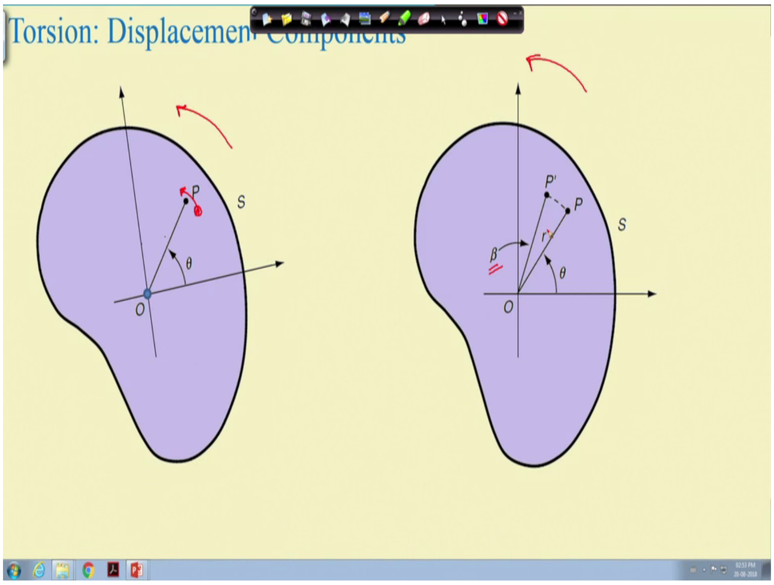

So, if we have, if we take a section like this suppose on this section on this section consider of any point P which is at an angle theta and this is the O is the along the it is the longitudinal axis. Now or in this case it is the center of twist along this axis it rotates, the rotation takes place.

Now if we apply some torsion like this, now because of this torsion like this the rotation of this cross section takes place and suppose the rotation is this. Now what happens due to this rotation in the P which was initially here now the P moves in this direction? So, P which was initially at this point, at this point now this P moves in this direction and this is the new position of point P; Suppose that new position is new position is P dash.

So, this is the this is the point this is the cross section where P is the position any position before actually deformation took place. And then because of this rotation in this direction and suppose the rotation is beta, because of the beta amount of rotation the position P, the point P goes to a new position and suppose this is P dash and these P is taking at a distance of r. Now let us see what we will do now is once we define the rotation like this now let us see what are the displacement components we have and what are the expression of those displacement components, how does displacement components are related with each other how their function of XYZ let us find out that.

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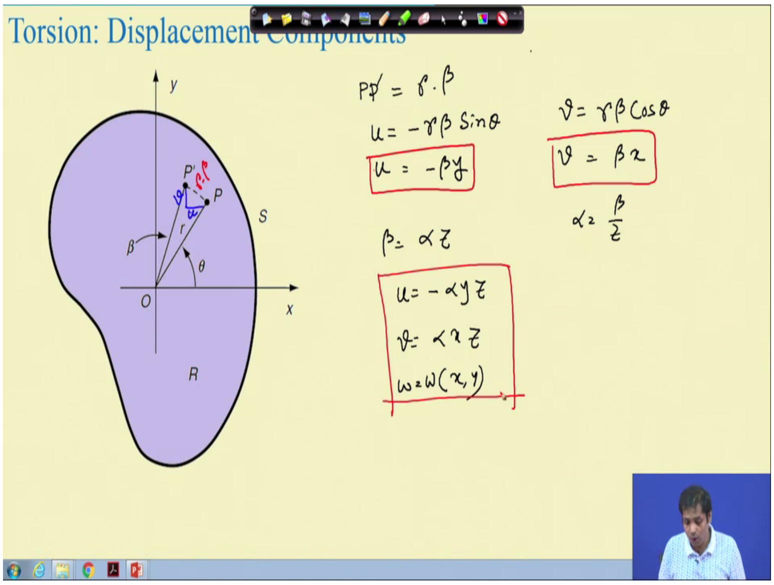

Now, so take this is the diagram, now if we check if we see then P P dash, we can write P P dash as P P dash in this expression P P dash will be r into r into beta.

Now, now if it is r into beta so this is r into beta, this distance is r into r into beta. Now this is the i into beta is essentially the total component is the total distance that is traveled by P or the deformation at point P. Now, if we get the, if we take the component of the deformation in a x and y direction the components will be if we take the component a this is u component and this is v component v is in y direction. Then those components will be u will be u is equal to that will be minus r b r beta then sin theta and v will be the component y r into beta cos theta.

