Theory of Elasticity

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Lecture - 33
Problems in Flexure

Welcome this is module 7 and we are going to start the solution of boundary value problems in elasticity. And in this module, we will discuss about the Problems in Flexure. Flexure is means basically the bending. The bending problems, we are doing from our knowledge of mechanics from the basic mechanics, we have done the bending problems.

So, the objective of this module is to find out what are the basically the difference between the theory of elasticity approach and a usual strength of material approach where we have already solved bending of beams or the flexural and found out the flexural stresses. What are the values of shear stresses and all those things we have already found out, But if you remember the bending problems for beams, we have solved for one dimensional body.

So, where the cross section and the cross sectional geometry is very less compared to the length direction length of the beam. So that means, less means the effect of stresses and all other things are very negligible. So, that assumption actually also the standard Euler Bernoulli beam assumption for pure bending that is the plane sections remain plane; before bending and after bending and all those things we know from our strength of material knowledge.

But what here, what we will do? We will just extend it to for two dimensional beam if the beam is two dimensional; that means, the thickness direction or the width direction or any of the direction. If we relax that condition or if we take the changes or the proper values of the stresses in that direction so, what will happen in that case for our beam theory?

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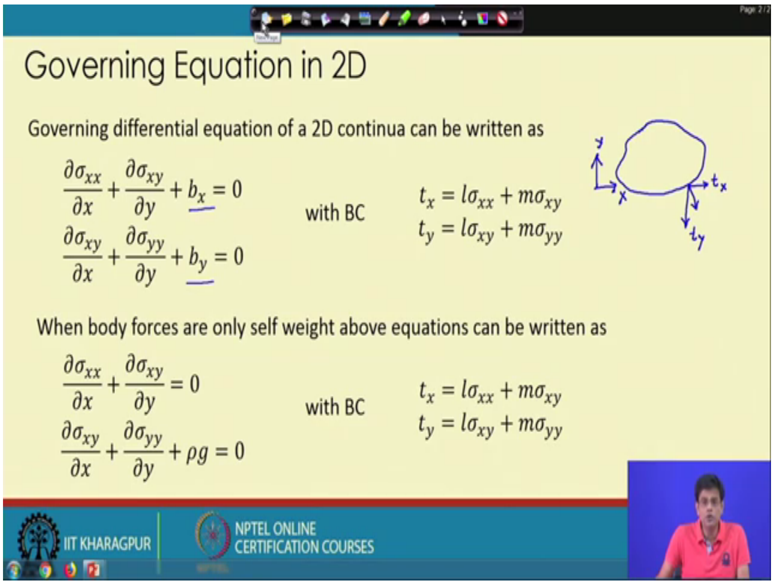

So, to start with we will go for some basic recap which already you have done for plane strain and boundary value problems. So, governing differential equation for a two dimensional body is generally this. So, these quantities are b x and b y are body force and with this is a you have to have the boundary condition which is the traction boundary condition something like this.

So, if this is the normal direction then you have the d x and d y it is the traction forces and this is your x y system. So, we will be basically doing for 2D system right now. So, the differential equation or the basic governing equation for the solids will be two dimensional and so, we know the what are the these two differential equation.

Now, these b x and b y these are actually the body force. Now, if we only take the self weight of the beam as body force; that means, the or self weight of the continua is the only body force; that means, it is rho g is the self weight and rho is the density. So, this will look like this. So, b x component will be 0 and by will be rho g. So, with this and two boundary conditions this l of sigma x x m of sigma x y; so, l m we know that is the direction cosines. So, we all aware of these boundary conditions right.

So, now, we know these governing differential equation. So, as well as we also know from our previous lectures that is the compatibility equations.

