Theory of Elasticity
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Lecture - 26
Formulation of Boundary Value Problems (Contd.)

Hello everyone. This is the last class of this week. This class topic is Displacement Formulation and Solution Strategies. If you recall where are we now is we discuss that depending on how the governing equations of elasticity written, we can have 2 kinds of formulation one is stress formulation, another one is displacement formulation.

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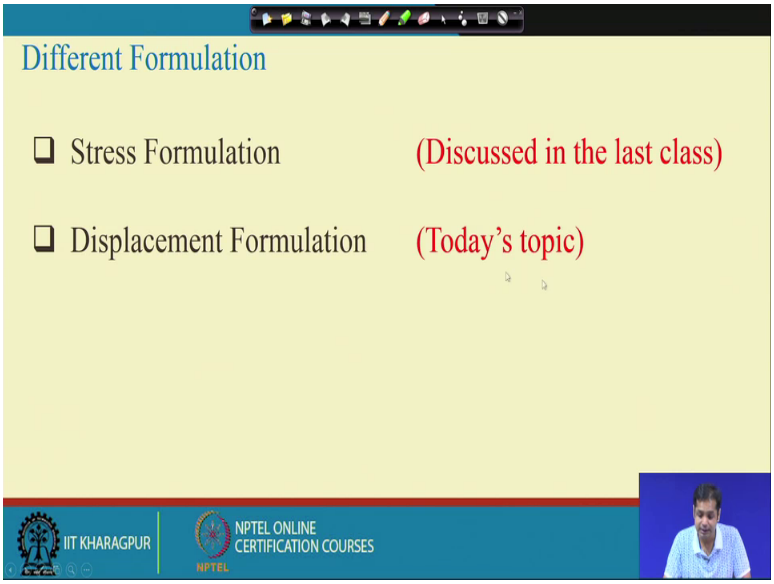

The stress formulation is discussed in the last class where the equations are written in terms of stresses. And then another formulation the equation the same equation can be expressed in terms of displacements as well and that is called displacement formulation and we will be discussing that displacement formulation today.

And once we discuss the displacement formulation, we summarize all these equations that we have that we developed so far. And then we see what are the different solution methodology techniques available and from there we will choose some techniques and then proceed ok.

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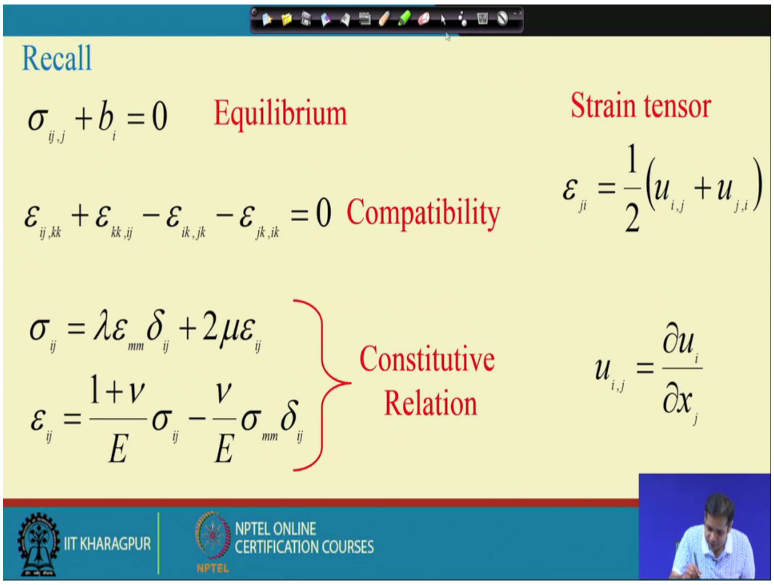

So whatever formulation we use either displacement formulation or stress formulation, it is the same set of information are used right. Same set of information means it is the same equilibrium equations, same compatibility equation or strain displacement relation or same constitutive relation.

Now, if you recall when we derive displace; stress based formulation, then we equilibrium equation anyway we have to use, equilibrium equation is there is no other way the equilibrium equation we have used. Then we use compatibility equation you see the compatibility equation and the strain displacement relation, they essentially tell you the same story, but in a different way. In this case when you write the compatibility equation and the displacement terms are completely eliminated.