Now, this is r into sin theta is essentially y, so this can be written as minus beta into y and this can be written as beta into x, so v is equal to this and u is equal to this. So, this is how if we take any point this is how because of the rotation of that section, this is how the displacement can be defined as a function of that rotation. Now, you recall we still stick to that assumption that that the rotation is a linear function of linear function of the length axis.

So, a rotation is beta length axis is z, so beta is a linear function of beta is a linear function of z alpha z where alpha is called alpha is essentially. So, alpha is essentially beta by z which means it is essentially the angle of twist per unit length, so if alpha is the angle of twist per unit length then this is how the rotation is related to each other. Now then what we have? We have if we substitute beta in these two equation in equation u and v then essentially what we have is u is equal to minus alpha y z and then v is equal to alpha x z.

So, this is how u and v is related to z, now w which is the outer plane now one component is still left that is the w which is the deformation in z direction a deformation of that plane in z direction what we assume is w as a function of only x and y. So, w essentially function of x and y, so this is another assumption we make that outer plane deformation on a plane that is function of x and x and y coordinate which defines that plane. So, this is these are the 3 different geometry 3 different displacement components.

So, we will use this 3 d displacement components I mean in your in our subsequent formulation, now we will come to this point again.

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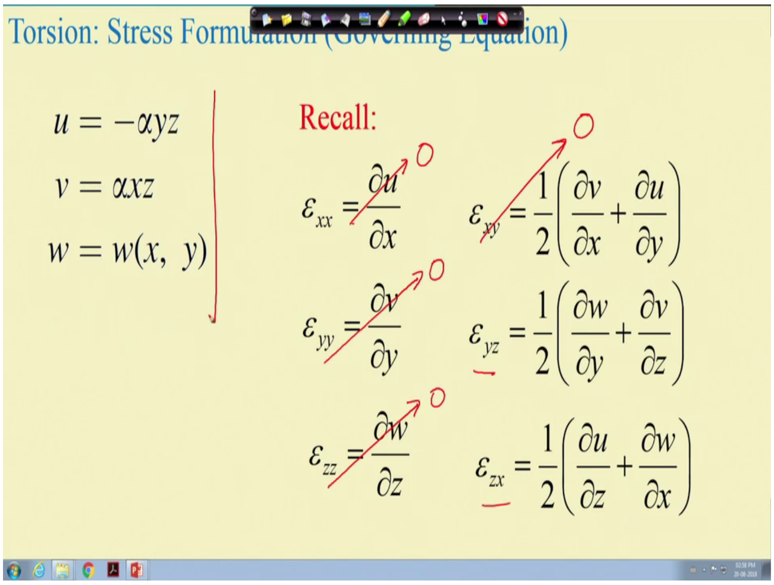

Now, if it is if this is the deformation component that we have just now this three deformation component now, let us find out what is the corresponding strain components and if you recall the strains are related to displacements as this, so this is a linear displacement.

Now, if you look at this, so straightaway we can make if we substitute u and v and w in this expression straight away we can make this is equal to 0. And then del v and then also we can make del y this we can make 0 this we can make 0 this also 0 because w is a function of x and y only. And then we have del v a del v del x into del v del u del y and if you substitute the; if you substitute u and v here we get minus alpha and alpha and this also become 0.

So, all these 4 strain components straightaway become 0 so only nonzero strain components we have then epsilon z and epsilon zx. So, it is a strain component on a plane in z and x; that means, the shear component of strain all the normal components are 0. So, once we have so this is essentially consequent of this displacement form.

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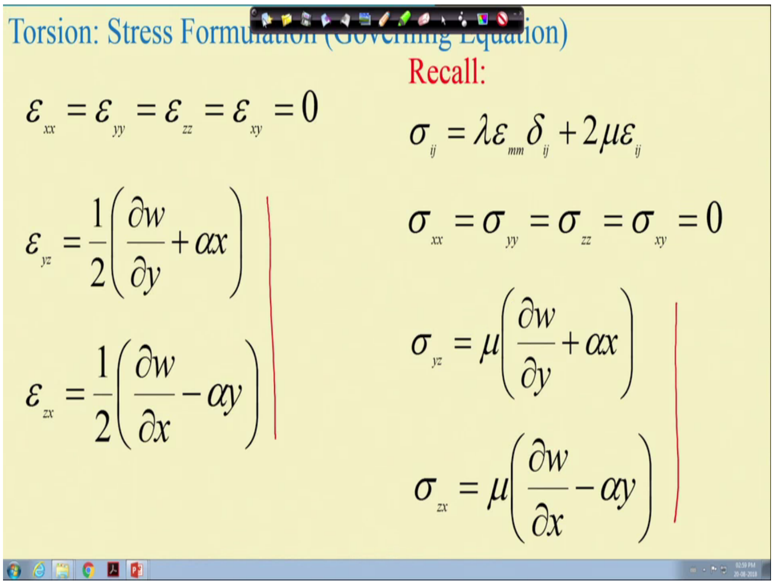

