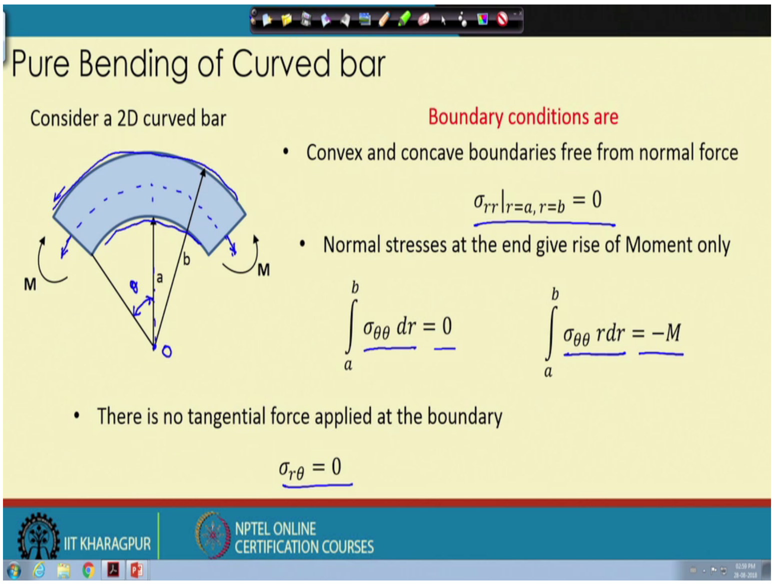
Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 35
Problems in Flexure (Contd.)

Welcome, so we are in the module 3, where basically we are discussing the boundary value problems in elasticity and we are discussing flexure Problems in Flexure. So, in this is lecture number 35, so in the previous two lecture, what we have discussed is the rectangular beam or a 2 dimensional rectangular body or 2 dimensional beam.

Where basically we use stress functions to find out the solution of the beam problem and also we have compared some of the results with the usual strength of material solutions. And in doing so, we also showed that the theory of elasticity solution gives you a more accurate result compared to strength of material approach.

(Refer Slide Time: 01:41)



Now, here in this lecture actually we will be doing for this same thing we will be doing for curved bar or curved beam, essentially under pure bending.

So, bending this problem is also solved in solved via strength of material approach where essentially we derived Winkler Bach formula which is a which we know from our knowledge of a strength of material approach that this Winkler Bach formula gives the hyperbolic stress variation. And we will see how elasticity approach or the theory of elasticity approach how this stress varies or what is the stresses if we calculate via elasticity approach. Now, this beam this is a curve beam and it is acting on a couple or the moment at the end. And this is usual pure bending formula that we have encountered in other straight beams.

So, what are the boundary condition for these beams? Boundary conditions are essentially there is no forces in these two surface; this surface and this surface there is no forces which is actually represents the sigma rr, if this is the radius at the centre of the 2 arc then sigma rr that stress along the radial direction at r equals to a and r equals to b is 0, which means that concave and convex boundaries are free from normal forces.

Now, normal stresses at the end give rise to moment only that means, the normal stresses here that is sigma theta theta, actually theta is measured from the this centroid line. So, if you draw a centroid line, so theta is at any point theta is measured this is a theta. So, this is theta, so the normal stresses.

So along these it will be the radial line; this does not create any normal force which essentially if I integrate from a to b sigma theta theta dr has to be 0 and it only gives couple because we have a moment acting at the end or the in free in surface. So, sigma theta theta r dr has to be the minus M, so this is another boundary condition.

And there is no tangential force applied at the boundary. So, there is no shear force actually acting or the essentially this kind of tangential force is not acting, so sigma r theta is 0. So, with these boundary conditions we want to solve this problem.