So, we use these equation when while deriving the equations in terms of stresses. And then in the constitutive relation we can have 2 form; one is the strain as a function of stress and other is stress is expressed as a function of strain, we use these expressions right. So, these 3 expression these 3 equation when you combine; it gives us the equation in the stress form and that is the stress formulation. Now when we write the displacement formulation, equilibrium equation we have to use these equations will be used.

Now since the equations are written in terms of displacement we will not use the compatibility relation. Because the displacement is completely eliminated instead we use this equation the strain displacement relation, but please note it is the same set of information we are using, we are not compromising with information. And then in constitutive relation, we will not use these expression; we will be using these expression ok. So, this and this together these 3 equations will combine and this gives us the equation which is in terms of displacement.

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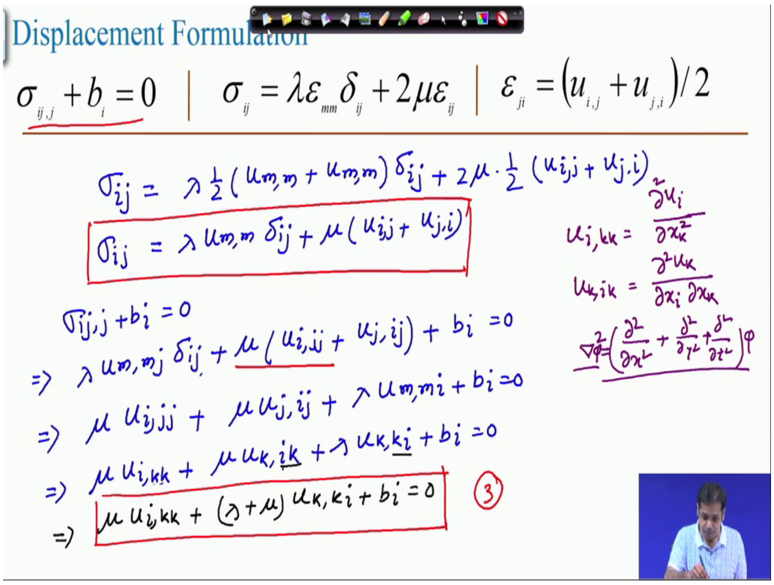

Let us do that exercise since you will see as we derive the displacement; deriving displacement formulation relatively easier as compared to this stress formulation. And we will also see that in terms of solution that is even in terms of solution, this is relatively easier in terms of a stress formulation. So, what we do is we will straight away derive this equation of course, using indicial notation for 3 dimension. We will not do this exercise for 2 dimension and then for 3 dimension; straight away we will derive it for 3 dimension.

Unlike, we did it using stress formulation ok, so as I said these are the equations that will be using, this is the equilibrium equation and this is the constitutive relation and this is strain displacement relation. Please note one thing it is implicitly assume here that your stress tensor and strain tensor; they are symmetry and that is why somewhere you will see that interchangeably i and j index are used. For instance, whenever we write sigma i j j and sigma i j; they essentially gives you the same set of same set of information same set of component.

Similarly, epsilon is it in a epsilon j i you can correct, you can make it i j, but essentially it may it is same epsilon i j is equal to epsilon j i ok. So, first what we do is first substitute these equation, this strain displacement relation the equation number 1, this is equation number 1; sorry this is equation number 1, equation number 2 and then this is equation number 3.

So, first what we do is substitute equation number 3; equation number 3 into equation number 1. If you do that then what will happen? Then you can parallelly do this exercise with me. So, sigma i j this becomes plus, it is sigma i j comma j this sorry you substitute this equation in equation number equation number 2.

Then this becomes to start with sigma i j, this is equal to lambda epsilon m epsilon m means i is equal to m, j is equal to m. So, this gives you half of half of u m comma m plus another u m comma m; u comma m means del u m by del x m ok. So, then we have lambda, then we have delta i j; delta i j and plus 2 mu into half epsilon i j is this if u i comma j plus u j comma i. So, this is your sigma i j; so you can slightly you can slightly and simplify it.

So, this becomes essentially lambda u m comma m delta i j; delta i j and this becomes mu into u i j plus u j i ok. So, this is equal to sigma i j; so, these equation we have this. So, this is the combined equation when you combine the constitutive relation, and the strain displacement relation; so, these equations we have. Now next what we do is next we substitute this is the expression for sigma i j we have.

