Theory of Elasticity
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Lecture – 28
Solution of Boundary Value Problems (Contd.)

Hello everyone this is the lecture 28 of this ongoing course in Elasticity and today's topic is generalized plane stress. See in the last class we have already discussed what is plane stress problem, what is plane strain problem and what are the corresponding formulation for different idealization.

Now, what we will do today is; what we learn today is we will have, we will we will see a slightly different and alternative formulation for plane stress. We will start with the motivation of that alternative formulation and then see what the formulation exactly is.

At this point please note that rest of the course we will not be using this formulation, the purpose of today's lecture is first to give you for the completeness of the entire discussion that. So, that you know there exist a formulation which is called generalized plane stress formulation. And also when you later when you take plates and shells analyzed plates and shells; then some of the exercise that we do in this formulation, similar exercise you will be you will be doing there. So, this will give you some idea to start with.

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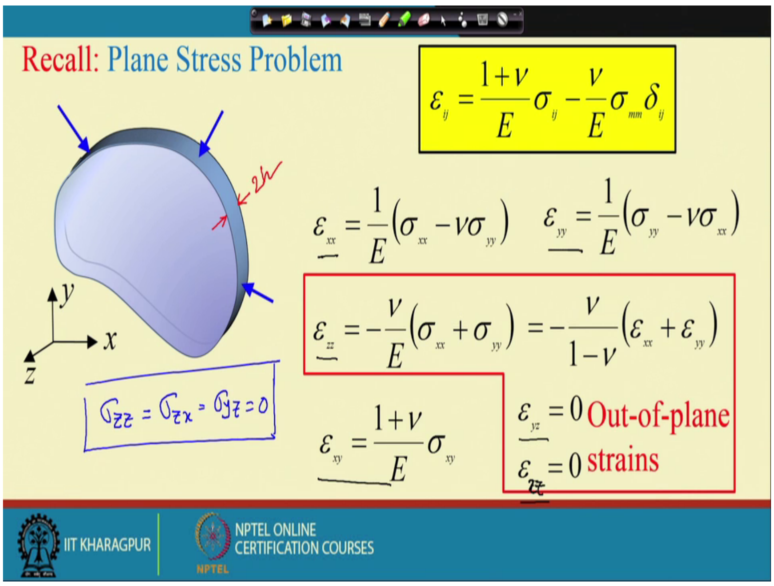

Let us first start the motivation understand the motivation of these alternate formulation for plane stress. Recall for plane stress problem what we discuss is the one dimension is very small such that suppose in this case this dimension is very small; the thickness direction is very small. So, thickness is 2 h.

So, that the radiation across this thickness, the variation of the stresses across the thickness is can be neglected. Therefore, the in these direction in this case the z direction; the component of stresses are 0. The component of stresses are 0 it does not mean automatically that the component of the strain in that particular direction is also 0. Now, so for a plane stress problem what we have is just we have already discussed this in the previous classes that sigma; we have sigma z z which is normal stress in z direction is equal to sigma xz or zx os equal to sigma y z they all are 0 that is the plane stress problem.

So, all these stress components we have they are on xy plane. Now let us, this is the general expression for strain general relation for stress and strain now if you substitute this all these 0 and non zero strain stress component into this relation, then we have this is the expression for epsilon x x. Then this is the epsilon this is expression for epsilon y y and this is the expression for epsilon z z as I just now said this stress in z direction is 0 does not mean the strain is also in z direction 0.

And then we have; so, this becomes essentially this if you substitute sigma xx and sigma yy from the first 2 equation we get this sigma epsilon z z it can be written in terms epsilon x x and epsilon y y. And then epsilon x y is equal to this shear strain and the other shear strain epsilon y 0 and epsilon epsilon z; it should be epsilon x z please correct it; this should be epsilon x z. Now they all are 0 right; all these stress components are 0, now you see epsilon x x, epsilon y y and epsilon x a x y they are on this plane right.