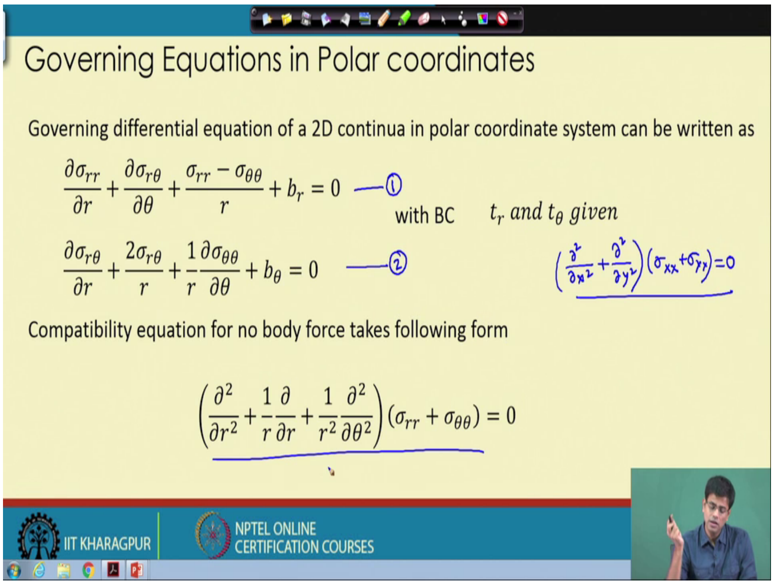
Now, look carefully that I am using here sigma rr sigma theta theta sigma r theta; all these are polar coordinate system, which is a essentially we have discussed 2 dimensional 3 dimensional cylindrical system.

When I reduced it to 2 dimensional it becomes polar coordinate system and in the polar coordinate system we also know that how it is related with the rectangular coordinate system for instance x is r cos theta y is r are sin theta.

And r is root over x square plus y square and theta is r to 10 y by x and all those things we know. So, in the general curvilinear system for instance if you remember that h 1 and h 2 that h 1 is here 1 and h 2 is r. So, that e theta e r is the e r and e theta r our basis vector here.

Now, since this is a polar coordinate system so, er and e theta is essentially not constant vector and its derivative will have some variable in terms of theta or r that we have seen in the while doing the cylindrical coordinate system. So, those formulas are essentially we need now to find out the solution of this problem. So, let us see how these things work.

(Refer Slide Time: 06:55)

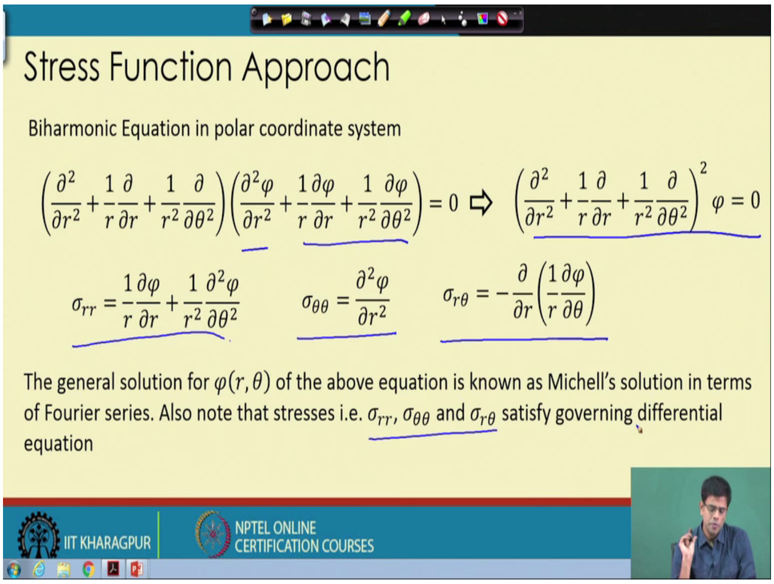


So, for instance in a general polar coordinate system or polar coordinates the 2 dimensional continuum can be written is this form this is the balance law these tools are balanced law where b r and b theta are general body force and if the body force are taken as 0, so, then this becomes the compatibility equation. See this equation is very much similar to what we get in the Cartesian coordinate system. For instance, in the Cartesian coordinate system we get del square del x square plus del square del y square sigma x plus sigma y equals to 0.

