Theory of Elasticity
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Lecture – 36
Problems in Flexure (Contd.)

Welcome so we are in the module 7, so we are discussing the solution of boundary value problems and we are considering the Problems in Flexure. So, in the last class we have discussed the pure bending of a curved bar essentially and then we found out that the normal stress sigma theta, theta we are considering in the polar coordinate system.

So, normal stress is essentially quite good with the hyperbolic form of the stress what we obtain from the strength of material approach. Now, in doing so we have also used stress function approach.

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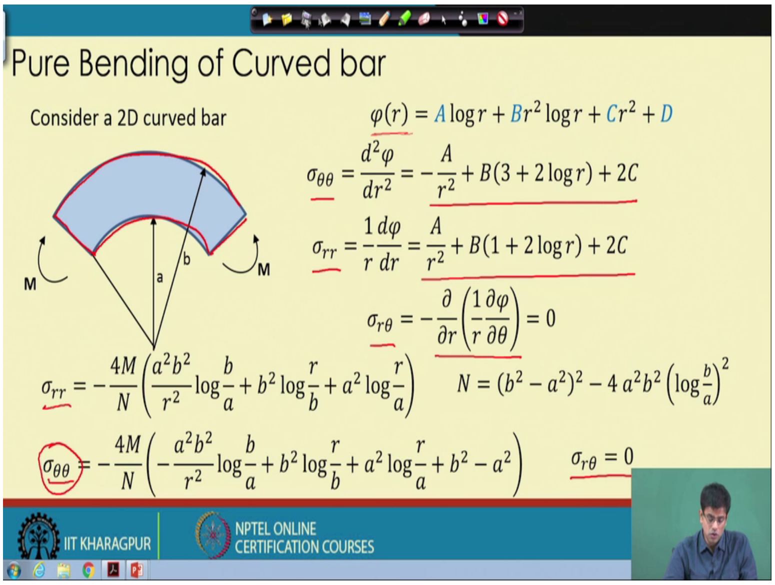

In the stress function approach we assume that our stress function is a function of r only there is no theta here. So, there is no theta here and this leads to our normal stresses are of this form. So, sigma theta, theta is of this form and sigma rr is also in this form and sigma r theta which is a requirement for the pure bending is also 0.

So, now what we did? We use the three boundary conditions that we have seen that is here in the radial convex side there is no normal force. And the this side there is no these two sides there is also no force generated only couple that it and sigma r theta there is no tangential force. So, sigma r theta is 0. With these three boundary condition we could find out sigma rr sigma theta theta and sigma r theta equals to 0 is already satisfied.

So, with this sigma r r and sigma theta theta, so this sigma theta theta; what we calculate with the strength of material solution by Winkler that formula. Now our present objective is to find out what are the displacements as we did it for the rectangular or that straight case, we found out from straights to displacement.

So, you see as we have seen earlier that these sigma r r and sigma r theta theta and sigma r theta this does not depend on the material. So, this beam now can be made of any material, so when we try to find out the displacement then we need to invoke the material laws or material constitutive behavior, then the material parameters comes into picture. So, for a specific material the material parameter values take specific values. So, we will be again considering the isotropic linear elastic material.

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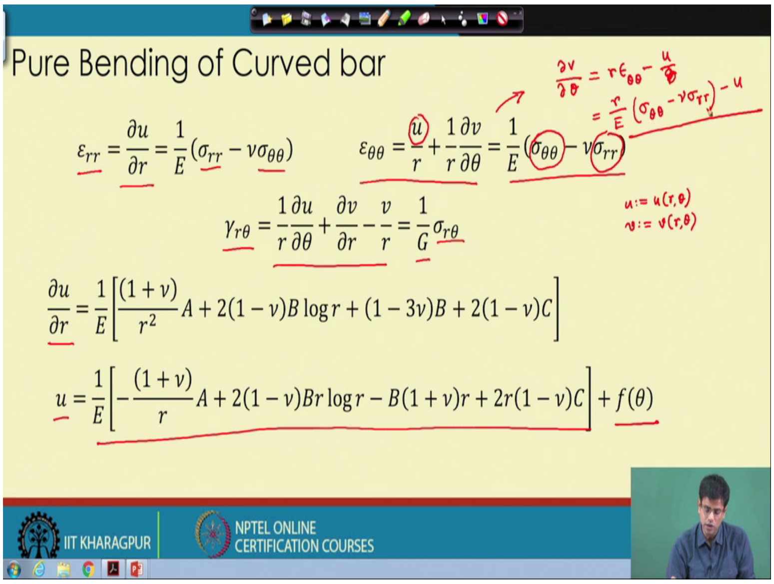