Now these expression of sigma i j let us substitute in these expression and if you do that; you try this the equation is if you substitute this then what we have is let us try this. So, this becomes; so the equation is equation is sigma i j; comma j plus b i is equal to 0. So, this gives if we have to differentiate entire thing with respect to j and if you do that then becomes lambda u m comma m j delta i j, then plus mu u you have u i comma j plus u j comma i j then plus b i that is equal to 0.

Now let us slightly simplify this, you see here one thing you check. First you take this term mu into this term mu into u i; j, this becomes what? This becomes mu into u of i that is j j ok, then plus. Now here check one thing; so this becomes what? This becomes mu into mu into u j comma i j plus. Now, here check one thing this is delta i j right and this is true this is delta i j is the kronecker delta and this is true when i is equal to this is

1, when i is equal to i is equal to j and in other all other cases this becomes this become 0.

So, in that case what we can do is we can lambda this become then lambda u m m. So, j becomes i then only; so if you expand it you have some of the term are nonzero and some of the terms are 0 or the non 0 terms will be when i is equal to j; so, this becomes m i ok. Then plus b i is equal to 0 ok; now we are almost done. So, this is mu u of i comma j j plus mu of; you see here is the j is the repeated index.

So, j is now the dummy index here m is also repeated index. So, m is dummy index here also j is repeated index; the j is dummy index. So, now, let us all these repeated index the dummy index you let us write this as k. So, this is k k ok; so now k k is the summation over k is equal to 1 to 3 and k is equal to 1 to 3 which is j j and k k are same. This becomes mu k; u k u k comma i k, again it is same because now k is the dummy index; then lambda u k comma k i plus b i is equal to 0.

Now then we are almost done what we have now is; now let us write this. So, final expression will be this is equal to; so, mu u of i k k plus, now if you check if we take since k i; here i k and k i both can be interchanged as long as the u is continuously defined over the domain. So, you can take u k lambda plus u common.

Lambda plus mu common and then these becomes u of k comma k i; plus b i is equal to 0. So, this is the equation ok; you can derive this and please for convince yourself now. So, you see how many it is how many questions essentially we have? We have k k is a dummy index; k k is a dummy index i. So, essentially we have 3 equations one is one is for i is equal to x equation in x direction, i is equal to y equation in y direction and i is equal to z equation in 3 direction.

So, essentially we have total 3 number equations and which is very very logical because when you are writing equation in terms of displacement only unknown displacement there are 3 unknown displacement; displacement in 3 direction. So, we have 3 unknown, but remember when we wrote the equation in terms of stresses; we had 6 equations which is again very logical because we have 6 stress components, 3 normal components and 3 shear components; so, we have 3 equations for stresses.

Now, so you see one thing please note when you write these expression; when you derive this expression in indicial notation this derivation becomes very straight forward. You try yourself writing this expression, deriving this expression using all these differential operator in another. And then you can see that the expression the derivation is slightly tedious as compared to this.

Now, once we have this if I have to write this expression, this expression now written in term in indicial notation. Now let us write this expression in differential in a operated term; now what is u i k k? When I write this when I when you write u i; u i k k what exactly it means? It means del u i; del x k del; del 2 u del x k 2 it exactly means that. When you write u k comma i k this means what? This means del 2 u k, u k is the kth component of u either u x u y u z that depends on what is the u; del x i and del x k right.

Now, so, what you have essentially, now you remember what was your del 2 operator? This del 2 operator, del 2 operator if you remember that that operator was that operator was del 2 del x 2, then plus del 2; if you will take any function, any phi if del 2, del x 2 plus del 2, del y 2 plus del 2 del z 2 of phi that is Laplacian operator. Now if I substitute this; if I write this expression in terms of this del 2 and del operator, then eventually the expression becomes this.