So, this is the compatibility equation we get in the Cartesian coordinate system and this is sigma xx sigma rr plus sigma theta theta and this is the Cartesian this thing.

So, these del square x by del x square is replaced by the polar coordinate system operator. So, now the final we also need to have the final form of this equation which is essentially this form.

(Refer Slide Time: 08:26)



So, the bi harmonic equation that we want to have, so if we this if we substitute now sigma rr and sigma theta theta of this form that is 1 by r d phi dr plus 1 by r square d square phi d theta square and sigma theta is d square phi dr square. Then we get the final biharmonic form of the compatibility equation or the popularly known as the biharmonic equation.

Now, you have to see or you can verify that that these form this sigma rr and sigma theta theta also satisfies the governing equation in 2 dimensional polar coordinate system. That is the sigma and this phi is also known as the stress function. So, this satisfies these two equations directly. So, you can also just substitute this and check its check whether it satisfies or not.

And then if you substitute it in the usual compatibility equation, so, this is essentially sigma rr and sigma theta theta, so this is sigma rr and this is sigma theta theta.

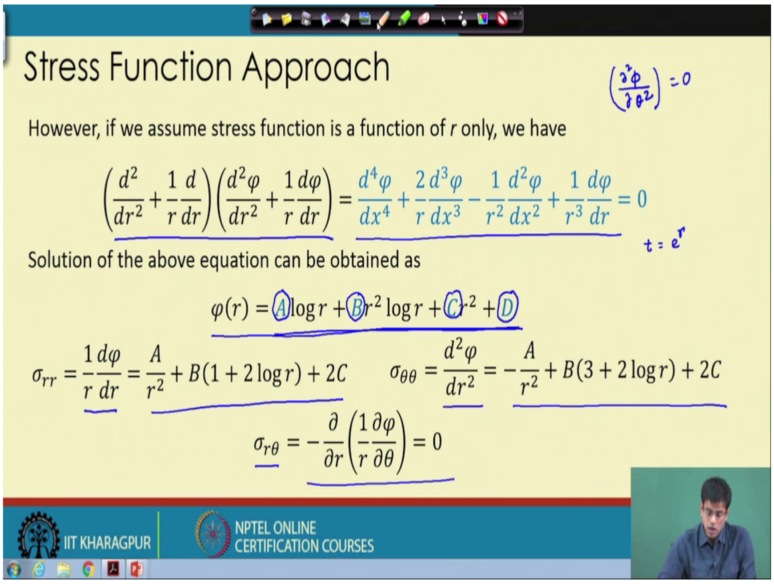
So, this if we substitute and then we take this out phi, so, this is the final biharmonic equation which is very similar to the Cartesian coordinate system except this is the in the polar coordinate system. Now, this is sigma r theta is also this.

So, this we can substitute in the governing equation and check whether this satisfies or not. Now, the general solution of this biharmonic equation is very lengthy formulation and it is known as the Michell solution, so in terms of Fourier series. So, this solution is very long solution and it contains the infinite terms and this Michell solution these Fourier series will be in terms of theta.

So, so we do not really required these things here or we will really discuss these things here rather we will take a modification of this Michell solution or specifically the first terms of the Michell solution, so, first few terms of the Michell solution.

Now, this is what we what I have already discussed that stresses of this form satisfy the governing differential equation. Now, here the phi is the stress function that we need to now concentrate on. So, let us see how this phi will looks like.

(Refer Slide Time: 11:49)



So, now this stress function if we just check this stress function from the previous thing that phi is a general function of r and theta.

But however, if I take if I or if I assume that phi is a function of r only that means, all terms related to theta the derivative with respect to theta will vanish. So, then my phi becomes the my biharmonic form of the equation will take this form.

