Theory of Elasticity
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Lecture – 37
Problems in Flexure (Contd.)

Welcome, so we are in the module 7, so this is the last lecture of the module 7 where, we are essentially solving some Problems in the Flexure involving the elasticity approach and specifically the stress function approach.

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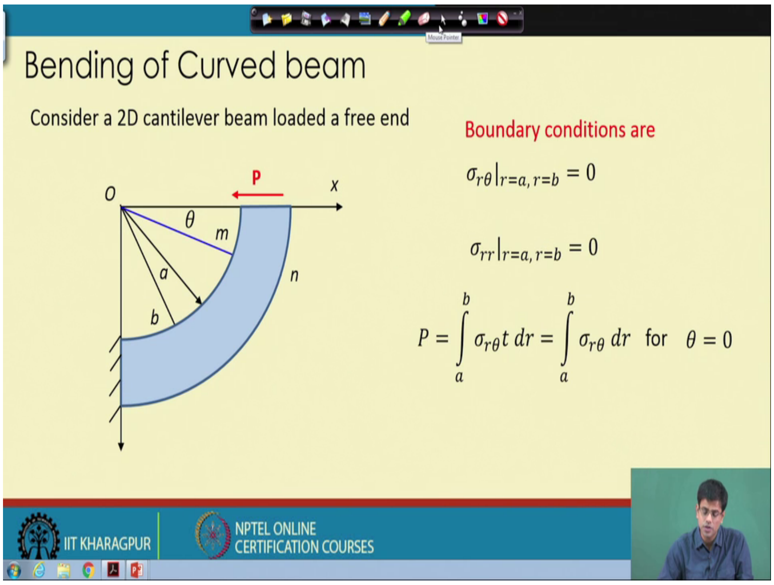

So, in the last class we were discussing about this problem where the stresses stress boundary condition using stress boundary condition we have found out the stresses. And the stress boundary conditions are those which is sigma r theta r equals to a and b equals to 0, and sigma rrr equals to a and b equals to 0 which means that along these two radial surfaces radial boundary surface there is no shear stress as well as there is no normal stress.

So, and also the third condition that we are using, so t is essentially 1 we are we consider it is a rectangular a narrow cross section. So, this becomes P the force the resultant shear force at here is essentially sigma r theta dr for theta equals to 0. So, with these three boundary condition what we have achieved in the last class is the finally, approximating a stress function.

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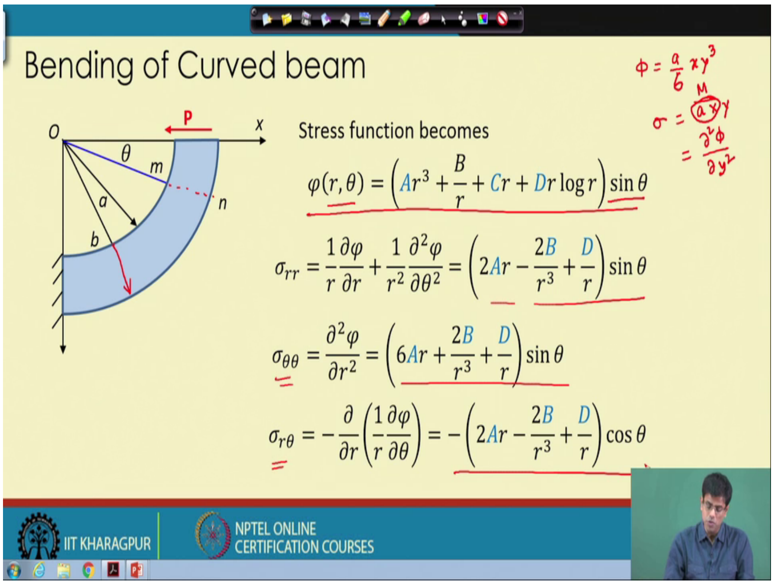

So, the stress function is are of following form. So, this stress functional is of the following form and then which is a function of r and theta the general r theta, but we have used separation of variable where this is a function of r and this is a function of theta which is the function of theta becomes sin theta. This is motivated from the bending moment equation and bending moment formula at any cross section of that beam.

So, now this is motivated from the strength of material solution which is similar to the rectangular case. If you remember the rectangular case we were using stress function of something like a by 6 xy cube or something like that. So, essentially the stress is a xy something in this form. So, essentially the stress is actually del square phi by del y square in the rectangular case. So, these similar to so this is the px is the moment. So, this becomes the moment this part becomes moment and my by i something like that this form was chosen. So, similar with this similar concept here also we choose the sin theta.

So, now with this approximation of the stress function phi we found out sigma rr which is of this form and sigma theta theta which is of this form and sigma rr a r theta which is non-zero which is of this form. Now, these two thing once we have found out our next job is to find out the strains and displacements.