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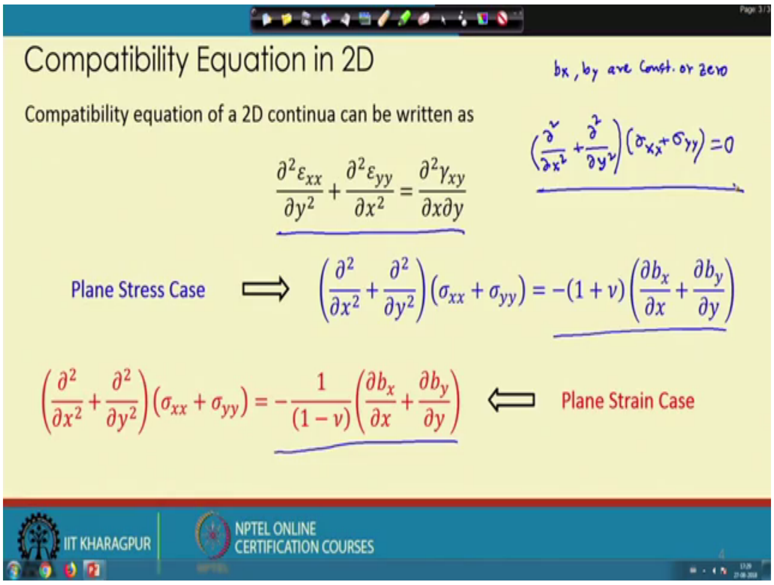

So, the compatibility equations for we have discussed the in detail this. So, I will just recap this compatibility equation which we will be using here or. So, this compatibility equation for two dimensional body is related to this which is del square epsilon xx del y square del square epsilon y by del x square equals to the del square gamma xy del x del y.

So, these equations if we now substitute with the stresses that is if we invoke the Hooke's law for isotropic elastic solids with poisons ratio and young's modulus. And if we write these strains in terms of stresses, then we get for a plane stress case and plane strain case two different right hand side and which will lead to this.

Now, interestingly if you look carefully that if b x and by at 0, then these two conditions becomes or even bx and by are constant, these two these compatibility country conditions are same for the plane stress and plane strain and which will simply be like this that is del square by del x square plus del square by del y square into sigma x x plus sigma yy equals to 0. Because the even if b x and b a b y are constant or 0; b x by are constant or 0. So; that means, the if it is constant the first derivative will be 0. So, these quantities essentially will be 0. So, these quantities will be 0 and this will be the compatibility equation.

So, essentially you have this compatibility equation and the boundary the governing equation which we have discussed in the previous slide. So, these two, these three equation along with the boundary condition, we have the governing equation for the two dimensional solids. So, since here we will be talking about the stress formulation. So, this compatibility equation is important for us. So, this also has been discussed in the previous lecture.

Now, if we have remembered what is the stress volume formulation or the stress based approach for solving the two dimensional problem or three dimensional problems. So, we will be mostly talking about the stress function approach.

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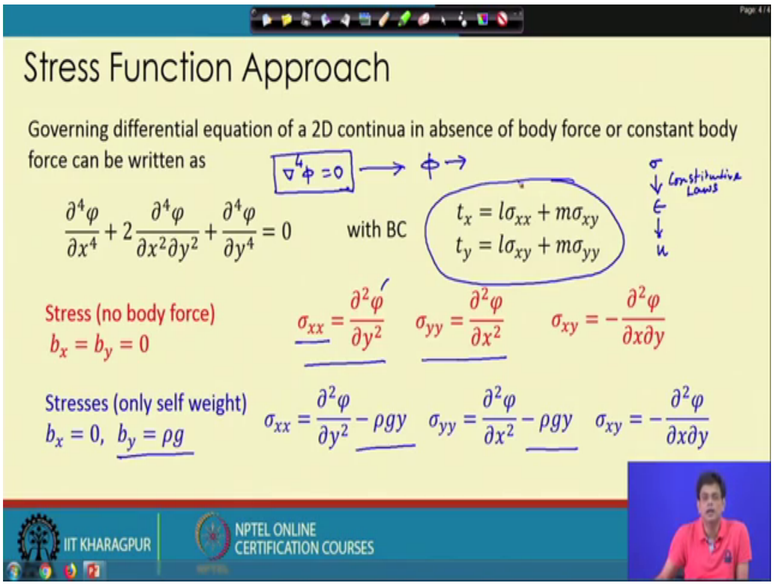