So, they see there is no derivative with respect to theta, so they essentially d 2 phi by d theta 2 cancels becomes 0. So, that is d 2 phi by d theta 2, if you put in the previous biharmonic equation this becomes 0.

So, this becomes this equation and if we if I just multiply or expand this thing and then this is my final differential equation. Now, see this differential equation this is the ordinary differential equation, this is not a partial differential equation and this differential equation can be.

So, it is solvable actually if you substitute a new variable for instance t equals to e to the power r then this differential equation can be converted to a differential equation with linear differential equation with constant coefficient. Now, this solution of this differential equation can be thought of can be written in terms of this that is A log r B r square log r plus C r square plus D.

So, this is this exercise you can also try and c whether they satisfies this differential equation or not. So, our phi is of this form finally, our phi is of this form and the stresses naturally will be 1 by r d phi dr, because 1 by r square d square phi d theta 2 d theta square are 0.

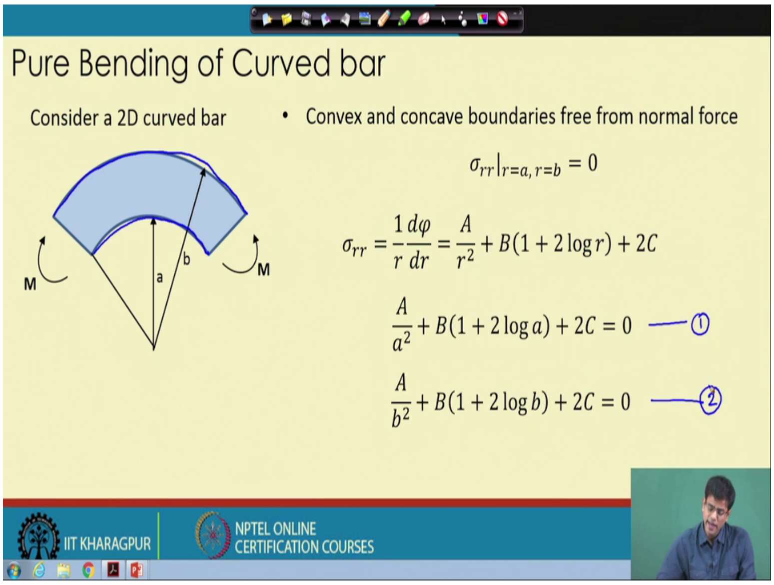
So, my stresses are a B c here three constants and sigma theta is del square phi by del r square which is also this and sigma r theta which is essentially 0 because d phi by d theta is 0. So, see we have a four constants, so similar to the previous approach these four constants we need to identify from the stress boundary condition that we have seen for the pure bending.

Now, our first job is to identify these constants what are these values. Now, this is the boundary condition that we are talking about we have three basic boundary conditions here which is sigma rr at r equals to a and r equals to b equals to 0.

And there is no normal stresses at the end and no normal forces at the end only moment is there and there is no tangential force. So, that is sigma r theta equals to 0 which is already satisfied by our stress function where we have already seen that sigma rr is essentially the sigma r theta is essentially 0 this is due to the form of the stress function we have considered.

So, we have the what is the form the basic thing is that our stress function is function of r only there is no variation in theta. So, that is the basic thing which leads to sigma r theta or that shear stress in r theta system is 0. So, which also satisfies our third boundary condition now from the first to boundary condition then we have to find out the constants four constants abcd.