Now, but epsilon z z epsilon y z and epsilon z z; they are the strain in the direction of in the in z direction they may be acting on different planes. So, essentially epsilon z, but the direction of the strain is in z direction so therefore, but all these all these if you look at these strain component this stress strength are 0 fine, but the normal strain in z direction is not 0. So, this is called they are called out of plane strain component because these causes the out of plane deformation.

If these 2 strains the shear strain at 0, but epsilon z z is not 0 is epsilon z z is not 0 means that you have strain in z direction normal strain in z direction and therefore, you have displacement is also in z direction.

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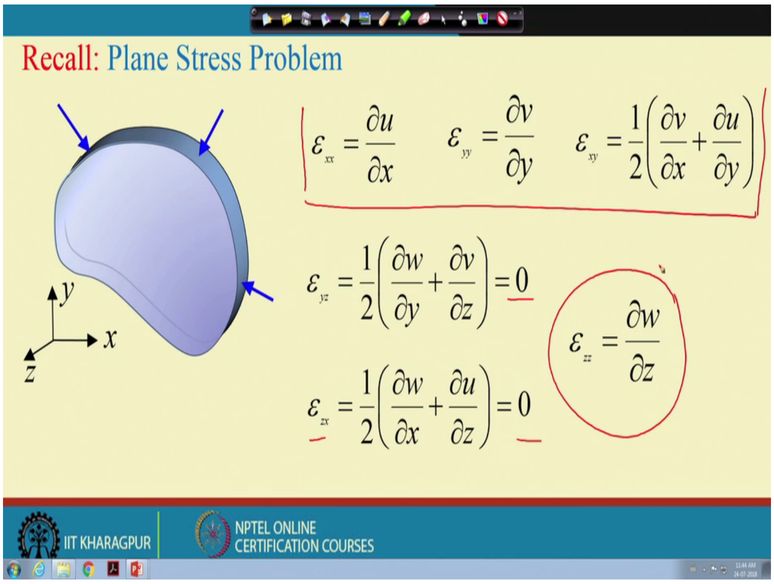

Now, this should be more clear when you write the strain component in terms of displacement if you write that then what we have is again we have; these are the strain displacement relations these all strain components they are all in x y they all in x y plane.

Now, epsilon y z is 0 epsilon z x is 0, but you see epsilon z z is del u del w del z. Now just in the previous slide we have seen that epsilon z z is epsilon z z can be written can be written in terms of epsilon x x and epsilon y y. It means that these quantity is not if you integrate it, then we may have w in z direction displacement in z direction. For this formulation if you substitute that we may see that the epsilon z z or displacement is a linear function of z.

Now, what it means? It means that we are working our idealization is the problems are if the all the problems or idealize is a plane problem means all the field variables are confined in a plane, defined in the plane. But here we have seen that the displacement we though it is a that is how we intended to idealize the problem; we will still have the component of displacement in z direction which is not compatible with the assumption for plane problems right.

So, there is an incompatibility now influenced stress problem; now generalized plane stress formulation is what? Just take care of this incompatibility; and slightly do some averaging to have an approximation do some averaging to have some alternative formulations so, that this incompatibility goes away; now that is what we are going to learn today ok.

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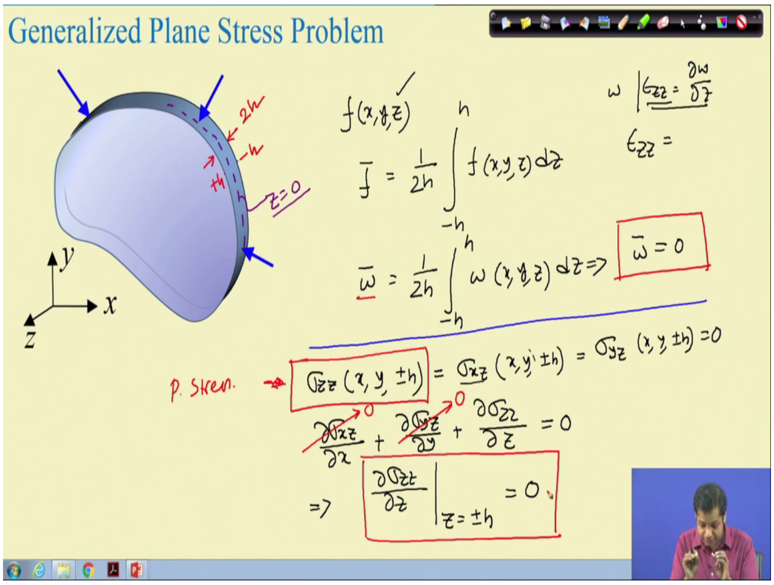

