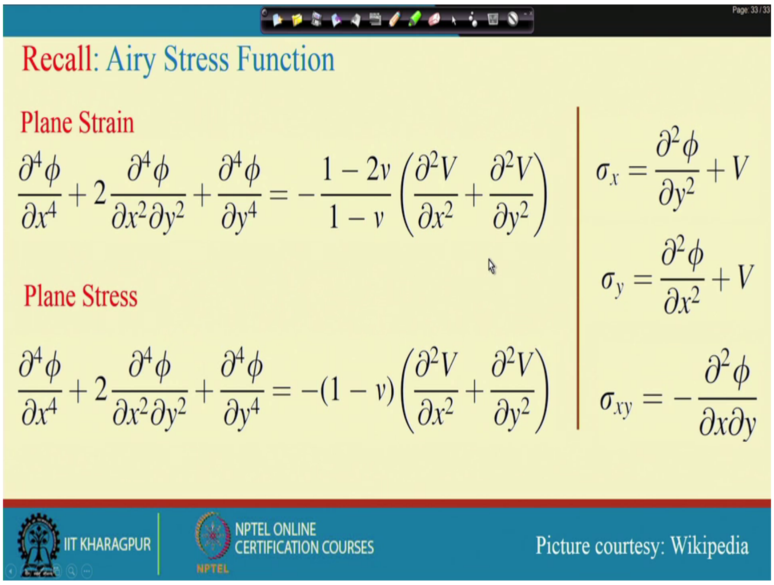
Theory of Elasticity

Prof. Amith Shaw
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 30
Solution of Boundary Value Problems (Contd.)

Hello everyone, this is the fourth lecture of this week what we discussed last class is we just introduced airy stress function. Today we will see that how the airy stress function can be used to solve boundary value problems.

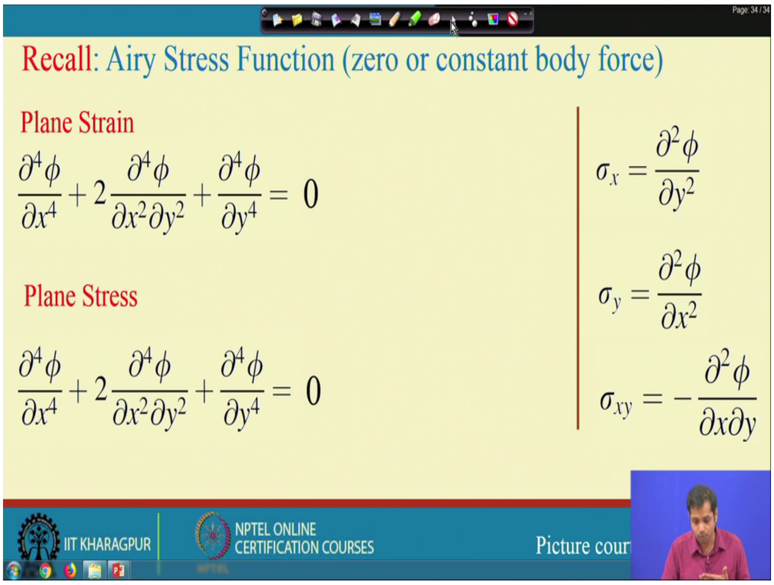
(Refer Slide Time: 00:34)



Now, you recall in terms of airy stress function phi, we have the phi satisfied that this is the way phi is related to different component of stresses. So, with this relation the equilibrium conditions are already satisfied and then if we have to make phi satisfy the compatibility equation and the condition that phi must satisfy if this ok.

This is the when these are the by harmonic operator; now this is for plane stress and this is for plane stress problem we have discussed this.

(Refer Slide Time: 01:14)

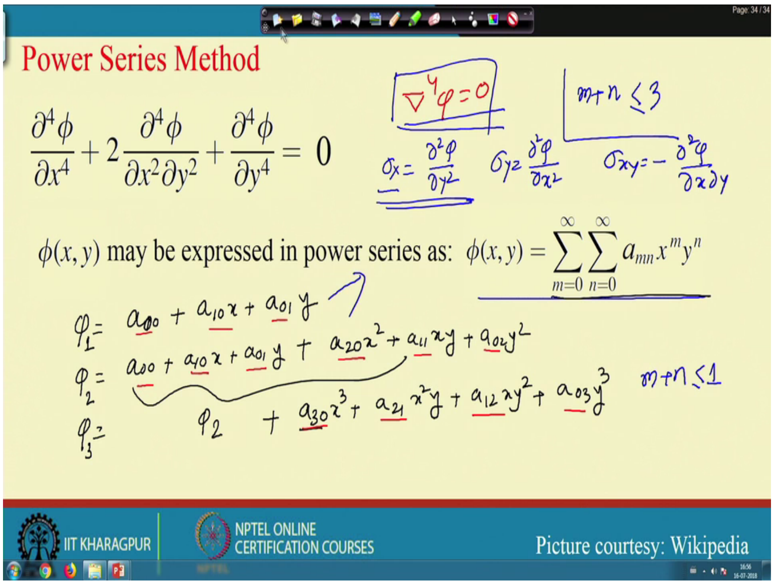


Now, in absence of body force the right hand side becomes 0 and essentially we have this equation and this equation is for plane stress and plane strain problem with both the cases this equation is same and this equation can be written as del 4 phi is equal to 0; this is a fourth order equation ok.