So, stress function approach also we have discussed which is a governing equation for two dimensional problem will lead to a by harmonic equation which is a del to the power 4 phi equals to 0. The short form is essentially del to the power 4 phi equals to 0. So, this is the bi harmonic equation we solve and phi is essentially the stress function and this stress function when the body force is 0 is defined like this. So, sigma xx, remember this is in terms of y and sigma yy is in terms of x. So, del square phi del y square is sigma xx del square phi del x square is sigma yy and sigma x y or the shear stress is essentially del x del del square phi by del x del y. So, this phi is essentially known as the stress function

Now, this can also be suitably modified for the cell (Refer Time: 09:04) of the body. In case in that case by will be rho g and this will be just minus rho g h and minus rho g h in sigma xx and rho g y and rho gy in the sigma xx and sigma yy. So, this by harmonic equation if you look carefully, there are a few important thing here. So, if we want to find out phi or if you have taken phi properly you see, there is no need to invoke material constant or material knowledge of the material.

So, essentially the stresses when we find out stresses, becomes independent of material. So, this is very interesting for this formula because formulation because this bi harmonic equation if we solve, we essentially get the phi. Now phi, from this phi we take the appropriate derivatives of phi and then we get into this sigma xx sigma yy and sigma xy.

So, naturally we can approximate phi as a polynomial and then find out the all stresses. So, this sigma stresses will be essentially independent of material. So, this is very interesting fact that we do not need to know the material a priori to find out the stresses. So, that is one important issue here, but one should also remember that that does not mean that material does not come into the phi material does not come into picture.

So, essentially phi is you know the stresses are independent of material constant, but when you start finding out the displacement; there the material laws or the material behaviour comes into picture. For instance, once you find out the stress sigma, then you probably want to find out the epsilon or the strains and when you do find out the stains you need to incorporate the constitutive behaviour. So, constitutive behaviour, constitutive laws.

So and when you find out the strains then essentially you find out the displacement. So, that is a complete cycle. So, first you find out stress by solving the bi harmonic equation and then from that stress you essentially find out strains where actually you are plugging the constitutive behaviour or constitutive laws. And then you solve for the displacement or the uv and w for the our case here uv. So, remember that these solution of phi also incorporates these boundary conditions.

So, the boundary conditions needs to be satisfied. So, this is in a nutshell the stress function approach. We have already discussed this for a stress function, this stress function approach. Now what we will do? We will give some example of stress function.

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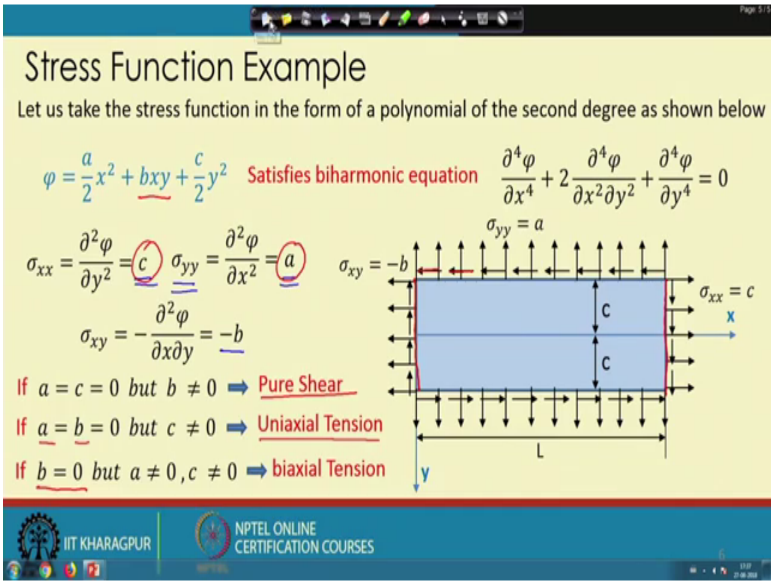

For instance let us consider a second order polynomial as a stress function. Now if you look carefully, this bi harmonic equation is a 4th order equation. Now any second order polynomial will automatically satisfies this equation. So, not only that any third order polynomial it will also satisfy this equation.