So, for which the stresses and strains we know in the linear regime linear elastic regime which he which follows by the Hookes law. So, now, we are doing it in the polar coordinate system for which epsilon r r is essentially del u del r which is 1 by E sigma rr minus nu into sigma theta theta n epsilon theta theta is u by r plus 1 by r dv d theta. So, this also equals to this and then shear strain in r theta system is gamma r theta, which is this which is equals to the shear modulus and this.

Now, if I just substitute the our stress expression that we have got from the stress function and then if I write it, if I compute epsilon r r which is du by dr which looks like this.

Now, after integration simple integration if I integrate this equation I get the expression for u and since u is a function of r n theta u is a function of r n theta. So, here u is a function of r n theta and v is also a function of r n theta. So, we are integrating with respect to r, so then the integration constant will be a function of theta right.

So, this gives us the first expression for the u and similarly if I substitute this u here if I substitute this u here. And then if I substitute the stresses sigma theta theta sigma rr from the previous approximation, I can now calculate this dv d theta. So, if I write it in a little proper manner like this dv d theta will be your r epsilon theta theta epsilon theta theta minus u by r.

So, or minus u so if you look carefully that we have just used this equation. So, now, epsilon theta theta I can substitute r by E is essentially sigma theta theta minus nu sigma rr minus u. So, this is my dv d theta, so in this expression I know u from here right and sigma theta theta we know and sigma rr we know.

So, if I integrate this equation now I can get the v, so now let us do that thing and check what happens.

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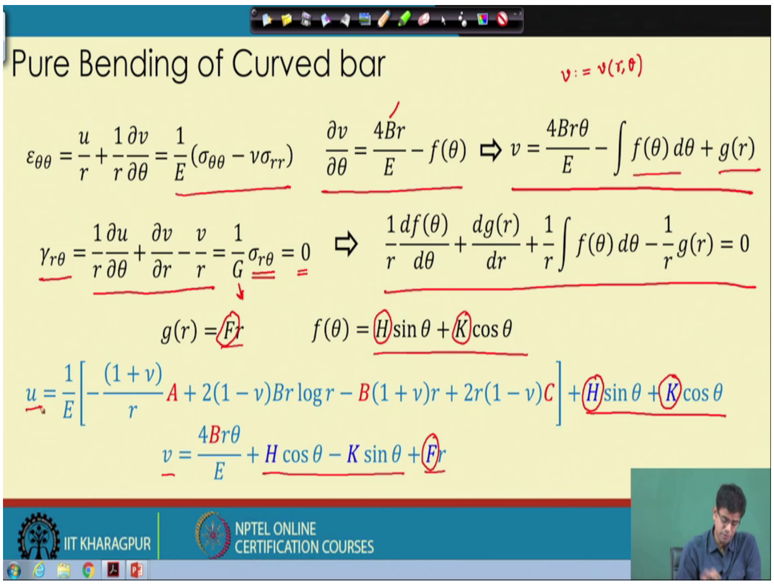

So, essentially this is the case, so dv d theta is r epsilon theta theta minus u. So, and r epsilon, epsilon theta, theta is essentially this quantity, so if I substitute and after doing some manipulation which I have obviously, not shown here. So, you can do this manipulation and this becomes dv d theta becomes this which involves only one constant which is B and this f theta and so f theta is coming from the u solution of u in the previous slide.

So, now this dv d theta if I integrate, so it becomes 4 B r theta by E minus integral of f theta d theta plus g r. Now see v is also a function of r n theta, so and we are integrating with respect to r and then with respect to theta. So, the integration constant will be with respect to r which is a function of which is a g function of r.

Now, what we have we have v and we have also u from the last a slide. So, if we substitute gamma r theta that is del u by del theta and dv by dr minus v by r which is essentially the shear strain in r theta system. And then this shear strain is essentially the shear stress by the shear modulus G.