Now, let us start the first thing what we do is suppose take this example. So, suppose this is this width of this plate is 2 h 2 h. Now, suppose in this case this is plus h this is minus h minus h and this is plus h ok. So, this plane is defined as minus h where z is equal to minus h and this plane is defined as z is equal to plus h. What generalized plane stress formulation the core concept of generalized plane stress formulation is to average all the field variables; whatever variables we have suppose we have stresses, strain, displacement all these variables average it across the thickness of this plate.

Now once we average all the variables across the thickness in the write out all these equations; all these governing equations like your equilibrium equations, compatibility equation, the Navier's equation all these equation write in terms of those average field variables.

Let us see how it is to be done; first suppose f is a function f is a field variable that f could be your displacement, stress or any other variable. Suppose f is a variable f is a variable now this is a function of suppose x y and z ok; now the average of f say f bar is the average f which is defined as 1 by 2 h.

And integrate minus h to h f of x y z and then dz this is how suppose we do some averaging. And if is this averaging is for any variable it could be displacement stress or anything. Now, when we do this exercise then what happens? The z dependency in f goes away; so in that case the average quantity f may be the function of x y z, but f bar will not be function of x y z; f bar will be function of only x and y right. Now, for instance if does this w is the displacement on z direction then what would be w; w bar will be 1 by 2 h integration minus h to h w of x y and z dz ok.

So, this is the now in the previous slide if you recall epsilon z z epsilon z z was del w by del z and epsilon z z was non zero and which gives us some non zero value of w some displacement in z direction if we directly integrate this quantity. So, w maybe non zero and now what happens let us see what happens to w bar w bar is the average of w across the thickness right.

Now, you see if you relate this equation with equation of with the equation of strain component epsilon z z if you remember; epsilon z z can be written as it can be written as if you go back this slide epsilon z z can be written as a function of epsilon x x and function of epsilon y y and some constant this constant.

Now, so therefore, from these we can say that epsilon if we integrate it w would be the linear function of the linear function of z. Now if w is a linear function of z was a more general we can say the w will be the odd function of z. Now if w is an odd function of z then what will happen the if you do this not only that if we take if we take a if we take a mid plane.

Suppose you define a mid plane like this where z is equal to 0; this is the mid plane. In this plane z is equal to 0; so, in this plane z is equal to minus h, in this plane z is equal to plus h and at the mid plane z is equal to 0. Not only that suppose if w is a linear function of z that w is an odd function of course, and then also your w is equal to 0 at this mid plane. Now if it is then if you do this integration then automatically we will see that this integration will give us w bar is equal to 0; this is very important.

You see our motivation of our motivation of doing this exercise has been the fact that we; though we are working on a plane problem our idolization is a plane problem, we still have some deformation across the formation across the thickness some out of deformation right. Now which is non zero now in order to make the problem, we have to remain in the in the frame work of plane problem we have to make that approximation that displacement out of plane displacement 0.

Now what we have done is now we change definition of displacement; instead of the displacement now let us work with the average displacement and what since it is average across the thickness. Now, now our field variable will not be w our field variable will be w bar which is 0. Now, if we write if we similarly do the averaging of all these field variable and write the equation in terms of all these average field variable, then we will see that our the assumption that the no out of plane deformation, no z dependency that assumption is still valid in that case.