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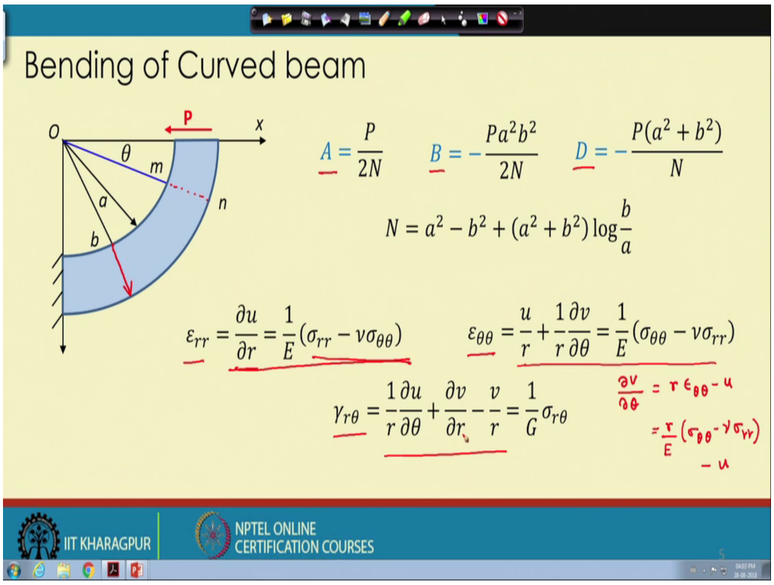

So, these are the constants which we have used to find out the which you have used to find out the stresses, we can now find out the stresses with the known value of this constants. And these constants are essentially found out by using the stress boundary condition. Now like in the previous cases in this case also we have to find out the displacement. So, the how to find out the displacement; we have to use the stress strain relation which is specifically the strain stress relation or using the compliance matrix or Hooke's law.

So, here the strain epsilon rr is del u by del r is essentially 1 by E sigma rr minus nu sigma theta theta. So, this is epsilon rr similarly epsilon theta theta, so epsilon theta theta it is better to write it in this form that del v del theta del v del theta equals to r epsilon theta theta minus u. And epsilon theta theta again I can write r by E E into sigma theta theta minus nu sigma rr minus u. So, this form a because will be integrating these equations to find out v.

Now, here del u by del r is essentially this so now, what we will do we will substitute sigma rr sigma theta theta in these two expressions and then integrate to find out u integrate to find out v and then we substitute this here or these shear strength likewise in the previous case. So, we will do once we know the expression of u, once we know the expression of u we substitute it by taking the derivative with respect to theta of u and derivative with respect to v of r. And then substituting v by r and then sigma r theta we know from our stress solution.

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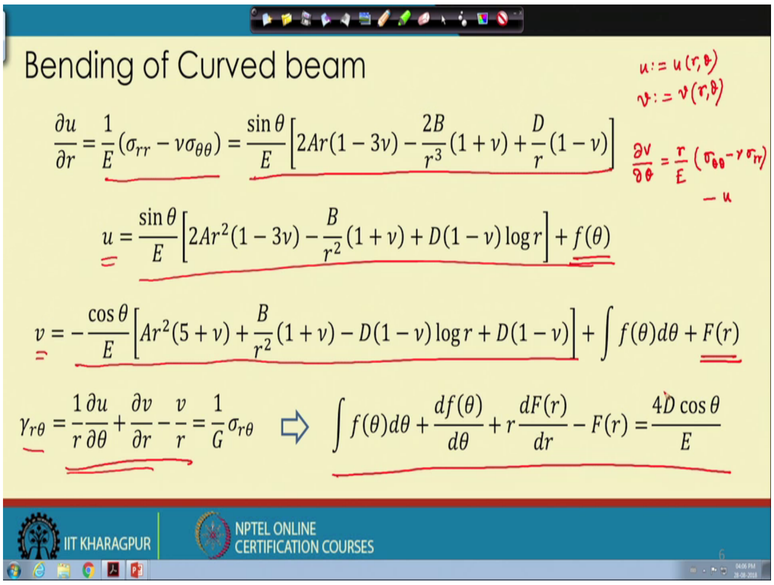

So, with this we can find out the conditions, so now the same thing we are doing here. So, here del u del r I use this and we can now after some manipulation. So, these manipulations we have not shown here so this you can just do and check this manipulation so this becomes the del u del r again similar to integrating this equation we can get the expression for u.

Now, integrating with respect to r so this is f theta, so because as in the previous case also u is a function of r and theta and v is a function of r and theta. So, we are integrating with respect to r, so it will be constants will be f theta similar to that we get a v which is del by integrating this equation del v by del theta is r by e into sigma theta theta minus nu sigma rr minus nu sigma rr minus u.