Now, if you remember the for pure bending case the shear stress this a sigma r theta is essentially 0. So, then this implies that shear strain will be 0 now if I substitute u and v and then equate it to 0 this will be my final equation.

So, as we did it for the straight case or straight beam that these some of these are will be function of theta some of these will be function of r only and some of these to for some of function of r plus some function of theta will be equals to 0 this is the same thing and if I use now g r equals to Fr and f theta equals to H sine theta plus K cos theta where again F F H and K are constants we need to find out.

So, this satisfies this equation right, so if you can just substitute and check whether this satisfies or not. So, now this with these functions of this form F r and a f theta and g r we can write now u and v. So, u becomes this, so this is a u and then this is f theta so f theta is H sine theta plus K cos theta and v becomes this plus integral f theta d theta. So, which is H cos theta minus K sine theta and F r which is g r.

Now, here if you look carefully that already we have found out ABC, so these three constants we have already found out from the stress boundary condition. So, this called even though I have written it here so these constants we know. So, we have to find out only three constants which is H K and F. So, only three constants if I can find out we can find out displacements u and v so let us see how we can find out these three constants.

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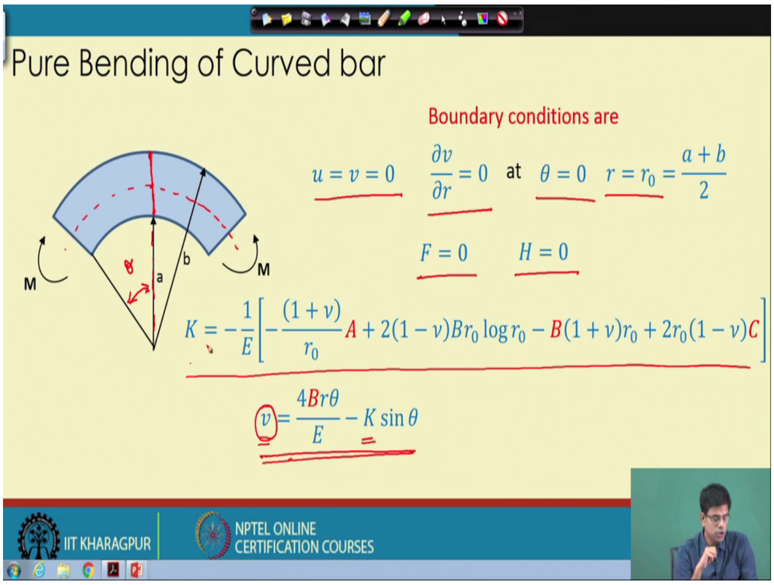

So, to do that, we have to again use the boundary conditions in terms of displacements. So, what in the boundary conditions we can in the radial direction that means, that the centroidal axis. So, this is a radial direction in this radial direction we can fix this point. So, in at these points we can fix this point this point at u and v will be 0 and dv dr that means, the vertical line along this fiber vertical element will be essentially fixed.

So, it prevents the rotation of the vertical element on this point, so which is implied by dv dr equals to 0. So, at theta equals to 0 remember that theta is measured from here so theta is this so this is theta. So, now, at theta equals to 0 r equals to r 0 r 0 is a plus b by 2 so this is a and this is b so this is a plus b by 2.

So, now this if we invoke into the K that is the u and v then we can see that if I invoke this first two condition and then dv dr then we can get F equals to 0, H equals to 0 then K is norm 0 which is of this form.

So, now if we write to v which is v here which is only K will be nonzero so v looks of this form. So, what is v if you look carefully that v is the fiber vertical axis the displacement. So, now if you see that this v represents actually there is a translation here which is a K sine theta and there is a rotation which is 4 B r by E 4 B theta by E.

So, these rotations and these things allows these fibers to be remain plane after bending, so this is actually the condition of pure bending that we use. So, this also proves that this is a solution of a pure bending, which where we assume that fibers remain plane before bending and after bending.

So, this equation actually he says or verifies this so this actually is the displacement for the pure bending case. So, similarly with F and H and K we can find out u also so this completes the pure bending part what we are discussing what we have started discussing in the last class.