Now, if we compute the stresses which is essentially del square sigma xx is del square phi del x square at del y square which is if you take the derivative you get the constant which is c. Now similarly sigma yy you get a so, and sigma x y is minus b. So, this three constant represent different type of stresses in the boundary of the solids. For instance if there is a rectangular body two dimensional rectangular body and in this direction this is a x direction and this is y direction and then, essentially you have sigma xx is c; that means, there is a constant stress in this phase and constant. So, this is essentially the tension and compression the sign does not matter because I can take the constant negative or positive whatever it is.

So, here so,; that means, tension and compression both are covered through this thing and similarly the sigma xy, the shear stress is essentially this these are shear stress in on the body. Now if I take specifically the a and c both of them are 0 and then if b is not equals to 0; that means, my stress function is just only b x y, then it will represent the case of pure shear. So, we know the pure shear from our strength of material knowledge. So, this will be there will be no axial force on this thing and it will represent a pure shear condition.

Now, similarly if we take a and b is 0, but c is not equals to 0, then it will represent a uniaxial tension. So, it can be reversed also if c and b and 0 and then a is nonzero, then also it will be uniaxial tension, but uniaxial tension or compression both can be modelled.

Now, if b is 0, but a and c are not 0, then it is a biaxial tension or composition. So, one of them can be negative also. So, this actually these stress functions second order second order polynomial as a stress function allow us to modify or to capture the stress test in the body in a very flexible manner.

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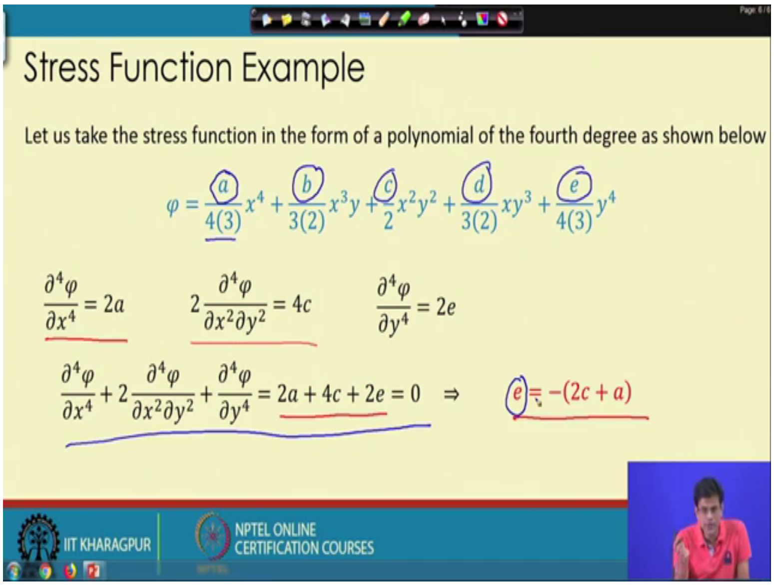

So, this will use it now another. Another stress function, we can discuss which is a 4th order stress function which does not satisfy the bi harmonic equation automatically because it is a 4th order polynomial.

So, now if we can compute these derivatives that is del square phi del x to the power 4 and all these quantities, then it can be seen that the final quantity becomes this; this has to be 0. So, from there actually we can derive that there are e is dependent on two constants. So, by looking carefully that these stress functions contains 4 such constants a b c and d and e. You see the way I have written it in a in this manner this 4 and 3 if you take the derivative this can be cancelled. So, it is not necessary you can write this a is a dash which is a by 4 into 3. So, the this is for the ease of our calculation.

So, now these 4 these 5 constants essentially are not independent. So, one of them is dependent if it has to satisfy the bi bi harmonic equation this. So, finally, we have a 4 unknown constant which is a b c and d and e can be written in terms of c and a. So, and this is the relation. So, now this if we c means what this 4th order polynomial produce. Let us see that.