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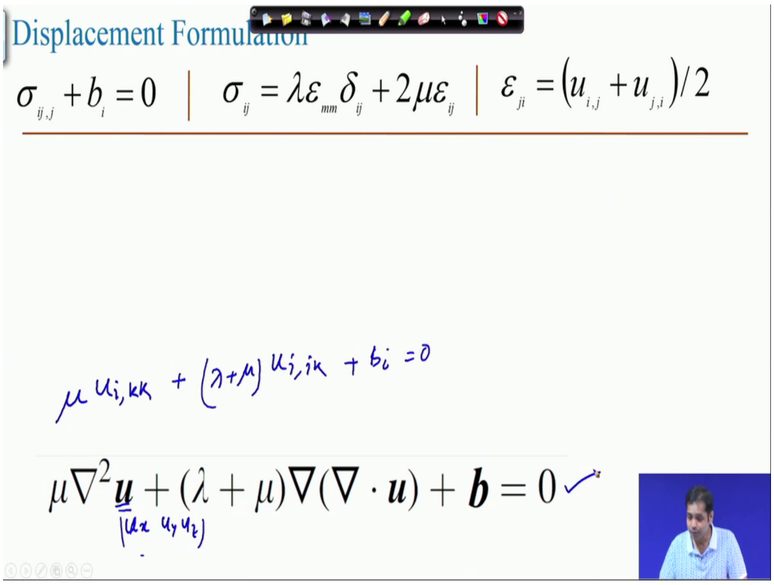

This expression becomes this you try yours; you please verify yourself. So, this is the first term which was this term was if you recall, this term was this term was mu into u i k

k. This is the term and then we call term was lambda plus mu u i i k; this term gives you this and then plus b i this is this was equal to 0.

Now if you combine this and write in a vector form; so, u is essentially now this gives you 3 equations u is now u x; u i and u z. So, what term you are taking depending on that you get first second or third equations. So, this is the final equation retained in displacement written in terms of displacement ok. Now, once we have derived the equation in terms of stresses and then equation in terms of in terms of displacements; let us now summarize all these equations. And then from there we will see how that equations can be solved.

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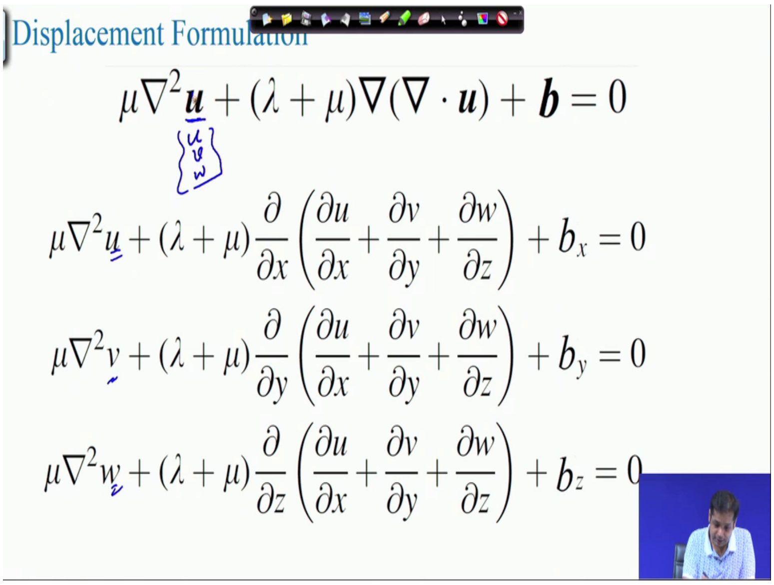

Now. so, finally this equation becomes this equation become this if you a this is the general form. And then if you write component wise then these are the components in x y z direction you get this 3 equations ok.

Please note here u v are the u v w are the components of u; where you write this u this u now here it is u v w this is a components and all these u v w are the scalars, but here u you can easily see it is written in bold letters; so, it is a vector.

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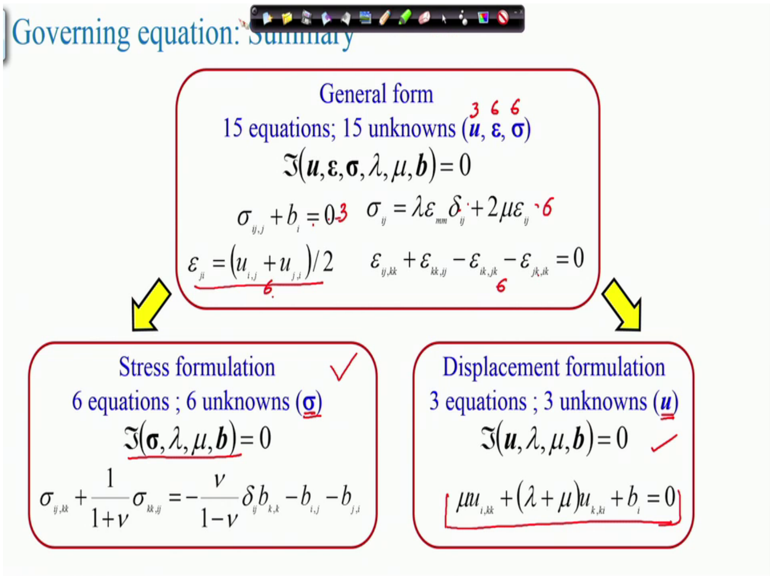