Now so, this is the governing equation now we have just only one equation and that equation is this. Now, how it is to be now solved? You see what we the method we use will be using as power series method where we start we do not know what is the expression for phi; only thing we know that phi has to satisfy this bi harmonic equation or if there is a body force then the other equations. What we do is we assume some expression of phi written in terms of polynomials. And in that polynomials there are some constants what we do is now we substitute that polynomial in this in this equation and also the boundary conditions available, based on that we try to find out some relation between those constants and determine those constants using those relations these steps will be clear as we.

Let us demonstrate that through one example.

(Refer Slide Time: 02:35)



At now before that the general approach will be supposed the phi can may be expressed in terms of power series (Refer Time: 02:43) ok; this is the power series any. Now this is essentially polynomial; now it is the sum over infinity does not mean that we have to take the entire infinite series. We do not have to take depending on the problem we may have to take third order or fourth order or fifth order that depends on the problem ok.

Now, and even we will see that the conditions available; some of the boundary conditions and the some of the information about the problem that are available which are obvious and based on that information some of the constants we can straight away we can make them 0 ok. Now for instance if it is if it is linear case this expression becomes; this expression becomes this expression becomes phi is equal to say a 1 0; a 1 0 first is a 0 0; the constant term a 0 0 plus a 1 0 x plus a 0 1 y; so, this is a linear polynomial.

Similarly, if you have to take phi as quadratic then a 0 0 plus a 1 0 x plus a 0 1 y; this is the linear term plus the quadratic term a 2 0 x square plus a 1 1, x y plus a 0 2 y square. Similarly, if we can take phi as cubic term where all these term will be; there all this term plus edition term will be a 3 0 x cube plus a 2 1 x square y plus a 1 2 x y square and a 0 3 y cube. Similarly, if you can have fourth order it is phi phi say it is linear term phi 1, phi 2, phi 3 and this will be phi 2 plus this ok.

So, similarly if it is fourth order, fifth order we can have so many we can we can express this power series in term like this these are essentially polynomials. Now, we know all these polynomials these are constant, these are constant these are constant ok. And that constant needs to be determined based on the conditions that is available with us base on the condition explicitly stated or based on the condition that we know about the problem. Now, you see look at this equation this equation is a fourth order equation right del 4 phi is equal to 0 ok.

So, naturally all these up to phi 3 up to phi 3 means m plus n if for m plus n for m plus n less than 3 less than equal to 3 this equation is automatically satisfied because less than equal to 3 these equation is automatically satisfied because it is a third order polynomial ok. So, naturally when you differentiate it 4 times then the this has to be 0. So, for any polynomial up to third order this condition is automatically satisfied. So, if for a problem we are assuming third order polynomial then we do not have to really show that this compatibility condition is satisfied.

Then in that case what will happen? The compatibly condition is automatically sati compatibility condition is satisfied because our polynomial already less than the less than the governing equation. And the equilibrium condition is equilibrium condition is satisfied by phi because it is satisfied based on the construction, but only thing we have to find we have to then use at the conditions the boundary conditions to solve to find out the unknown constants.

Now you may argue at this point; if the equation is fourth order equation then what is the point in having or in assuming the solution of third order; which is very rational logic, but you look at our objective is not to determine phi.

What is our objective? Our objective determined sigma and how sigma and phi related with each other? Sigma and phi is related as this if you recall that sigma x is equal to sigma x is equal to del 2 phi del y 2, sigma y is equal to del 2 phi I am not I remove the body force term x 2 and sigma x y is minus dels 2 phi del del 2 phi del x del y. So, sigma is sigma and phi that for that relation phi has to be at least quadratic. If phi is linear suppose if you take phi is linear then what happens? Sigma x sigma y, sigma x y all are 0; see in order to have non zero stress field phi has to be at least quadratic. If you take phi quadratic then what happens? All these stresses will be constant if you take phi cubic this stresses will be linear.

But phi the minimum requirements is phi has to be quadratic ok. Now if you take phi if you assume phi as say cubic polynomial compatibility condition del 4 phi is 0 anyway, it is satisfied and then in that case the unknown associated with that polynomial; cubic polynomial needs to be determined by the boundary conditions. And then you have expression for phi that expression gives you non zero stress field because that expression is cubic and your sigma your the relation between sigma and phi is quadratic ok.