Now, let us see once we have this strain components what happened to the corresponding stress components so these are the strain components we have. Now, this is if we substitute w here in these expression from this expression if we substitute u in this u and w from this and uv and w from this then what we get is we get this expression. We get this expression of these are the these are the two nonzero strain components these you can try you can you can verify this.

Now, once we have these strain components then what are the stress components then let us see this stress you recall this is the this is the Hooke's law this is how the stress and strain are related to each other, lambda is a lambda constant and mu is the shear modulus.

Now, next what we do we substitute these strain components into this equation and then correspond then get the corresponding stress components. And if you do that then straight away we can make all these stresses are 0 all normal stresses at 0 which is straight away and these shear stress sigma xy is also 0, only nonzero strain component is this and then this will be sigma ys it will be this and sigma zx will be this. So, then we have this is the nonzero strain components and these are the nonzero stress components.

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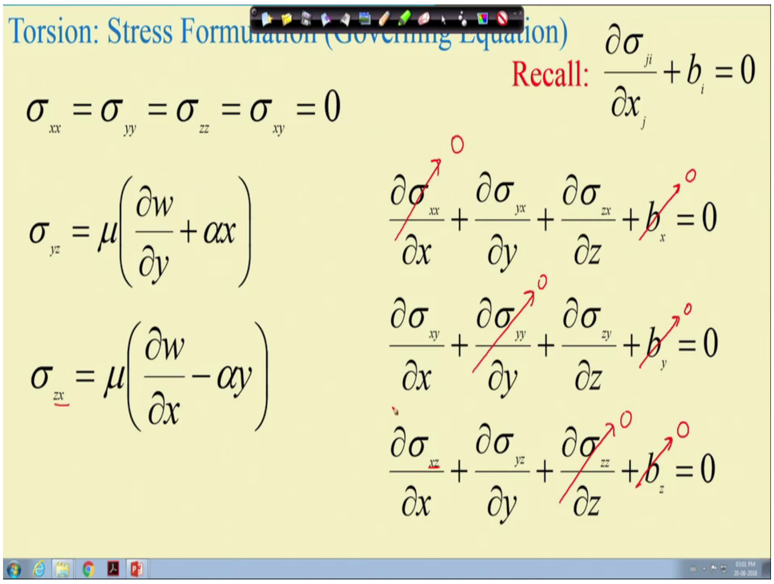

Now, let us move on next, so this is the non 0 stress components. Now once we have these stress components, see we are we are applying the equations we are essentially trying to solve different kinds of problem, but the equations which are the governing equations, compatibility equation, those concept remains same.

So, here the governing equations already we derive the same governing equation we have to use here this is the governing equation if you recall. Now, if we make if we write this equation this essentially three equations and if we write all these 3 equations; these 3 equations are this. So, now, please note that we already since that this Cauchy stress tensor is symmetric so somewhere it is written xz somewhere it in is written z x both are same.

So, then another condition we impose now that since it is only subjected to only subjected to torsion we are only focusing on torsion and we assume that there is no body force in the material. So, is if there is no body force then all these body forces are straight away we can make 0. Now so from this straight away we can say all these equations or these equations are as if you substitute this x this equation. So, this term will be 0, all this term will be 0, this term will be 0. So, similarly if you substitute all these expressions we are left with 3 to one expression and that expression is this.