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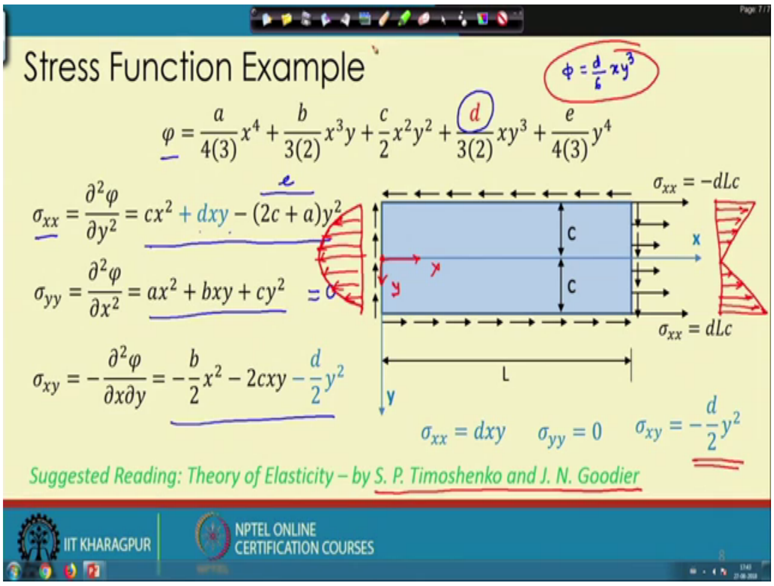

So, the stresses will look like this which is very simple because you just take the derivative of the stress function phi. So, now, this will be my sigma xx. Similarly sigma yy is x square b xy plus cy square and sigma xy is this.

Now, what this complicated stress means is that that sigma xx varies quadratically and sigma yy is also varying quadratically and then sigma xy is also very quadratically. Now for instance, if we now take the special case where the only the coefficient d is nonzero and all other constant that is a b c and essentially e; e is also 2 c plus this is actually the e which is also 0.

So, if we take only d as nonzero. So, my stress function will be d by 6 x y cube. So, now, my phi will be essentially d by 6 x y cube. So, now, if this is my stress function, then my sigma xx is essentially d x y and sigma yy this will be essentially 0 because there is no such d here and sigma xy that shear stress will be minus d by 2 y square.

So, now this if we want to check what sort of stress it produce. So, we will see that this produces a quadratic variation of the shear stress which is add the sigma xx is like this. So, if we now draw a distribution of this shear stress so, it will be like this. So, sigma xx if we draw then this is this kind of stress it is producing so, this is my.

So, if you remember this is very much similar to the bending stress that you observed from the strength of material knowledge and the shear stress is essentially quadratically varying. So, if you now look the shear stress distribution along the y, it is a parabolic and this is the parabola for the shear stress. So, where the shear stress is maximum at y equals to 0 and then we get the 0 at the fibers. So, this is the my x x equals to 0 y equals to 0. So, this is my x axis and this is my y axis. So, essentially the distribution of shear stress is like this.

So, now you see that these captures the our knowledge of strength of materials. This function essentially captures one of our knowledge of stress function. So, there are several other types of stress functions also are there which you can go through by this book, where there are several derivations are several definitions of stress functions are there. So, now let us go for a simple beam problem where the beam is loaded with a point load or essentially the shear force at the free end.