So, we can assume for a problem, we will be just demonstrating that through an example we can assume quadratic polynomial, cubic polynomial; even for quadratic and cubic polynomial del 4 phi that bi harmonic equation is automatically satisfied ok. But remember if you have a body force then right hand side is not 0, then that is not automatically satisfied then you have to find out the conditions. So, that that gives you another equation, but for this problem where the body force is 0; body force term is 0 cubic and second order polynomial automatically satisfy this bi harmonic equation.

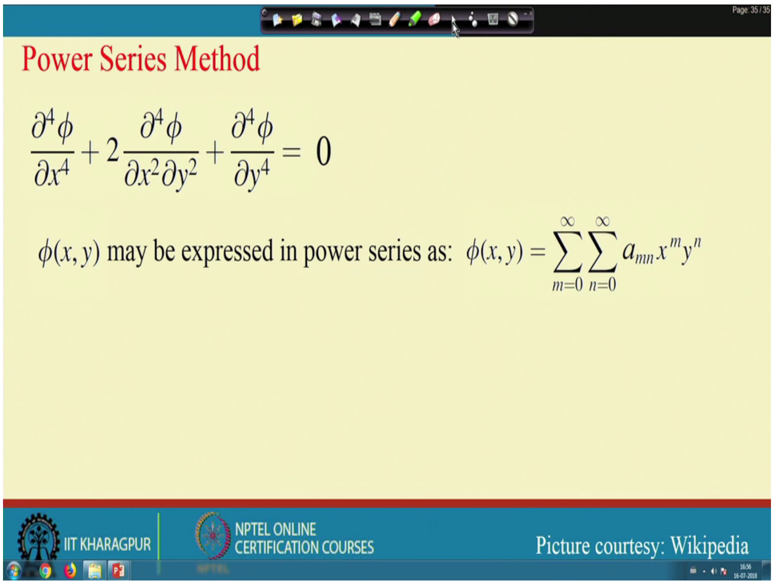
Can we take linear polynomial? Means for m plus m plus n less than equal to 1, can we take a polynomial? For instance can we take this? This satisfy the compatibility condition this satisfy this equation, but then can we assume the expression for phi as this? We cannot because this does not give me any non zero stress field because in order to have the stress field, non zero stress field if phi has to be quadratic ok. So, when we assume phi we do not have; we do not know what is the expression for phi one thing we know that phi cannot be linear.

Another thing we know that phi can be quadratic or cubic, but in that case this del bi harmonic equation is satisfied, we do not have to show that they are that is satisfied, but for any other order polynomial we have to show that the that polynomial satisfy the bi harmonic condition as well ok.

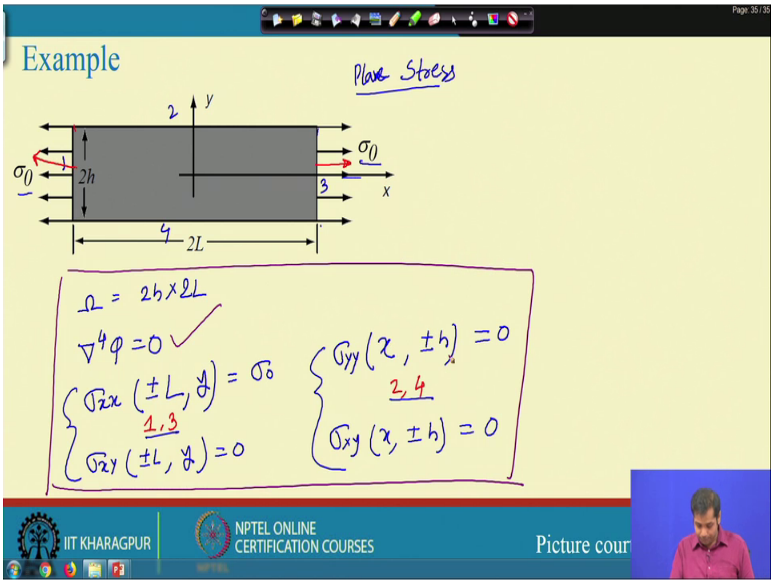
Now let us demonstrate this to a let us start let us first understand the procedure through an example we will be demonstrating this procedure again and again in the context of different problems. So, let us in this class since if the first problem you are going to we are going to apply this Airy stress function method, take a very simple problem very simple stress field.

So, that we understand the steps that need to be followed in order to apply this method ok.

(Refer Slide Time: 11:05)



(Refer Slide Time: 11:07)



So, let us take an example like this; this is a plane stress problem assume plane stress problem plane stress problem ok.