(Refer Slide Time: 16:10)

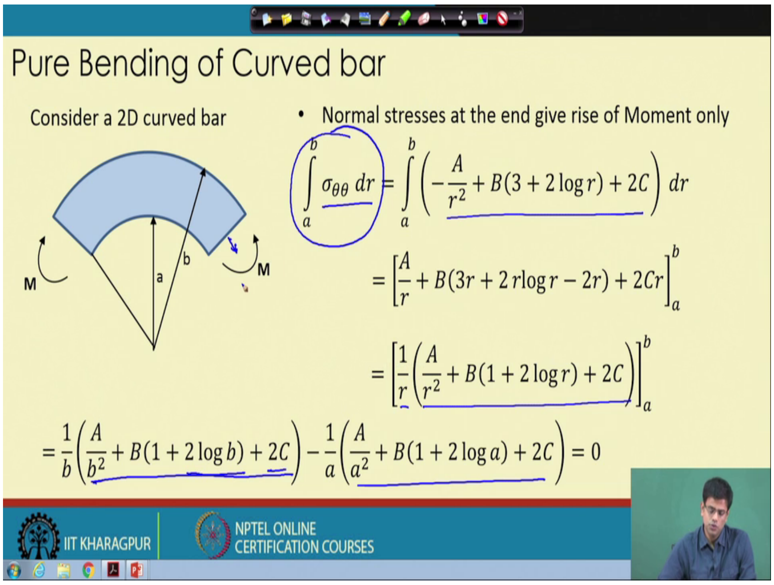


So, let us see the first boundary condition tells us that convex and concave boundaries are free from normal forces, now that is sigma rr if I right.

So, r equals to a and r equals to b it has to be 0 so that means, this end this end and this end there is no sigma rr. So, rr means along this the along the radius radial direction.

So, this leads to 2 equation, this is the first equation and this is the second equation. So, then again we have a third equation we need to find out a third equation which is through the normal stresses at the in give rise to moment only.

(Refer Slide Time: 17:03)



So, the moment means first of all we have to show that it does not give you any stresses or normal force in this axis. So, there is no normal force it is only the moment it gives.

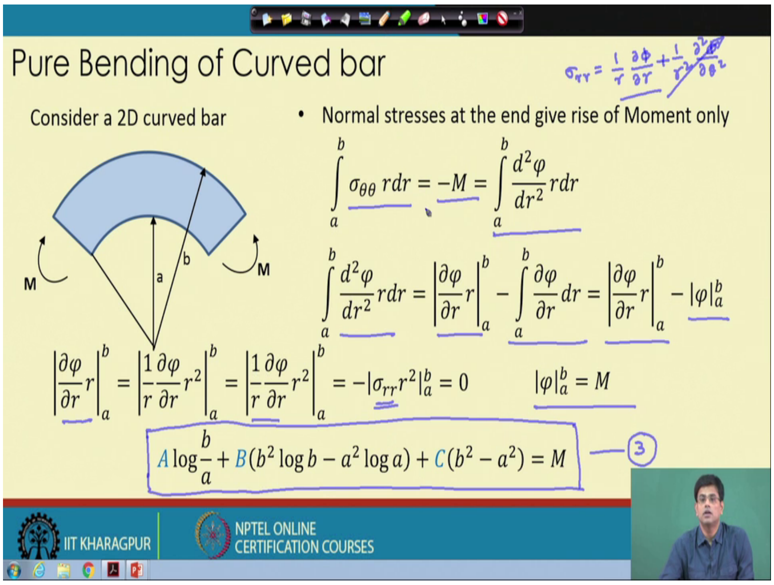
So, if I just integrate now sigma theta dr and substitute this then if I just do simple integration which is this is the form of sigma theta. And then if I do the simple integration from a to b then it seems that we have this form and this form is actually we have deduced in the previous slide that is if I substitute a and b in place of r.

So, this A by b square plus B into 2 log b plus 2 C and then 1 by a by a square plus B into 2 log a 1 plus 2 log a plus 2 C. So, now, this is 0 from equation number 1 in the previous slide and this is 0 from the equation 2 of the previous slide.

So, this quantity finally, this quantity becomes 0, so these also satisfies our stress function also satisfies this boundary condition. Now, this once I satisfy this boundary condition then it proves that our stress function does not give any rise to the normal force. So, it satisfies the equilibrium also, so that means, we need to satisfy the other thing now.