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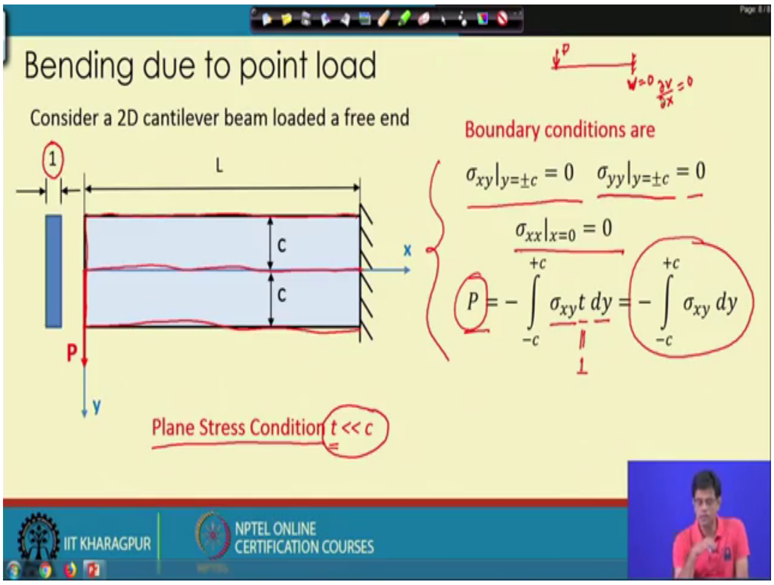

Let us consider a cantilever beam where this cantilever beam is the shear force resultant is essentially P. So, this shear force resultant is acting on the beam and the beam is fixed in this side. So, we will evaluate this condition in a later stage. Now, you see here the beam is essentially three dimensional it should be ideally three dimensional, but we are considering two dimensional case. And so, the thickness of the beam is much less than the depth of the beam or which is essentially 2 here 2 c here. Now, to make our calculation easy we take t is 1 unit. So, this is 1 unit. So, thickness of the beam we take the one unit which is which can be generalized for t non equals to 1 or any value of t, essentially what it means that the depth of the beam is much greater than the thickness or the z direction dimension of the beam.

So, which invokes the plane stress condition? So, we will be talking about here now this plane stress condition or the plane stress beam. So, what are the boundary conditions if this beam is the this beam we need to solve then, what are the boundary condition? See the boundary conditions for these from the strength of material concept we were doing it for one dimensional case and one dimensional case the beam is fixed here which is essentially like this. So, this was our idealized strength of material beam.

So, now here what the boundary condition we used to have. So, we used to have that here the displacement and slopes are 0, but right now what we are doing here? We are doing the stress based approach. So, stress based approach means we need to find out the boundary condition in terms of stresses. So, here the original boundary condition if you remember that u equals to 0 and or v sorry v equals to 0 and del V del X and everything equals to 0 at this point. So, slope and deflection is 0 at the built in end or the fixed end. So, this condition is in terms of displacements. So, V is the vertical displacements of the beam which is along the y axis, but here what we need is the stress based boundary condition.

Now, you see the boundary of the surface in this surface, there is no load. So, there is no load. So, that implies that sigma yy and sigma at plus minus c it is 0 and sigma xy also at plus minus y equals to plus minus c; that means, this zone it is 0 and sigma xx at x is equal to. So, this phase is there is no horizontal or the along the axis there is no load this implies this. And then, p is the total resultant shearing force along the y axis which means that that sigma x y t into dy should be integrated from minus c 2 plus c and this has to be equals to p. So, this t we have taken as 1. So, this becomes a final boundary condition which is P which is equals to P.

So, now this is these boundary conditions are very different from what you have done here. So, this is based on the stress or stress boundary conditions. So, now, we will see how we can solve this problem efficiently.

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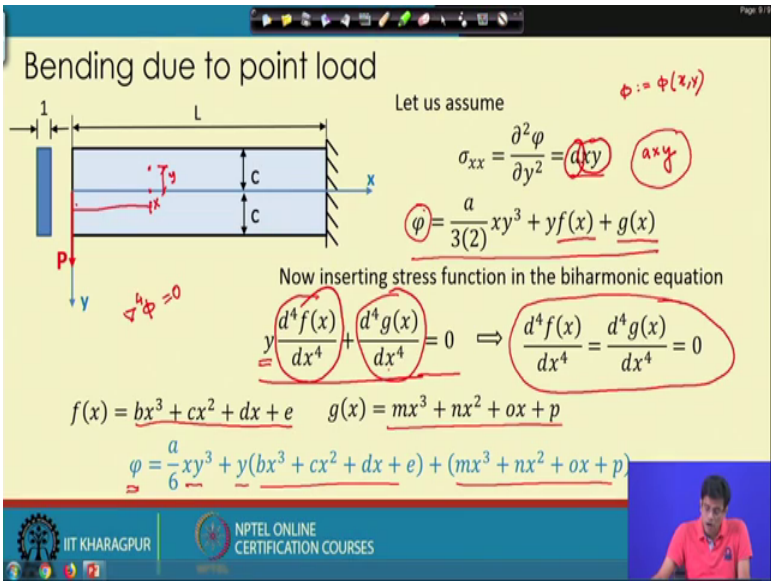