Now, this problem is what? This problem is we have a rectangular domain of 2 h and a 2 L size and at 2 n you apply some intent tension sigma 0 if that tension. Now, but remember the problem this is the physical problem we have; there is no body force now from this physical problem we have to translate this we have to write the mathematical model for this problem. And what is the mathematical model?

Mathematical model constitute 3 parts if you remember; one is the out the boundary value problem constitute 3 part; one is the definition of the domain. And then the governing equations and then the boundary conditions; now definition of the domain is now omega the domain here is say is the different color. Now omega here is 2 h cross 2 L that is the domain ok; now governing equation governing equation is del 4 phi is equal to 0 that is the governing equation ok, there is no body force.

Now, in addition to that we need the boundary condition now let us see what are the boundary conditions we have here. You see we have 2 boundary, 4 boundaries here this is 1, 2, 3 and this is 4 boundaries ok. Now boundary one what is the what is the boundary 1? Boundary 1 we have sigma 0 is see we have normal stress sigma 0; if this is the normal this is the normal direction, here also this is the normal directions and normal stress is sigma 0 and there is no shear stress on this boundary. Similarly normal stress is sigma 0 there is no shear stress on this boundary.

It means can we say that sigma x at the normal stress is in the direction of x; all this unit normal is the direction of x. So, can we take that sigma x x at x is equal to say plus minus L; plus L and minus L if you assume all coordinate system like this at any y means on the surface at any y and on the surface at any y that is equal to sigma 0. So, this is the boundary condition boundary on the condition on the normal stress on this boundary and this boundary. Now what other stresses what are the stresses we have on the normal stress we have on this boundary 2 and boundary 4?

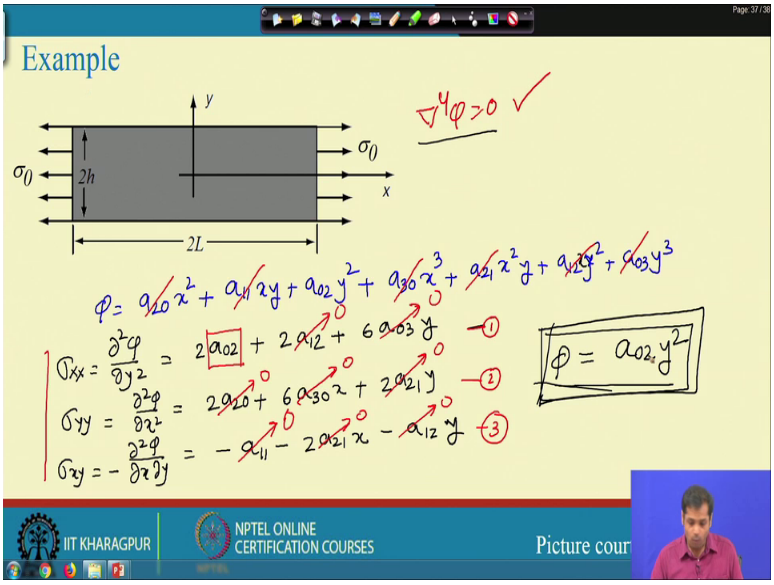
Say sigma y y because that is the direction of unit normal at boundary 2 and boundary 4 sigma y y for plus minus h means minus h at boundary 4 and plus h for any x that is equal to 0. So, this is at boundary this is at boundary 1, boundary 1 and 2, boundary 1 and 3 this is as boundary 2 and boundary 4 ok; these are the conditions on the normal stresses. Let us see what are the conditions on shear stresses; now you see on this boundary there is no shear stress, on this boundary there is no shear stress. Similarly, on this boundary there is no shear stress, on this boundary there is no shear stress therefore, then sigma x y plus minus L at any y that is is equal to 0.

And in this case sigma x y at any x plus minus y h that is equal to 0; so these are the 4 conditions specified at 4 boundaries. These are the boundary conditions specified as boundary 1 and 3 and these conditions are specified at boundary 2 and 4. Now this is my complete description of the problem, complete description of boundary value problem ok. Now, we have to solve this equation such that the following conditions following boundary conditions are satisfied ok.

Now once we have this is the very important very important step one ones you one once a problem is given to you encounter a problem like this, you have to solve it governing equation anyway you have to you have already derived that is known that is general that is this equation is general for any problem, any problem we can apply this equation.

But problem to problem what is the major what is the main difference at the boundary conditions ok? So, it is very important to identify the boundary conditions; now you know other boundary conditions given here. If you recall we had 3 kinds of problem; one is where your conditions are boundary conditions are specified in terms of stresses, boundary conditions specified in terms of displacement at the boundary conditions specified in terms of both displacement and stresses; this is the problem where boundary conditions are specified in terms of stresses.