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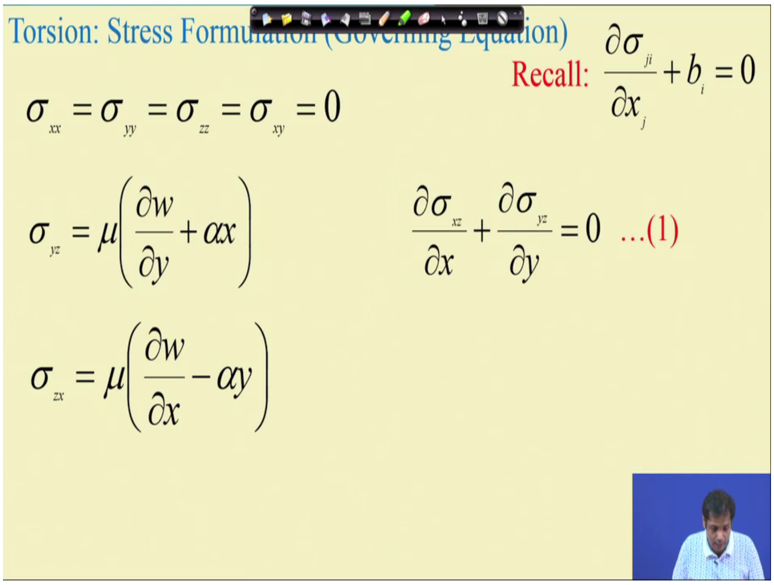

So, this is essentially is the third equation of this of this set all two first 2 equations straightaway you can make them 0 because of this condition. Now let us call it let us say this is equation number 1. So, what is this equation essentially this is an equally this is the equilibrium equation, now for the assumptions and for the case we are interested in this equation takes the form this.

Now, once we have the equation equilibrium equation, then you recall essentially we had 2 equation; one is equilibrium equation another one is compatibility equation and then we combine these 2 equation through a stress function, so let us now derive the compatibility equation. So, compatibility equation becomes you see this is the how we got this equation compatibility essentially tells you compatibility equation you can write in terms of strains you can also write in terms of stresses.

How do we write when we write in terms of stresses then essentially gives you how the different strain components are related to each other that we directly get from the compatibility of displacement. Now, once we have that if we substitute, if we replace strain by stress as the expression that we have is the compatibility equation written in terms of stresses. So, if we recall the Beltrami Michell equation is essentially that, but we would not do, we mould not let us not right that equations directly; let us derive this compatibility equation from the information that we have on the screen.

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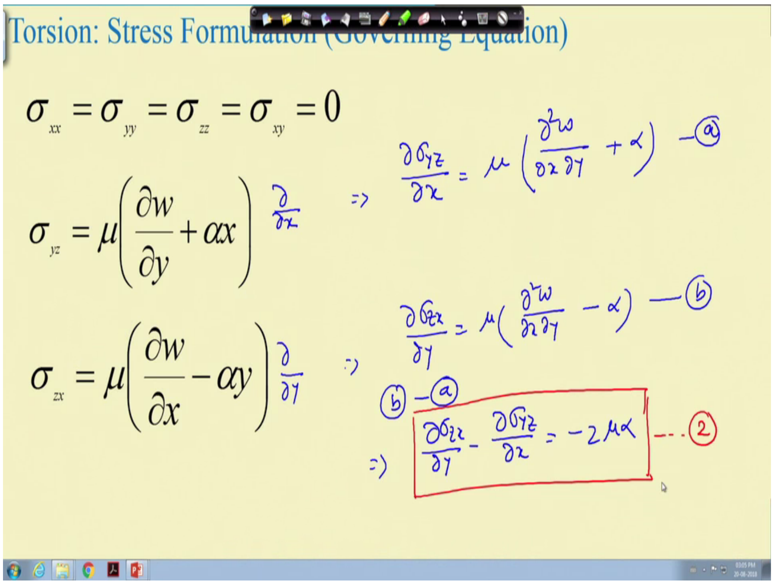

Now, the if you look at these two equations; these two equation is essentially what these two equation essentially we have the definition of strain and in that definition of means definition of strain displacement relation and then in that relation we substitute a strain we replace strain by stresses and we have these two equations.

Now, let us combine these two equation and if we combine them let us do that exercise if we combine them then what we have is first you take the first one and the yz let us differentiate this one by del del x del del x and then differentiate this one by del del y. So, if we do that what we have is, we have del sigma y z del x that is equal to mu and then del 2 w del x del y plus alpha and similarly this equation gives us del sigma zx del y or xz that is equal to mu into del 2 w del x del y and then minus alpha.