So, here what we have done in case of a pure bending we found out the stresses and then from those stresses we to find out those stresses we have used stress functions the stress functions are a function of r only. And these stress functions the involve some constants and those constants we found out stress boundary condition.

Now, once we done we have done with the stresses then we found out the displacements and those displacements are essentially the from the stress strain relation wherein we invoke the material laws which is essentially Hookes law. And then with this Hookes law we find out the displacement cu and v using that displacement boundary conditions. So, this is the complete solution of the pure bending problem of a curved bar.

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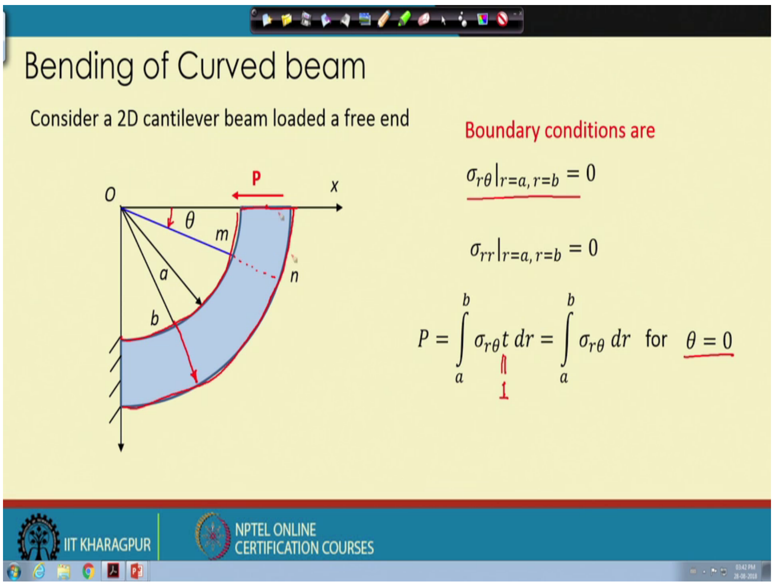

Now, in the next what we have planned is the 2 D cantilever beam which is a one quarter of a circle. And this cantilever beam is essentially fixed at this end and then there is a force or the shearing force resultant along this line which is P.

Now, this is similar to the cantilever problem that we have solved in the rectangular mean straight case, so this problem will also try to address here. So, this problem if you remember that you have already solved this problem by strength of material approach and you know the solution for this, for instance the displacement here at the tip of the cantilever you know that what is the formula.

So, we will see whether this formula actually works here or not, so now the boundary condition for this. So, will follow usual theory of elasticity approach where the we need to have the boundary conditions, but the boundary conditions are sigma r theta r equals to a and r equals to b is 0. So, these are there is no shearing stress along these part.

So, at the free this these two part so r equals to is this and r equals to b is actually this r equals to b is this so there is nothing, so sigma r theta. So, there is no force here, so these are free and sigma rr is actually r equals to a and b it is 0. So, there is also no normal force there is no a tangential force also and P the load here is essentially sigma r theta d r.

And we have likewise in the previous case we have taken it is 1, so for theta equals to 0 remember this is only for theta equals to 0. So, theta is measured from here, so this is theta. So, now, this if you remember strength of material solution this bending moment is varies with sine theta right if you remember the strength of material solution. So, bending moment at any cross section of that curved beam varies with the sine theta. So, this fact will use while approximating the stress function.

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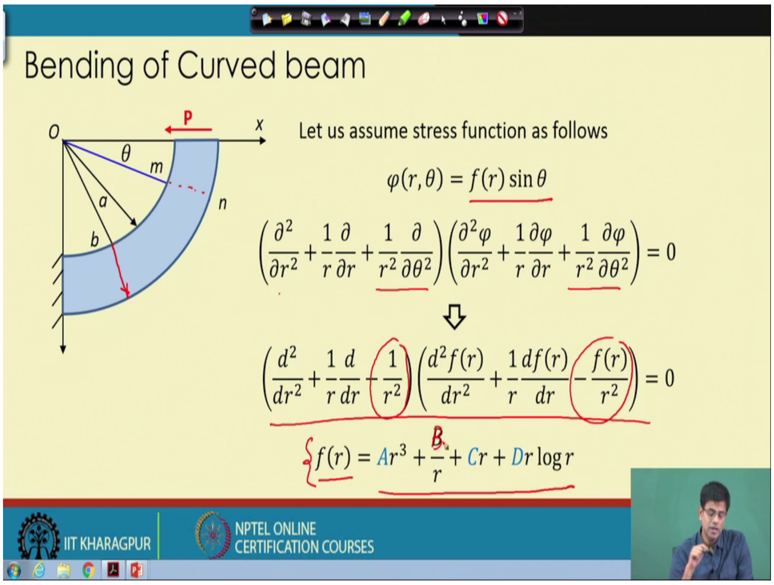