(Refer Slide Time: 16:38)



Now, ones we have that let us; let us start with the very general expression as I told you there is a; if you this example is taken is this example is taken from the book. The list you get you can get that in the reference, but when you or many other books this example is given.

But when you see that example probably in many places you see that the what is the expression for expression of phi is directly given ok. But since this is the first problem we are solving we have not yet gather experience in doing this exercise. Let us start this by assuming a very general expression and then see; we may have many components in that general expression, but some of the components we can we can make them we can make them 0 by applying the conditions available and then see how solution can be obtained ok.

Let us start with let us let us assume phi is equal to assume phi is equal to take a cubic polynomial phi x y that is equal to; in cubic polynomial the first the linear term there is no point in having linear term because linear term does not contribute anything in computation of stresses. So, let us start with the quadratic term and that is it if you recall. So, if phi is equal to a 2 0 x square; this is the quadratic term a 1 1 x y and then a 0 2 y square this part is quadratic take cubic term as well a 3 0 x cube plus a 2 1 x square y plus a 1 2 y square and plus a 0 3 y cube.

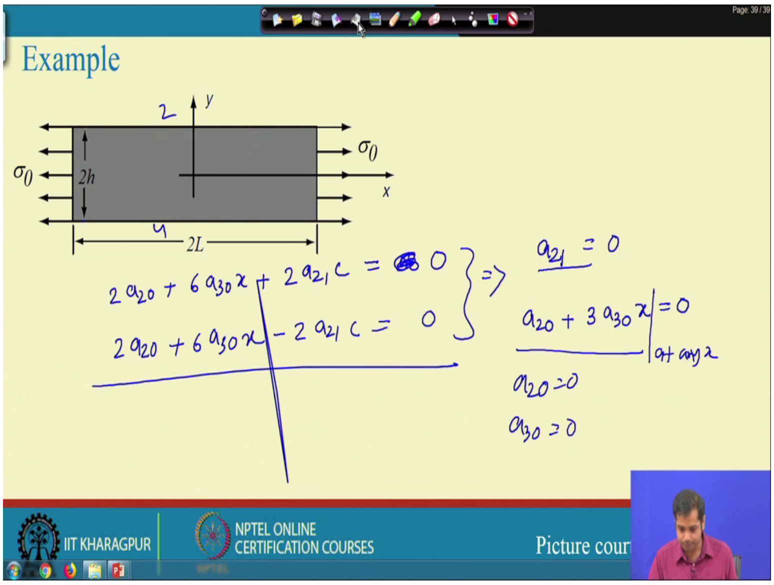
Suppose that is the expression of phi, as if expression of phi; in this expression how many unknowns we have? We have 1 2 3 4 5 6 7 unknown which are a 2 0 and a 1 and so on. Now, since this is quadratic this is cubic equation the bi harmonic equation is or to bi harmonic equation is satisfied y is equal to 0; so, that is satisfied. So, we do not have to show that these equations satisfies these this condition, but what we have to show is that these equation satisfy this boundary conditions ok.

So, what we do now is we find out what are the conditions is to be satisfied such that this boundary conditions are met ok. Now to that for that let us find out the expression for stress, now if you recall sigma x is equal to sigma x x is equal to del 2 phi del y 2. And if you if you write this then this becomes 2 a 0 2; you can do this exercise parallelly and check a 2 then we have this one; 2 a 1 2 plus then 6 a 0 3 y.

And similarly you have x x sigma y y is equal to del 2 phi, del x 2 that is equal to 2 a 2 0 plus 6 a 3 0 x cube x; x plus 2 a 2 1 y, 2 a 2 1 2 a 2 1 2 a 2 1 y ok. And then we have finally, sigma x y which is minus del 2 phi del x del y that is is equal to minus a 1 1, then minus 2 we are x x y term here 2 a 2 1 x then minus a 1 2, then we have this tell we x here 1 2 y ok. So, this is this 3 equations we have equation 1, 2, and 3.

Now, you see these are the expression of stresses in terms of this constant. Now this expression should satisfy all these boundary condition; let us see let us first start with these boundary condition, the boundary condition on this. Now, if you do that now sigma x x expression is this.

(Refer Slide Time: 21:47)



Now let us write let us let us satisfy this condition on this on this boundary on boundary 2 and 4. And, if you do that then what we get is you check that 2 a 2 0 plus 6 a 3 0 x plus 2 a 2 1; 1 is c and that is equal to sigma 0 there is there is some problem you check it ok.

That is equal to sigma 0 this is the stress sigma x x on this boundary similarly we have 2 a 0 2 a; 2 0 plus 6 a 3 0 3 0 x. And then this is minus this is 0 actually this is for boundary number boundary 2 and 4, this is boundary 2 and 4; so this terms becomes 0. And then it is minus 2 a 2 1 c; that term become 0 ok. So, this is stress sigma y y; sigma y on this boundary number boundary number 2 and this is sigma y y on boundary number 4 and from this what we get is you can check; from this we get a 2 1 is equal to 0.