So, you substitute u here and sigma theta theta and sigma rr you substitute from the previous stress expression and then you integrate with respect to theta and then you get a constant which is Fr, so capital Fr which is a function of r because v is a function of r and theta. So, now, this expression is very long expressions so now, or after substituting these two expressions in the shear strain. So, essentially this is the expression we get this kind of equation.

So, again this is some of them are function of theta only some of them are function of r only and this becomes a function of theta and r. So, you see this d is a constant which actually we need to plug in and see. So, in the same similar manner in the previous cases we follow, so we can just now solve this problem by in a similar manner.

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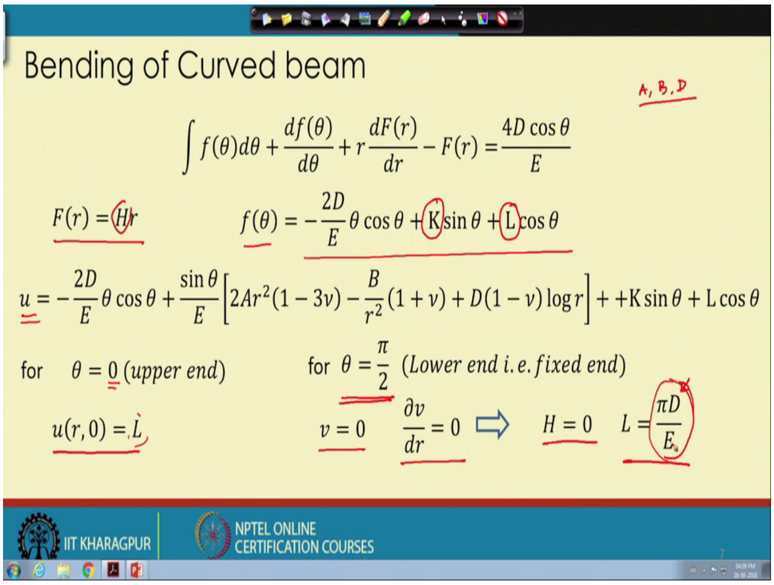

So, if I just now assume that Fr is Hr and f theta is of this form then we can directly solve this problem. So, a the D is a constant remember here D A B D we had three constants which we have already found out from the stress boundary conditions. But these stress boundary conditions are not a these stress through this stress boundary condition we found out A B D, but we have here three more constants which are H here K here and L here.

So, these three constants are the new constants we have in worked into it and this also gives us u and v. So, I have written the expression for u here which is pretty long expression and then similarly we can write it in the v also. So, the boundary conditions has to be used and those boundary conditions is essentially will be the displacement boundary conditions.

Now, these displacement boundary conditions if we if we use it here then we can find out these three constants that is H K and L. So, now, what is the displacement boundary conditions that we are going to use, so similar to the straight cantilever beam if you have remember that at the fixed end if theta equals to pi by 2 is our fixed end. So, at be the fixed end v equals to 0 and dv dr that is the any element in the vertical line will be 0.

So, which actually gives us H equals to 0 and L equals to pi D by E, so see D is known to us, so these are the and this is that same d that we have obtained from stress boundary condition. Now, this D we know so we know H and L, now it is of our special interest that if we want to find out u at the free end of the curved cantilever which you see is that theta equals to 0 at the free end. So, u the u displacement is actually L so L is actually pi D by E, so we have found out L so actually will compare this.

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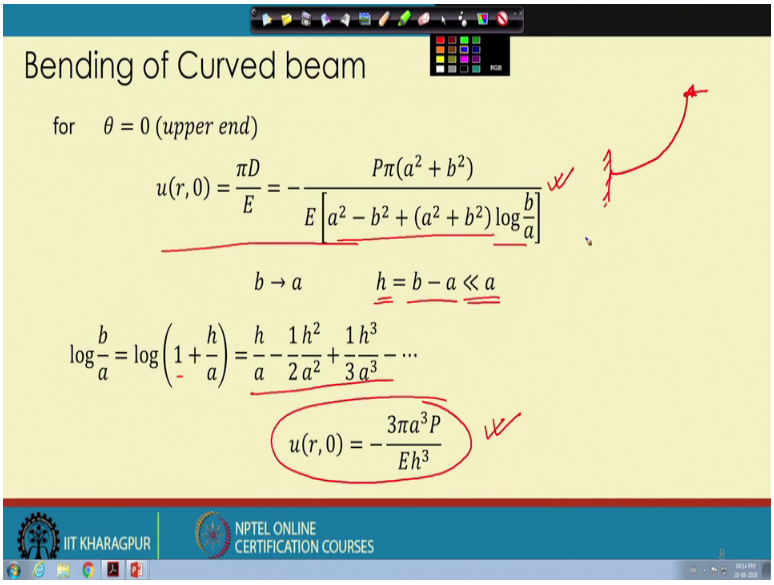

So, now if we see that the cantilever that we have seen earlier the upper end of the cantilever or the free end of the cantilever u r theta is pi D by E. Now, this D if we substitute from the previous expression, so D is our from the previous expression if I substitute D here this is minus P a square b by N where N is a square minus b square plus a square plus b square log of b by a. So, if I substitute at the upper end of the at the upper end this D so this becomes minus P pi a square b square E into a square minus b square a by this.