So, the stress function is essentially sure this is b and this is the cross section. So, here let us assume the stress function is f of r sine theta or form.

So, this stress function is motivated from the strength of material intuition the strength of material intuition where the bending moment where is at in a cross section is with the sine theta and it is a far is the, that we have seen in the previous case.

So, our five is not only dependent here on r it is dependent on theta also so f r sine theta now if we substitute this. So, our by harmonic equation will be full equation here so these terms are there which were not there in the previous case. So, this by a harmonic equation if I substitute this quantity so you will see that this becomes sine square theta and so, this goes there, there will be sine squared theta here.

So, this sine squared theta 0 means theta is equal to 0 which is a trivial case here so this quantity has to be 0. So, this is again you see this the differential equation is very similar to the previous case only these terms are extra. So, now this differential equation again with the substitution of t equals to e to the power r we can do a linear differential equation with constant coefficient and solution of that differential equation is of this form.

So, you can also again like in the previous case you can also check whether this constant this function f r satisfies this differential equation or not. So, we are not discussing this here rather we are interested what it produce here. So, this f r is actually the involving four constant ABCD so this is also another constant A B C D.

So, these four constants again we have to find out from the stress boundary condition which we have discussed in the previous slide. So, now, let see how this four constant produces stresses.

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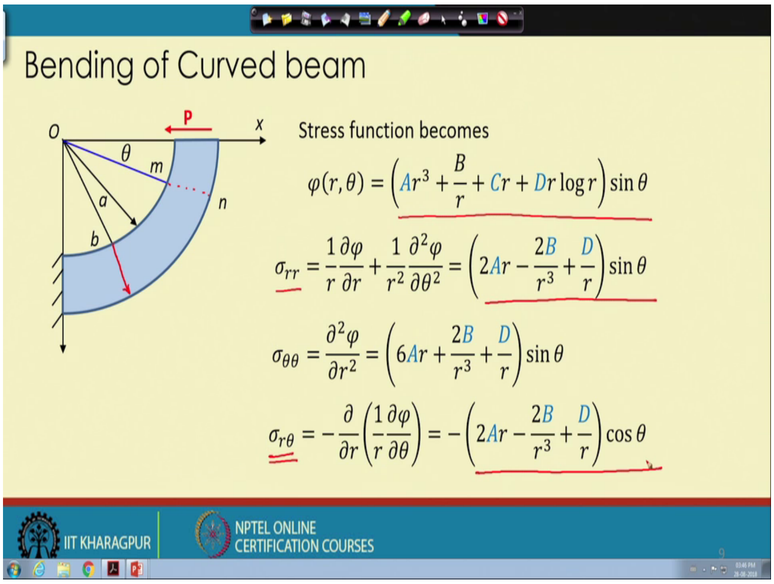

So, four this so my phi becomes this so my phi becomes this, so which is a function of r and theta even though these functions looks this function looks very simple because we have used separation of variable a vary a, so f f r into g of theta final thing and g of theta is essentially sine theta.

So, sigma rr he we can compute and sigma theta theta again we can compute and sigma r theta we can complete. So, sigma rr if we compute from this phi which is the stress function phi is of this form. So, the sigma rr is of this form which involves A B and D 3 constants and similarly sigma theta, theta which is del square phi by del l r square which in also involves AB and D.

Now, contrary to the previous case the sigma r theta is not now 0 so this is because phi is a function of r and theta also, so this quantity is non zero and then this quantity becomes this. So, you see all stress components are involved in three constants A B and D. So, we need to find out these constants A B and D with the stress boundary conditions, so let us see how we can find out the stress boundary conditions these constants.