Let us let us see that. So, this is one thing ok. So, w bar is equal to 0 now, another important thing is you see all the forces; if you recall all the forces acting on this body are on are can be can be all on the same plane right. For instance if these are the forces they are on the same plane; we do not have any force in z direction; we do not have any body force in z direction right.

So, all these forces are 0 in z direction then we can write that what we have is; so, on this plane, on the surface means surface at z is equal to plus and surface at z is equal to minus essentially there is no traction on the surface; there is at z is equal to plus h and z is equal to minus h. Now, if we look at the traction on the surface how many component of the traction has? 3 component one is the normal and 2 are the tangential component and the normal component will be what? Normal component will be say this is one thing.

Now, let us normal component will be sigma z z; you check x y and then plus minus h that will be equal to sigma x z which is x y plus minus h equal to sigma say; it may be z x or exact same thing y z which is function of x y and plus minus h, they all will be 0. It means that your only surface at x is equal to z is equal plus h and z is equal to minus h this all this tractions are 0 right. There is no forces in there is no external forces in this direction there is no body force as well. Now in that case; then what would be the corresponding? Let us write the corresponding equilibrium equation in z direction.

The equilibrium equation is z direction if you recall; this will be sigma del sigma x z or z x both are same. Del x plus del sigma y z del y plus del sigma z z del z that is equal to 0; there is no b z term because body force in z direction is 0. Now, this is important you see in sigma x z irrespective of x and y at the surface plus suppose equation is this is in equilibrium equation, this is valid for any x y z.

Let us evaluate this equation at 2 surfaces z is equal to plus h and z is equal to minus h. If we do so then what happen at plus minus h means on this surface irrespective of the value of x and y; sigma x z is 0. And again similarly irrespective of the value of x and y on the surface; sigma y z is equal to 0. Then straight away we can say that these term these term will be 0 and these term will be 0.

Now, sigma z z; see sigma z z is sigma z z at on this surface is equal to 0, but here we do not know that sigma z z here we have derivative with respect to sigma z z. Means these what it gives? These gives that del sigma z z del z at z is equal to plus minus h that is equal to 0 right; this is also important ok.

So, what we have? We have 2 things one is not only sigma z z at plus minus h is equal to 0 then its derivative its first derivative with respect to z on this surface that is also 0 this directly; this comes from the plane stress assumption; this directly comes from the plane stress assumption plane stress assumption. And this comes from the that assumption projected substituted on the equilibrium in the equilibrium equation and we get this.

Now, based on this observation what point we want to make? Let us see that let us arrived at that point ok. So, let us write what we have now is quickly that see this is one thing w bar is equal to 0. And another important thing is what sigma z z is equal to 0 and sigma z z del z is equal to 0; these 2 are 0 what information it conveys what that let us let us find out that.

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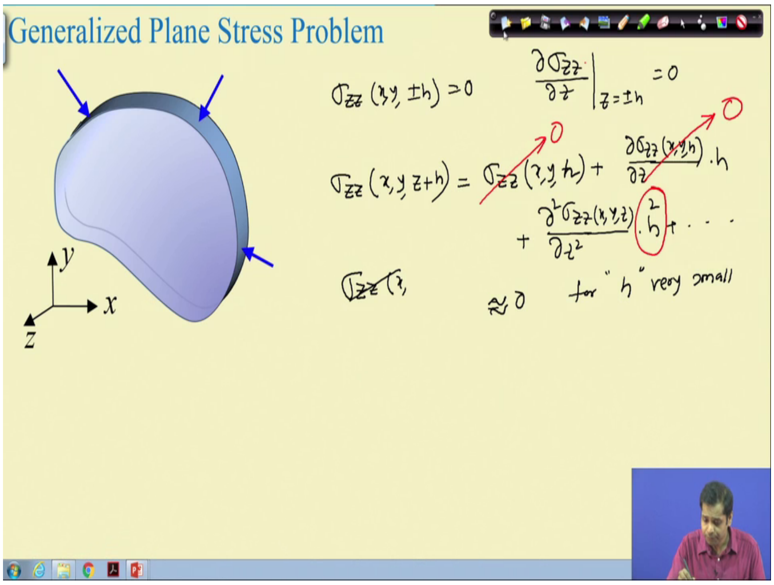

Now suppose ah; so what we have is sigma z z x y and plus minus h that is equal to 0 and also have sigma z z; del z at z is equal to z is equal to plus minus h that is also equal to 0 ok.