And a 2 0 plus 3 a 3 0 x that is equal to 0 ok; now a 2 0 equal to 0 we have. Now this is true for if you recall; if you recall the stress sigma y y this stress this is due this is for any x. So, any x sigma y y on this boundary x c is equal to at x y is equal to plus n minus e; at any exist condition is this stress is 0. And therefore, this has to be 0 this has to be 0 at any x at any x and which gives you a 2 0 is equal to 0 and a 3 0 is equal to 0 ok. So, then on this expression what are the things you have? You have a 2 1 is equal to 0 a 2 1 is equal to 0.

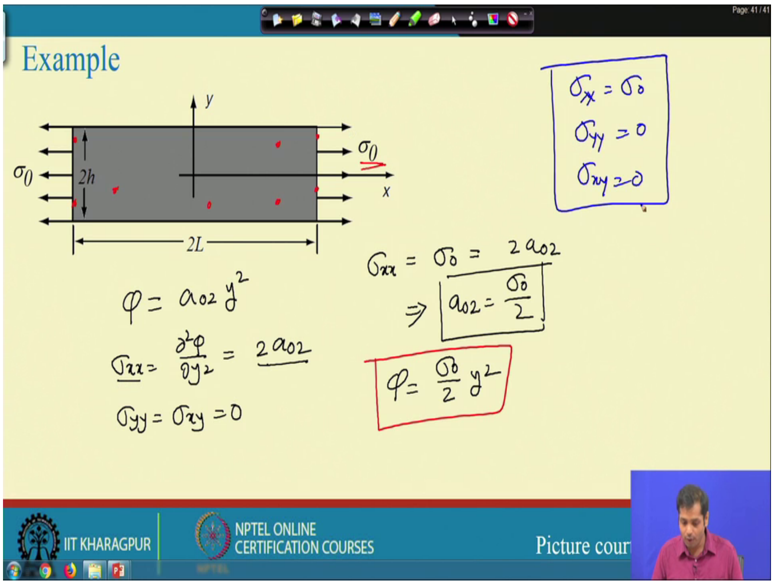
And then a 3 a is equal to 0; a 3 0 is equal to 0 and a 2 0 is equal to 0 a 2 0 is equal to 0. Now, similar exercise if you do on this boundary, this boundary normal stress as well as shear stresses; we will check that all the all these constant will be 0. Similar is all these constants will be 0 only constant non zero will be a 0 2 and if a 0 2 is non zero means all these are 0 all these are 0, all these are; all these are all are 0 and then essentially we have a expression of phi is equal to phi is equal to a 0 2 y square you see.

So, we could have started with quadratic polynomial then also we arrive we could arrived at this. We started with cubic polynomial to show you these exercise that how to apply these boundary conditions. Now one boundary conditions; I have demonstrated here, but similarly you can apply the other conditions on the stresses and get the relation between different coefficients and find out their value.

For this particular problem those values will be 0 other than a 0 2, you can try by starting with the fourth with the with the fourth with the fourth order polynomial as phi. And then do the exercise and then see what are the constants what are the information about that constant that you can derived based on the boundary condition available.

So, now, if phi is equal to this then this phi satisfy the equilibrium the compatibility condition is satisfy the boundary equilibrium condition. And then it also satisfies the stress the traction boundary condition that is given to this given for this problem ok.

(Refer Slide Time: 26:41)



Now once we have once we have this now let us let us move on. So, we have expression of phi as phi is equal to phi is equal to a 0 2; 0 2 y square. And then what we have is sigma x is equal to del 2 phi, del y 2 which is equal to 2 a 0 2 and sigma y y is equal to sigma x y is equal to 0 from phi we can derive this.

So, if you take any point sigma x x will be a 0 2 which is constant and sigma y y is equal to this. Now, sigma expression of sigma x x is this now this is constant and this sigma this expression is 2 for any x and any y. So, if you take a point here, if you take a point here then also the expression of sigma x x this, if you take a point here expression of sigma x x this, if you take a point here expression of sigma x x this.

And these expression tells you that the stress is constant everywhere. If you recall, if you recall when we when we introduce stress we said that stress is the point of description and therefore, at every point that we have this stresses are different. But there could be some situation where the stress is a uniform stresses at different point is same with this is one example. But still the stress is a point by description that is still valid here, but every point the stress is constant one example is this.

Now, if this stress is constant every point the stress at this boundaries also same constant ok. So, at this boundary stress is known stress is known which is sigma 0; it means that sigma xx will be. So, sigma xx will be sigma 0 and this is equal to 2 a 0 2 and this gives us a 0 2 is equal to sigma 0 by 2 ok. And, if this is then what is the expression of phi?