So that means, that these normal stresses are only giving rise to any normal stresses gives rise to a moment only there is no normal force. So, normal force we have checked it does not give.

(Refer Slide Time: 19:17)



So, our next boundary condition because its normal stresses at the end give rise to moment only.

So, if I just now integrate sigma theta r dr which is equals to m. So, now sigma theta if I substitute here which is d 2 phi by dr square. So, phi is a function of remember in our case that phi is a function of r only, so this becomes our this integral become this. So, we need to evaluate this integral what is the form.

So, now, in doing so we just do the by parts; so, integration by parts where this is the first term and this is the second term. So,, so essentially the first function is r and then second function is d phi d 2 phi by dr square.

So, this is the integration by parts now this integration by parts if I write. So, this becomes integral d phi dr the second term becomes phi so this is clear now this term is also we have to evaluate. So, this term if I write it in this form then 1 by r phi dr by r square.

So, I just multiply it r and 1 by r, so this term is essentially our sigma rr if you look carefully that sigma rr is our sigma rr is actually 1 by r d phi by dr and there 1 by r square into d 2 phi by d theta 2.

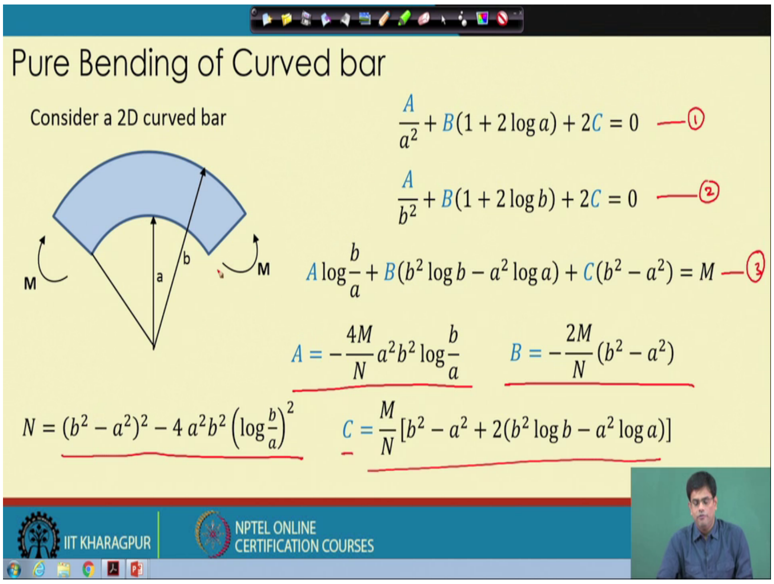
So, this term will be 0, so only this term is there so this becomes our sigma rr. Now if I substitute b to a sigma rr is essentially b to a 0 because that is our another boundary condition which we have already satisfied earlier so this term becomes 0.

So, now, only this term is left and so minus of phi a to b is equals to minus M. So, this becomes phi a to b becomes M so now, if i substitute my phi now.

So, then A log b by a plus B into b square log by log b minus a square log a plus C into b square minus a square equals to M. So, this is my third equation that we are looking for, so this is my third equation.

So, now we have a 3 equations and we have 3 unknowns you see the unknowns are ABC. So, if we can now find out these 3 unknowns we can compute the stresses very easily. So, let us see what are the form of these unknowns.