Now, let us write sigma z z; sigma z z x y z plus h let us write this let us express sigma z z in Taylor series about this at z. So, what this Taylor series will be? Taylor series will be sigma z z this could be plus h or minus h in both the cases same. So, let us write only plus h sigma z z x y plus or h and then plus del sigma z z del z this will be x y and h into h plus del 2 sigma z z del z 2 again x y z into h square and so on ok.

Now you can put instead of plus h; z plus h you can have z minus h in that case this will be minus and this will be plus and so on. Now, you see from these expression straightaway we can say this is equal to 0. From this we can say this is equal to 0 it means so what it gives you? How this sigma z z varies across the thickness right that what we express in Taylor series; now it says that it varies, it depends on h square. So, it is order of that is h square; so, what is h? H is the thickness; so 2 h is the thickness.

Now, you see if h is very small then h square is further smaller this a this is higher order term be neglected for a very thin plate when h is very small. And therefore, all this term including this including h square and all these higher order term we can we can assume they are 0 we can neglect them.

And if we do so then we have sigma z z, we have sigma z z x sigma z d quantity will be I mean all will be 0 when for h very small this is important. So, that is why you see that is the center point of plane space assumption; when we say that thickness is small therefore, the across the thickness variation is small and therefore, it is neglected, it is neglected this is the reason you see this order is h square and the they were very small very thin plate; the h square term will be very small. So, essentially these part can be neglected and that is why the sigma z z is equal to you can make them 0 which is the plane stress assumption ok.

Now, so once we understand that now let us do this let us write this. We have already seen what is the averaging this is the averaging and we can do this averaging for all this field quantities, stresses, strain or other displacement field. And then write the equation for that write the governing equation; let us let us do that let us do that exercise.

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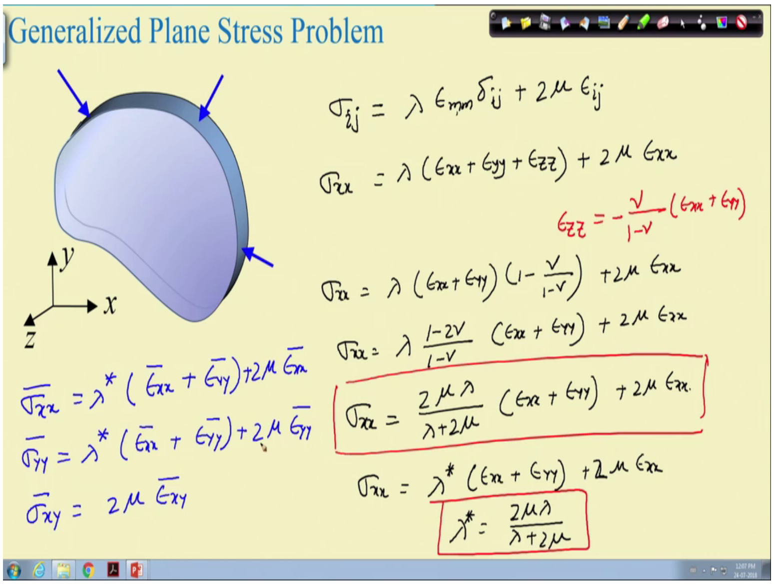

Now, if we write the governing equation you recall your governing equation was sigma ij; let us express sigma ij sigma ij that was lambda into epsilon ij epsilon not ij this is; if you recall this will epsilon m m and then delta ij plus 2 mu into epsilon ij ok.