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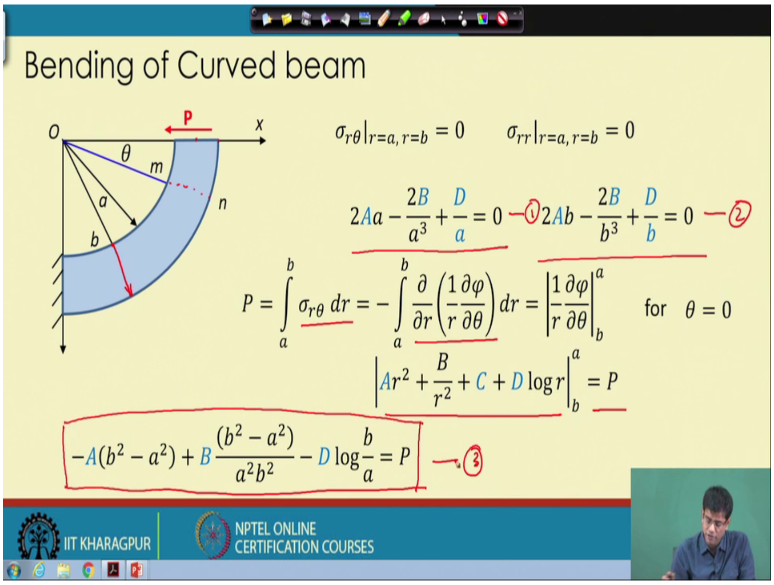

So, the first boundary conditions is essentially the sigma r theta r equals to a and r equals to b there is no tangential forces along these lines along the radial boundary and the sigma rr that is there is no normal forces is 0.

So, if I just put these two conditions so essentially we substitute in the previous stress expressions and then this stress expression becomes this. So, we substitute r equals to a and r equals to b so we get this now the third condition that is the at theta equals to 0 this stress resultant becomes p. So, sigma r theta integral sigma r theta dr from a to b so from a to b here this becomes P at theta equals to 0.

So, sigma r theta is this so essentially del r by 1 by r into d phi dr which is in r theta and this if you integrate then again this quantity will be from a to b. So, this is my this thing and then if I substitute d phi d theta d phi d theta and divided by r then I get the this quantity which is this and this is equals to ma. So, minus actually have interchanged the limits so from a to b, I did the b to a because it is a minus.

So, this becomes this now if I substitute now b and a here, so this is my final form of the equation where which I need to solve. So, I have three equations three unknowns again like is the previous case I have three equations three unknowns. So, what are those so this is 1 this is 2 and this is 3, so we can find out these three unknowns so and finally, compute the stresses.

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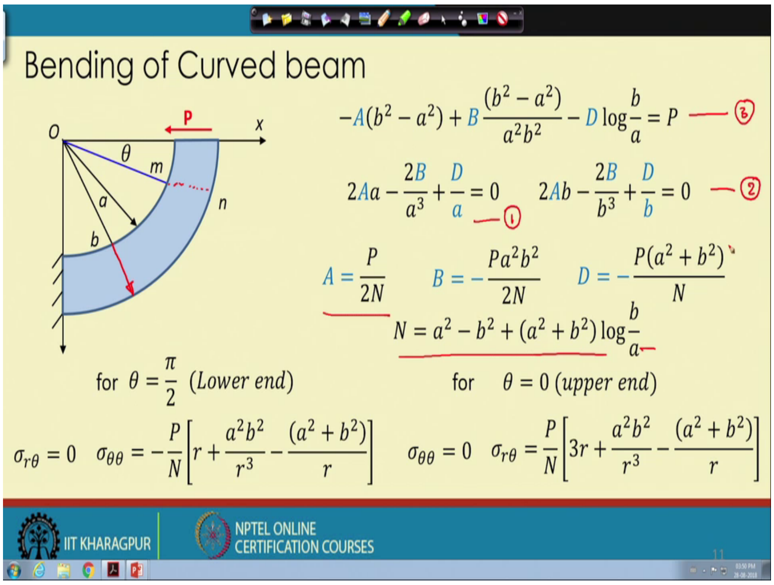

So, with this we can find out, so this is my first third equation essentially and this is my first equation and this is my second equation. So, we got this equation from the boundary conditions that is obtained from the boundary conditions obtained from the a stress boundary condition.