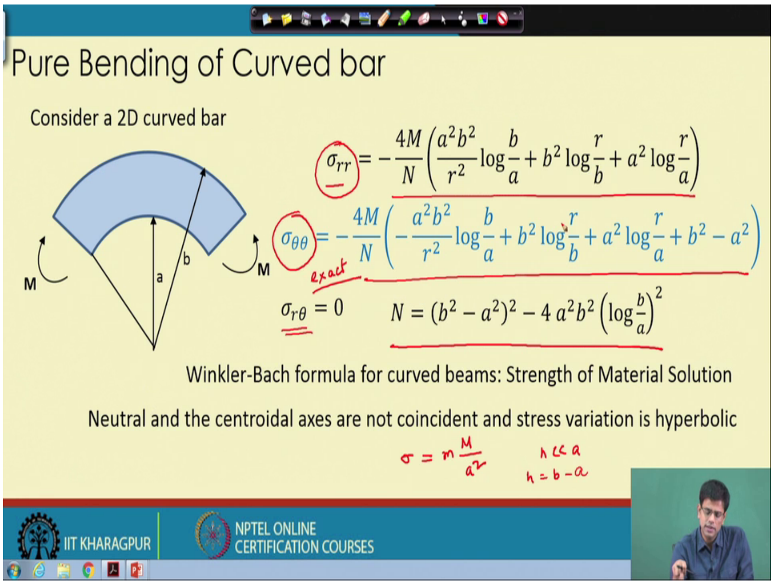
(Refer Slide Time: 22:44)



So, here the unknowns the equation 3 equations that we have seen is written. So, this is the first equation that we have seen earlier and this is the third second equation and this is the third equation and we have a unknown 1 2 3 unknown A B and C. So, now if I solve ABC, we get of this form; so these forms are very lengthy form.

So, C is of this thing and then we have also used N is essentially this. So, these with these three constants we can now compute stresses and this stress is essentially what we have seen is comparable with the Winkler Bach formula that we have seen in the strength of material.

(Refer Slide Time: 23:54)



So, let us see what are the stresses so the stresses look like this. So, in case of a pure bending this sigma rr is this. So, here N is also a constant actually which compose of b and a.

So, and sigma theta is also minus 4M I N which is also a very long expression. So, this sigma rr sigma rr sigma theta is essentially we can compute from the theory of elasticity. So, and sigma r theta is essentially 0.

So, if you remember strength of material solution of this problem which was the Winkler Bach formula. This Winkler Bach formula gives us the stress variation in general this stress variation is a hyperbolic equation.

And this stress variation is essentially sigma equals to some we can write it in this form some constant say M into capital M by a square is our this thing, now inner radius. So, this is hyperbolic stuff now hyperbolic kind of equation.

Now this sigma theta is equivalent to this sigma for the strength of material solution. Now if we compare these two solutions, so this if I term it exact because this is exact sigma theta and this is a Winkler Bach solution.

So, if we compared these two things then we can easily see that these two approach the Winkler Bach formula gives a very nice approximation or very accurate approximation to the exact result. While the linear variation when we approach that these thickness of these that is b minus a here the h is much less than the outer radius that is a.

So, h is much less than the outer radius h is a essentially h is b minus a. So, this b minus a is much less than a then we can use this kind of formula. So, where we directly use the straight beam formula pure bending of a straight beam, so that gives me the linear stress variation and which is pretty inaccurate actually.

And then if we take the curvature effect properly then we get the hyperbolic a hyperbolic stress variation. And then if we go a little bit more for the exact theory of elasticity solution then we get the actual stress variation which is of this form.

Now, as I have told earlier this hyperbolic stress variation is quite good approximate quite good with the theory of elasticity result now. So, what we have seen here, so the pure bending case we it is very popular with us. So, pure bending case for the curved beam which is generally done in the strength of material approach, which is a Winkler Bach formula.

So, this formula is very close to the exact theory of elasticity solution and theory of elasticity solution how we have achieved the theory of elasticity solution we have achieved with a stress function approach. So, in the stress function approach we consider the first few terms of the Michell solution which leads to our stress function is dependent on r only.

So, it is independent of theta and since this is a pure bending case. So, which is satisfying our boundary conditions, so by this procedure we achieved sigma rr sigma theta and sigma r theta you see only sigma theta we can compare with the usual Winkler Bach formula for the strength of material solution. But also you got the sigma r at the variation of the stresses along the radial axis.

So, this gives you better insight of the problem via theory of elasticity approach. So, I stop here today and I will continue with the derivation of the displacement for pure bendings in the next class.

Thank you.