Now, this is what? This is equal to lambda; lambda into epsilon x x plus epsilon y y plus epsilon z z right and then plus 2 mu into epsilon x x. If you write only sigma x x; let us write the expression for sigma x x, rest of the other expression can be derived in the same way. Now recall in the previous slide we showed that for plane stress problem epsilon z z is equal to epsilon z z can be expressed in terms of epsilon x x and epsilon y y and that expression is mu by 1 minus mu epsilon x x plus epsilon y y.

Now if you substitute this into this expression what we get is; we will get that is sigma x x is equal to lambda into epsilon x x plus epsilon y y you can you can try this and then it will be 1 minus; 1 minus mu by 1 minus 1 into mu plus 2 mu into epsilon x x. Now this is finally, becomes sigma x x is equal to 1 minus lambda into 1 minus; you can do this exercise 1 minus nu into epsilon x x plus epsilon y y and then plus 2 nu into epsilon x x. And finally, if you substitute nu in terms of lambda and mu we know the relation between lambda mu and nu; if you substitute that then the expression of sigma x x will be we will get as like this we will get 2 mu 2 mu lambda by lambda plus 2 mu into epsilon x x; you can try this epsilon y y plus 2 mu into epsilon x x ok.

This is the expression we get for sigma x x or this is can also be written as your sigma x x, sigma x x is equal to lambda star into epsilon x x plus epsilon y y plus 2 mu 2 nu mu epsilon x x; where lambda star is equal to this, lambda star is equal to 2 mu lambda by lambda plus 2 mu ok; so this is expression for sigma x x ok. Similarly, we can have expression for sigma y y sigma x y now as I said we have to we have to now write all this expressions, all these variables in a naught the variables we have to write their average we have to average them across the thickness of the plane.

Now if you do that averaging let say sigma x x can be written as suppose sigma x x is the average. So, let us use different color for this sigma x x which is average of sigma x x that will be equal to lambda star. This will be then epsilon x x average of epsilon x x plus epsilon y y; average of epsilon y y, then plus 2 mu into epsilon x x which is average of epsilon average of epsilon x x right.

Now, similarly we can we can write all these expressions; suppose sigma y y also can be written as sigma if you write sigma y y average that will be lambda star into epsilon x x average plus epsilon y y average plus 2 mu into epsilon y y average right. And similarly epsilon epsilon x y can be written as x sigma y can be written as if you recall 2 mu into epsilon x y average.

So, all these bar they mean the average across the thickness. Now, how to average epsilon x x? Epsilon x x if you recall epsilon x x was what? Epsilon x x was now quickly can do that epsilon x x ok.

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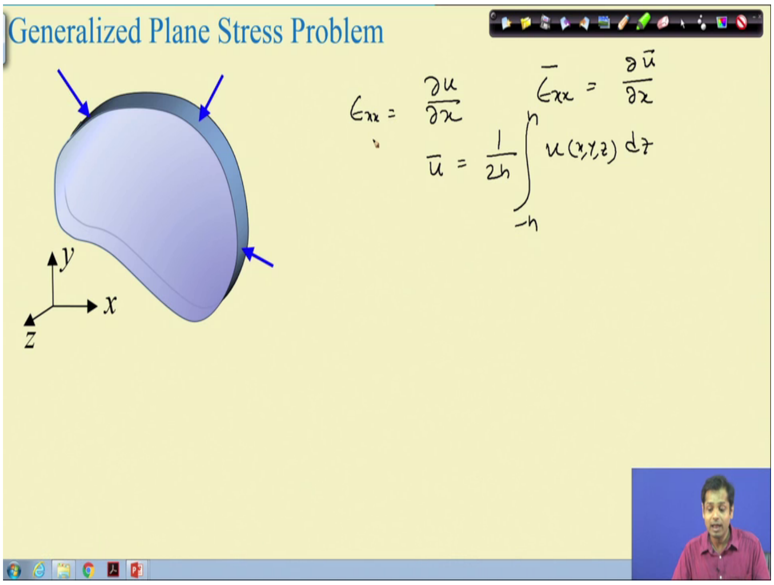

Used now epsilon x x is epsilon x x is what? Epsilon x x is del u del x right. Now, then epsilon x x epsilon x x bar will be then del u bar del x, where u bar is the average of u. Then what is u bar? U bar will be then 1 by 2 h minus h 2 h, then u of x y z dz; the way we average w in the same way we have to average all these variables ok.