For instance, let us see the stress function that we consider it is axy. So, why a x y because if you see the moment at any cross section of this cantilever beam say at point here which varies alongs x. So, P x is the moment now this P x we also know that moment varies along the cross section of the at a different point of the different point in x, this moment varies as P x.

Now, if this moment has to be found out here which is at a distance y, then that also we know from our flexure formula if you remember that sigma xx is my by i. So, m is the moment and it depends it varies linearly. So, that is why we take this constant a as the unknown constant and the variation of this stress is x y. So, we take sigma xx is a x y. So, this is from our earlier knowledge.

Now, if we now integrate this stress or the this differential equation del square phi by del y square equals to a x y, then we get this kind of function. Now remember here phi is a function of x y. So, phi is a function of x and y. So, if you integrate it. So, this will be simply integration will be in terms of y. So, a x y cube by 6 and then the first constant will be f and then second constant will be g both of these will be function of x because we are we are integrating with respect to y.

Now, if we now substitute this phi in the bi harmonic equation that is our original equation dels 4 del 4 phi equals to 0, it leads to this equation y into this del to the d to the power 4 so, and since it is a one variable. So, d d to the power 4 by d x to the power 4 fx and d to the power 4 gx dx to the power 4 equals to 0.

Now, see this quantity is purely a x dependent quantity and this quantity is also dependent on x and this multiplied by y plus a purely x quantity has to be 0. So, to satisfy this equation for x y this must be 0 and this must be 0; otherwise it cannot satisfy. So, this is the condition for which this equation is a valid. Now, if we individually take this equals to 0 and integrate to find out fx, fx will be cubic polynomial. So, this will be this and gx will also be this.

Now, finally, our phi which is which we assumed in this which we assume axy as a del square phi by del y square and then integrating phi here. And so, this phi becomes this where a by 6 x y cube and then y into this quantity and this quantity. So, this becomes our complete phi for the beam we are talking about. Now, let us see what happens to this phi if we find out the, if we invoke the boundary condition of this thing.

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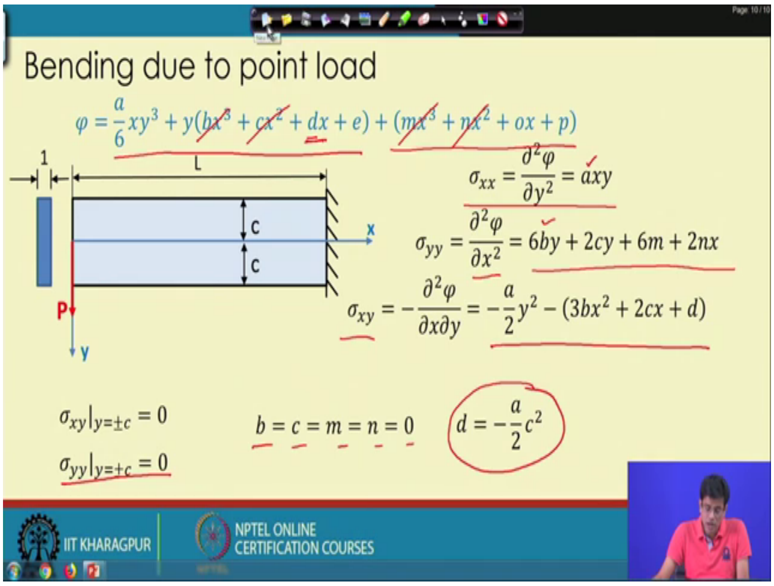

So, this was the phi we were talking about this was the five complete phi. Now this was the stresses and so, we can find out; once we get the phi, then we can find out other stresses for instance sigma y by del square phi by del x square which happens to be 6 by plus 2 c y 6 n plus 2 nx.