So, if we solve these three equations we get A of this form again there is an N, so N is essentially composed of a and b and then B is P a square b square by 2 N and D is P into a square plus b square by N. So, here once we know A B and D we can compute all stresses like is a in the previous case.

So, if you now look carefully what the stresses at the two ends of the hum beam. So, yeah if you look theta equals to 0 that is at the upper end of the beam if you look then at this upper end actually the sigma theta theta sigma theta theta will be 0.

So, sigma theta theta will be 0 so that means, these along these lines in the normal stress along these lines will be 0 or this will be 0 but sigma r theta. So, r theta which is the along this so this will be nonzero which is your P by N 3 r a square by b square r cube minus a square plus b square r.

So, which is involving all those constants A B and D and N is essentially the same thing here what we have seen. So, now see in the lower end on the contrary in the lower end where theta equals to pi by 2 is only the normal stress exists. So, this normal stress exist, but shear stress is 0, so this can be easily verified from the stress expression that we have obtained.

So, if you look carefully in the previous slides, so that the stress expression you see e for the theta equals to pi by 2 these this normal stress is actually this is in terms of cos theta.

So, this actually sigma r theta will be 0, because it involves cos theta. So, um, but sigma theta theta which involves sine theta will be nonzero, but at theta equals to 0 the situation is just reverse sigma theta theta equals to 0 and sigma theta is nonzero.

So, basically likewise for the straight case the cantilever case, we have also found out here the stresses compared to the stresses bia straight approximating the stress function or a specifically the solving the bi harmonic equation.

Now, while approximating the stress function here we used our intuitive knowledge of strength of material, where the stress function is function of essentially will use the separation of variable.

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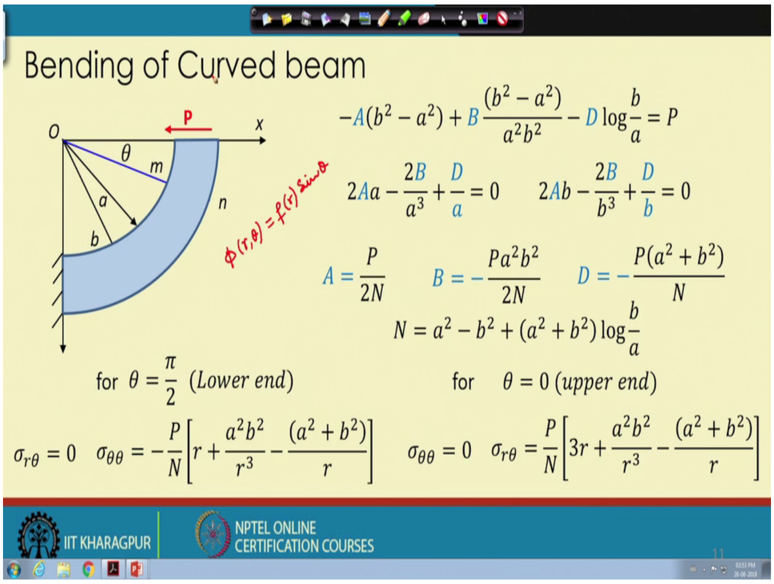

Where the phi is a function of r theta can be written as f of r sine theta. So, you can use in instead of sine theta you can use cos theta also if there is a couple you can use cos theta.

So, which will give you the solution in this manner I would suggest you to consult the book I have suggested in the previous lectures that is the demotion coin good year the theory of elasticity book. There there are several examples are there how such stress functions can be taken, and then how these these stress function represents certain type of solution so as we discussed in the previous class also.

So, how to find out these stress functions it is not just a there is no specific rule, it depends on the how the stress conditions for a specific problem. So, based on the specific problem, so how the stress conditions are we choose the stress function and it requires experience as well as your practice.

So, now in the next class what we plan is that likewise in the previous examples for in case of a straight beam cantilever beam, we will also derive the displacement from these stress expressions with involving the stress boundary involving the displacement boundary condition, where we also use the Hookes law for material. So, today I stop here, so we will meet in the next class.

Thank you.