Similarly we can get it we can write epsilon y y epsilon all the displacement we can have an average. And then we can average their derivatives and get the average strain and then that average strain can be substituted in this expression to get the average stresses. Now, once we have the average stresses or the once we have average all this field variables across the thickness; next step is to write the write the expression; write the governing equation in terms of those average stress and strain.

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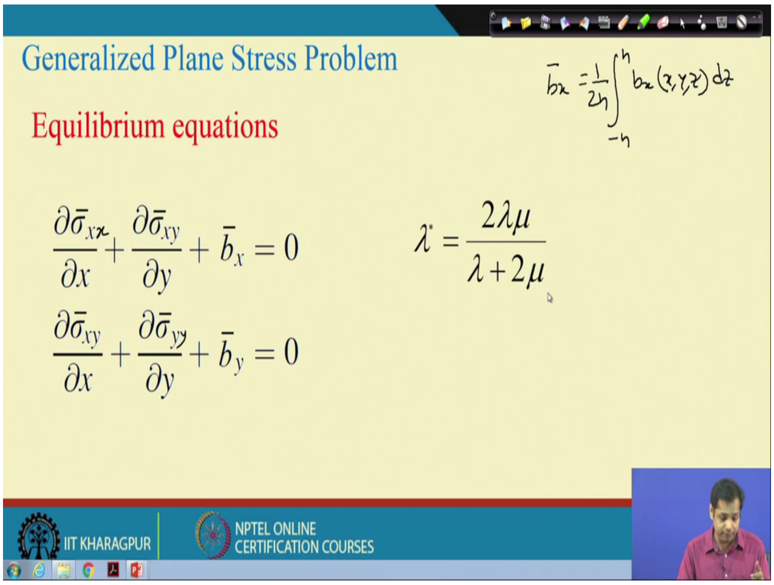

And if you do that your equilibrium equation becomes this; if you see equilibrium equation the form of the equilibrium equation remains same. But now here it is written in terms of average stress component average stresses; this is x x, this is y y and what is b x? b x average is equal to integration minus h 2 h; b x which is x y z dz 1 by 2 h. Similarly b y also can be average like this; so this is the average equilibrium, this is the equilibrium equation.

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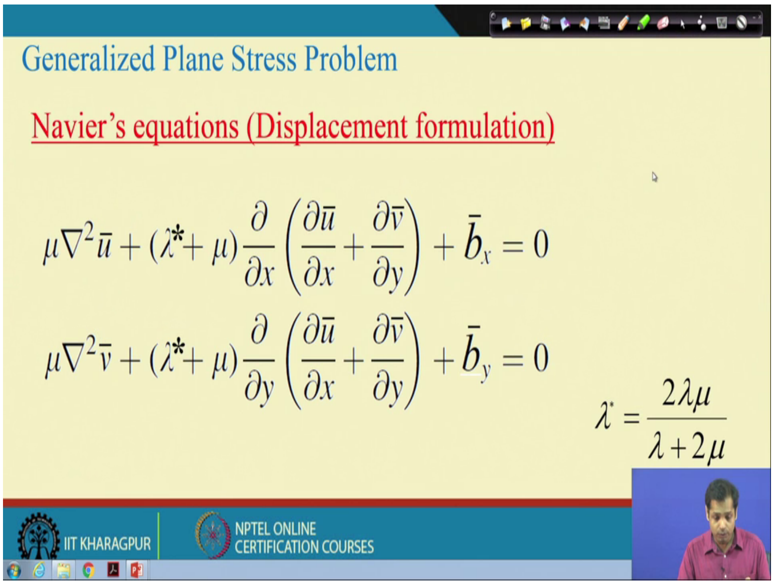

And, similarly we can have once we have the equilibrium equation; next we can have your this is the Navier's equation. Navier's equation is also written in terms of you see all these average displacement and then an average body force term.