So, now similarly sigma xy is also we can find out. Now these constants these a b c. So, here actually a b c d e m n o p; all those are constants. So, all these constants we have to find out from the boundary conditions. So, boundary conditions we have discussed the boundary conditions. So, if we invoke these boundary conditions into this, so, it leads to b c m n 0 is essentially 0. So, which again if we plug into these conditions, these will leads to this d equals to this minus a by two c square.

So, you see so, b c m n all quantities goes here bc mn these are 0. So, now, only d is there here d is there and d is in terms of a. So, this also we have substituted in terms of a. So, now, let us invoke the other boundary condition, what if we invoke the other boundary condition, how it looks like.

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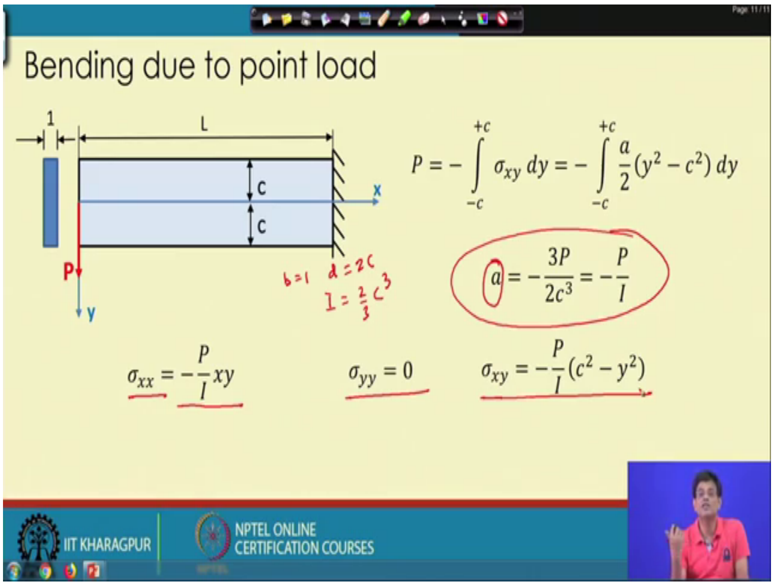

So, in the last boundary condition we had this. So, which is essentially which is essentially the resultant shearing force. Now this shearing force sigma xy dy. So, we substitute sigma xy if you remember the sigma xy, then we can if you remember the sigma xy which is minus a by two y square and bc all of them are 0 b 0 c 0. So, d is in terms of a minus a by 2 c square. So, sigma xy is essentially minus a by 2 y squared minus d by minus d. So, minus d is essentially minus a by 2 c square. So, that we substitute here and this goes to and if we solve this, then we find out the integration this constant a which is of this form.

So, now here this 3 P by 2 c cube and I is essentially bd cube by 12 that we know from our formula. So, b is essentially one here and d is essentially 2 c. So, if we find out the moment of inertia, then which will be two third c cube. So, this two-third c cube, we substitute it and moment of inertia I. So, it is minus P by I. So, now, if you see that sigma xx is minus P by I xy sigma yy is 0 and sigma xy is this. This actually coincides with the our strength of material knowledge of shear stress bending stress or the flexural stress sigma xx and the sigma yy.

So, this is the same result that we have obtained in the from the strength of material concept. Now this result we obtain from the stress function approach or the theory of elasticity approach. Now we will see what this in the next lecture we will see what extra things it provides the theory of elasticity approach what extra thing it provides to us. So, I stop here today and we will discuss the extra these concepts to find out the displacement and strains in the next class.

Thank you.