Now, this is the if you see this is the thing so this is your, so at this point this is the radial line. So, here this is the there is a force now if you have solved this form this problem you have solved it with this with h equals to um 3 pi a cube where a is the radius of this arc and three pi a cube P into E by h cube h is the depth and b is actually one for us because we have taken the unit thickness.

Now, which actually here are is this is the general formula from the elasticity solution where we have used two dimensional expression. Now if b tends to a for this solution that b tends to a means that the thickness of the beam becomes small. So, that thickness is b minus a is very small compared to a then we can write log b by a is log of 1 plus h by a. And this can be expressed in the series terms which is this and then we consider we substitute this here and after substitution here; if we after substitution here and we neglect the higher order terms of h that is h square or h cube in the top.

So, is so that means, the small because h becomes small, so q becomes small square becomes small and 4 becomes small. So, if we neglect those higher order terms then we get the solution this so this is same what we have got from the strength of material approach. So, in a sense what we have shown here that this is a elasticity solution and this is the strength of material solution. So, elasticity solution for a special case can be considered as a strength of a solution of the strength of material. So, this approach elasticity approach gives you more refined solution where strength of material solution gives in general more coarse solution or the approximate solution, but elasticity solution gives you more fine a refined solution or more detailed solution on the problem.

So, this essentially means that if we want to study a complicated body or the complicated stresses involve we need to solve the elasticity solution and it depends we need to go for the elasticity solution from detailed study. And then that does not mean that strength of material solution is no use because strength of material solution often very well approximate the exact solution. For instance the wrinkled back formula what we have seen in the strength of material solution is very well or very accurate compared to the elasticity solution of the pure bending of the curve being that we can we can verify.

So that means, the strength of material solution is also very important and this is a in a nutshell completes or flexure part of the problem. So, I will just briefly summarize what we have done here, so in a um in the first lecture what we have done we have introduced the basic two dimensional differential equation compatibility equation and then the stress function in the differential equation in terms of stress function. The stress functions we have also shown that if there is a body force constant body force or there is no body force that stress function what is the form of the body force, and what is the form of the compatibility equation and then the stress function.

Now, this stress function leads to a bi harmonic differential equation for a rectangular Cartesian system as well as polar coordinate system. So, these with these bi harmonic solution of this bi harmonic equation with the proper boundary condition gives us the solution of the elasticity problem. Now these the solution will be in terms of stresses, so boundary condition will be in terms of stresses. So, first what we have done we have solve a plane cantilever straight sided cantilever with a vertical force.

And then we checked several stress functions and first we solve it through intuition from the strength of material knowledge where we can compute the stress functions we can approximate the stress functions from our intuition. And then solve the stresses by solving the bi harmonic equation properly and then use the stress boundary condition to solve it to find out the constants.

Now, in the next class what we have shown is that it is not only you need not to only use your intuition or your strength of material knowledge you can actually superimpose the different stress functions which indicates different stress states of the material. And then using the property of linearity or the linear differential equation here the bi harmonic equation we can again solve the problem and we give the same answer.

Now, once we solve the stresses then we go for the displacement now it is interesting to point out once again here that in this procedure when we solve stresses the stress of the body does not involve any material. So, the; that means, the constitutive behavior is independent of the stress solution. So, that does not mean that it is not important because when we find out strains and then the displacement from the body a displacement from those stresses then we need to invoke the material laws or constitutive laws and then the material constants comes into picture.

So, essentially the strain and stress we find out and then from there we integrate for finding out the displacements u and v and then we again use the displacement boundary condition to find out the displacement the to find out this constants and then we find out the displacement. So, in the next what we have shown we have shown the pure bending case where the pure bending case is essentially the we have solved in strength of material wilding Winkler back for formulation.

That Winkler back equation gives us the hyperbolic stress variation and here with the elasticity approach we have solved this problem and it can be a proved that Winkler back actually very well approximate the elasticity solution. While the linear is stress pattern does not approximate at all, so the error is much more here. Now, in the next case similar to the first case we have also considered a curved cantilever and this curved cantilever is acting with a horizontal force P and this P is essentially this problem essentially we have solved in the strength of material approach.

So, we then compare this reflection of the free end of the strength of material approach and the elasticity approach we find that elasticity strength of material approach is a special case of the elasticity approach. When the thickness goes to a thickness is much less than the radius inner radius of the cantilever beam. So, this completes our flexure formulation and in a next module we are going to start the torsion problem so.

Thank you.