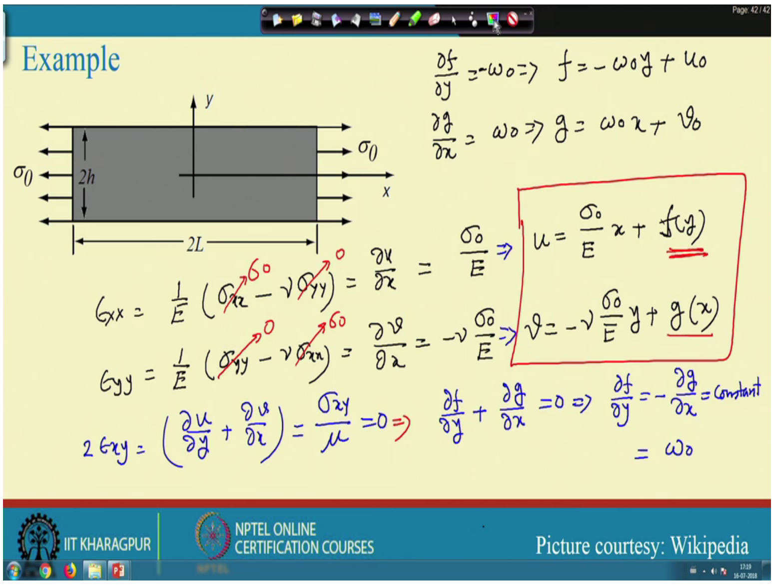
Then expression of phi is phi was actually sigma 0 2 by 2 into y square that is the expression of phi.

So, we are almost done; so we have calculated stresses. So, stresses are everywhere, so this stress field in this case is sigma x x is equal to sigma 0 sigma y y is equal to 0 and sigma x y is equal to 0 that is the solution of stress. Now once we have stress we have to now next determine what is strain and that there we have to bringing the constitutive relation; the relation between stress and strain ok.

And there though if while calculating this you take you take any problem any, any material for the for the same state of stress same bound, same conditions, same geometry every time you will get the same stress distribution. But now when we convert stress to strain and then strain to displacement material property will come in ok.

Let us find out how to determine strain now using the stress strain relation.

(Refer Slide Time: 30:08)



Now, if you recall for plane stress problem for plane stress problem; the stress strain relation was epsilon x; epsilon x x was 1 by E sigma x x minus nu into sigma y y ok. And epsilon y y is equal to 1 by E and then sigma y y minus nu into sigma x x and this is equal to del u del x; this is how the strain displacement relations are defined when u and v are the displacement in x and y direction.

Now in this case only unknown only sigma y y is 0 let us we sigma y y is 0. So, we can directly put sigma y y and then finally, we have this is equal to and sigma x x is sigma 0; this is sigma 0. So, essentially what we have then? Del u del x will be sigma 0 by E and then del v del y will be minus nu into sigma 0 by E ok.

Now, from this if we integrate it then what we have? We have u is equal to then u is equal to we have sigma 0 by E into x plus a constant of integration; actually now it is the partial integration is the partial derivative here. So, the integration will the constant the constant of integration it is not constant in true sense; it be it will not be the function of x, it will function of y right; it will be it will be function of y.

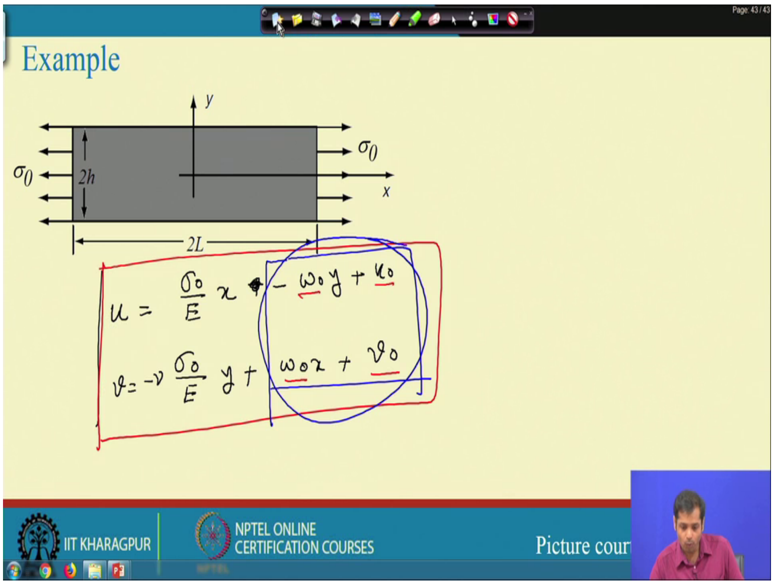
Similarly v will be minus nu sigma 0 by E into y plus g of x ok. Now still, so this is the expression for u and v, but this is not the complete expression of u and v because we have not yet derived when we need to obtain what is the g x and what is f y. Now, in order to do that the third definition of strain which is epsilon x y that we may need. And what is that? If you recall that epsilon x y is equal to 2 epsilon x y is equal to del u del y plus del v del x ok, that is how the this strain the shear strain is related to displacement. And that is equal to sigma x y by mu where mu is the shear modulus or sometime we use g as well.