Now, if we take this is equation number say a and this is equation number b equation number one, but one already deserved for come for governing equation equilibrium equation. And if it is equation number b then let us take b minus, b minus a and that gives you sigma zx zx del y minus sigma yz and then that is del x that is equal to we have minus 2 mu alpha. So, this is the compatibility equation is not it written in terms of stresses.

Let us say now this is equation number two, so equation number 1 is the governing equation, equation number 2 is the compatibility equation these are the 2 equations we have. So, just let us write equation number one once again here, so equation number let us go to the next slide.

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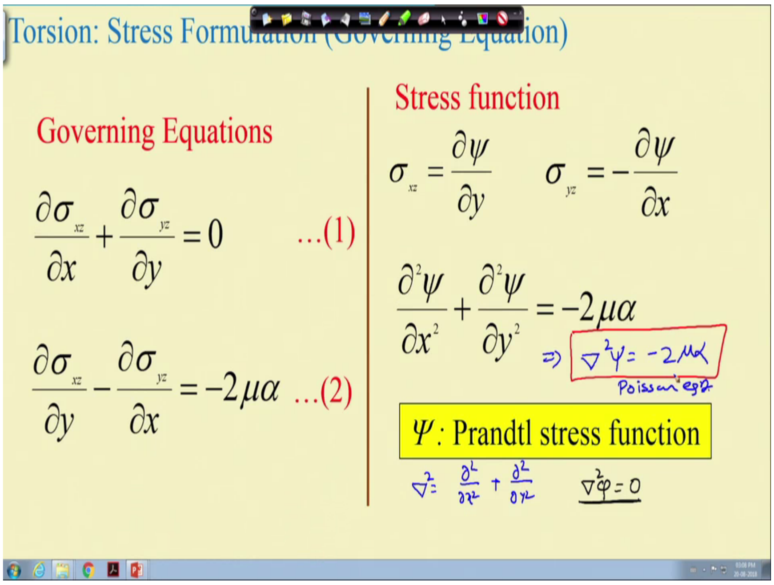

So, essentially what we have is, we have this is the equation these are two equations these the first one we derived from govern from the equilibrium equation and the second one derived from compact. This is essentially a compatibility equation written in a different form.

Now, if you recall then what we did that time we introduce a stress function when we solved a plane strain problem, plane stress problem or even been bend in bending problem in the previous week. We introduced we used we solve these equations through Airy's stress function, we essentially combine these two equation through Airy's stress function.

So, here we will not use Airy's stress function we will define we will define a separate stress, stress function and suppose that stress function is this. So, please note the difference between this stress function and the stress Airy's stress function. Airy's stress function when you define it was the second derivative of that stress function gives you stresses it is the first derivative of stress function that gives you stress.

In case of when we when we introduce Airy's stress function as a consequence of that we got a fourth order equation which was by harmonic equation, let us see what we get in this case. So, this is the stress function; now if we substitute that stress function in these two equation and combine them equation one and two then what we get is we get this. So, this is this stress function is called Prandtl stress function or it is also called warping function why it is called warping function we will discuss in subsequent classes.

Now, this can this equation can also be written as like this is essentially del 2 operator if you recall del 2 operator was del 2 del x 2 and then plus del 2 del y 2. So, this x essentially gives you del 2 psi is equal to minus 2 mu alpha, this equation is important. So, now, we have we combine all this equation and finally, we have this equation this is our final equation and you if you recall and the this is Laplace equation is written as del 2 take any function say phi del 2 phi is equal to 0. If we the if it is homogenous by the right hand side is 0 we call it Laplace equation, and this equation is called poissons equation P poisson equation.

Now, the thing is what we have here, you recall in case of when we introduced when we wrote this in the case of plane strain problem or plane strain problem or bending problem. When we use Airy's stress function our equation was like this equation was del 2 phi del 4 phi is equal to 0, but now we have equation this ok, so this is our governing equation right.

Now, essentially what we have to find out we have to solve this equation for psi and then once we have psi to the relation between psi and stresses we can compute stress and then from stress we can compute strain and subsequently displacement. But before that so, this is the governing equation now you recall when we talk about a boundary value problems governing equation is just one part of that problem and another important part of that problem is the boundary condition.

Now, what we do is, this is the governing equation we will be using this equation for solving different problems in torsion, but before that let us discuss what would be the form of boundary condition and that is the topic for topic for next class. So, next class we will discuss boundary condition how to write boundary condition for the given torsion problem. I stop here today next, see you in the next class.

Thank you.