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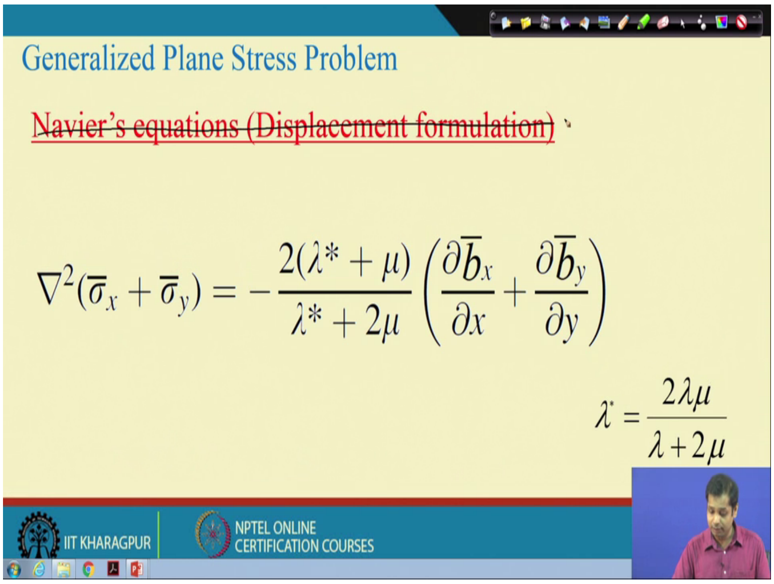

Similarly, we can have we can have the there is a mistake please correct it this is not Navier's equation this is the compatibility actually written in terms of stresses ok.

We will get Michel's equation now this is the equation I can written the stress formulation written in terms of the average field variables. Now once we have the an expression in average field variable then you instead of solving for in the previous problems; we have equation in terms of directly stresses displacement and we solved for displacement and the stresses.

But now here equations are written in terms of every stress; average displacement and the solution will be average stress and average displacement. And if we and an in an average sense if you see then your the plane the plane problem the requirement for a problem to be a plane problem can be to be an idealized as a plane problem that requirement can is it satisfy; the out of plane displacement w average is 0 and then sigma also we have seen sigma z z sigma z z is anyway 0 for thin plate.

So, this is a out of plane strain problem that was the motivation for our for to look into a different formulation explore different formulation. And this is the formulation which gives you which removes that incompatibility. Now having said that it is also important to note that in the rest of the formulation; rest of the problem that will be using the formulation that will be using for different problems, we will be we will not use these generalized plane stress problem we assume whenever you come across plane stress problem, we assume thickness is very very small and we solve for the stresses and the displacement not the average displacement and stresses ok.

But again this exercise when we when as I already started my discussion with the statement; the purpose of purpose of telling this is to have a completeness in the discussion. So, that you know there exist some formulation like this and also this averaging as far as the averaging across the thickness is concerned when you pull on plates and shells formulation; you will see the similar kind of averaging we also do there, but in a different way; So, this is just an exploration of that with this I stop here today next class; so, as far as the formulation is concerned this is almost done. Now once we have the formulation the next step is to; I mean formulation means once we have the equations ready with us does equation could be in terms of displacement, it could be in terms of stresses.

But then once the equations are ready we have to solve them we have to find the solution of that problem. There are many different methods to find out the solution of those problems, the method that will be extensively using in this course is add a stress function approach. And in the next class we will start with we will just introduce all these stress function. And then in subsequent classes we will see how that is stress function approach can be used to have solution for different plane stress and plane strain problems; with this I stop it here today; so, in the next class.

Thank you.