Now the sigma x y is equal to 0 we have seen for this case; so this become 0. Now this gives us; so if I substitute u and v from this expression what we get is; we get this if we get it if we get substitute from this expression what we get it del u del u del y. So, this part will be 0; so this becomes essentially del f del y then plus del g del x that is is equal to 0. So, this gives you del f, del y is equal to minus del g del x that is equal to some constant ok.

Now, suppose that constant is equal to w 0 or omega 0 not w omega 0 ok. So, if del f del y is equal to omega 0. So, now just we claim same here itself we can write if del f del y is equal to del f del y is equal to omega 0 or minus omega 0; if you can take minus omega 0 then this gives you f is equal to f is equal to minus omega 0 y plus a constant of integration say this is u 0. And similarly now del g del x will be your omega 0; this gives you g is equal to omega 0 into x plus constant of integration v 0 ok.

So, therefore, just if I have to write the entire expression let us write the expression for u.

(Refer Slide Time: 34:43)



Then expression of u expression for u will be final expression of u will be. So, u is u will be sigma 0 by E into x and v will be sigma 0 by E into y minus nu plus we have this term; this is for f del f del y it f y this term minus omega 0 y plus u 0 this term will be minus omega 0 y plus u 0, this will be omega 0 x plus v 0. So, this is the complete expression for displacement.

So, now we have stress how the stress field is stress with the solution of the stress field; now we have the solution of displacement field. Now here just before we stop one important point; you see what is w 0? This, what are this term? w zero, u 0 v 0 omega 0 and omega 0 u 0 v 0; it tells you what is the rigid body when x is equal to 0 y is equal to 0 at every where u 0 is the rigid body displacement; in x direction and v 0 is the rigid body displacement in y direction and omega 0 is the rigid rotation.

So, this part actually gives you rigid body motion. So, this rigid body motion does not give you any strain. If you see if you if you calculate strain from this displacement only this part will contribute to the strain, this part will not contribute the contribute to the strain which is very which is very obvious if I take any object like this and this object moves in this direction in this direction rotate as rigid body then this object will not experience any strain the strain will be 0.

So, therefore, now, but still they are unknown in order to find this w 0, u 0 and v 0 what we need is we need some more conditions, condition related to the displacement. And that displacement boundary condition if we have then we can determine this thing. So, if I just integrate strain and get the integrate this strain field and get the displacement; what we can have? We can have the displacement up to a up to an arbitrary rigid body motion and this is the arbitrary rigid body motion ok. In order to find out this constants we need to have more condition associated with displacement of the problem ok.

Now, so this is the solution of the problem solution of the problem is the various state simple example even whether stress field is constant and idea here has not been to not been to solve any particular example the idea here has been to demonstrate the different steps that we need to follow for using Airy stress function approach for solving boundary value problems in elasticity.

Next class what we do is next class we will derive the expression similar expression for Airy stress function expression in polar coordinate system and then see some examples on polar coordinate examples related to that coordinate system., But again next week when we talk about beam bending problem and then next to next week when we talk about torsion problem; there we will there we will try to there we will apply this concept similar steps, but in a different class of problems means different in problems where we have different kinds of behaviour.

And another important exercise we will do next 2 weeks is once we have once we solve this once we have the solution of those problems beam bending problems or torsion problem and then we compare those solution with the solution obtained from strength of material and then check what is the discrepancy and try to find out what are the possible reason for discrepancy ok.

Now at this point though it is not the last class of this of this week, but since we are talking about Airy's stress function you see I we discussed earlier also that in most of the problem that we encountered most of the problem almost all problems that we encounter in real life analytical model analytical solution, solution having in closed form is very difficult and in I mean almost all problems we have to go for numerical techniques. And numerical techniques is an indispensable to will any engineering analysis.

But still we are talking about Airy stress function, we are talking about analytical models, we are talking about solution closed form of simplified problems. Because though numerical solution gives us very rigorous very comprehensive understanding of the behaviour of any structure, but for what at the end of the day in terms of translate your analysis result in terms of design provisions and for quick understanding of sensitivity of different parameters; we do use we do need analytical models.

The exercise that we do in this next weeks and this week's essentially gives you a platform how to approach a problem; if we have to if we have to derived an analytical model for a physical system ok. With this I stop today see you in the next class.

Thank you.