Now, so let us summarize these equations what we have is we started with if you recall we started with a very general form where we had 15 equations and 15 unknowns and what are the 15 unknowns? We have 3 displacements and then 6 fends and then 6 stresses these; these are 15 unknown, this is 3, this is 6 and this is 6 total 15 unknowns. And then we have 15 equations 15 equations are what? The constitutive relations, we have 6 equation these constitutive equation; we have 3 equations and then not the equilibrium equations we have 3 then the constitutive equation we have 6.

And then either you take strain displacement relation or compatibility equation we had 6 equation here, 6 equation here. So, either you take these 3 first stage formulation and this 3 for displacement formulations. So, total relations available is 15 relations available, so this is for the general form. But that these 15 cannot be really unknown because we found there are some e some relations can be again established between these equations and we can combine them and then we can reduce the number of equations and that we did.

Now, then we have stress formulation and then stress formulation essentially is what we did? We combined the equilibrium condition, the compatibility condition and the constitutive relations. And eventually the questions we have is this, where sigma is the unknown and this is the equation of a level. And this is the final form of the equation written in terms of written in addition notations. And then we have another formulation displacement where again the constitutive relation, constitutive relation, equilibrium equation are not the compatibility rather the strain displacement relation we use. And we can have 3 unknowns and 3 equations these 3 unknowns are the components of u and this is the final form of the equation written in indicial notation ok.

So, either you use this form to; so, there in once we have any problem either when we write represent this; we have to represent this idealized system through some mathematical model. And this is the mathematical model written in terms of stress and this is the mathematical model written in terms of displacements ok. Now next see what are the ways to solve these equations?

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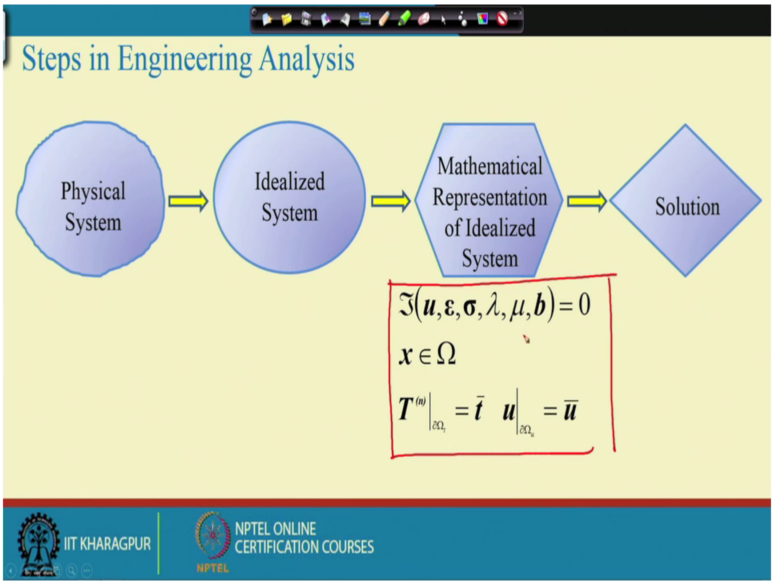

Now you see again I will; if you remember this is the same slide shown in one of the classes, where I why we discussed how what are the different steps we follow in any engineering analysis.

We idealize the physical system and then write the mathematical model for the idealized system and then solve it. What we have done so far is in this week we tried to write the mathematical model for an idealized system. Now remember one thing when we as we stated that time as well every steps from physical system to idealization of the physical system and then from idealization of the physics mathematical representation of the ideal idealized system. And then again for the solution every steps you make some assumptions ok.

And these assumptions are very important because they are the limitation of any formulation. Now when we write; so this is the general form of mathematical model, this is the general form we derived so far for elasticity linear elasticity problem. Now from there you can write in stress based form or displacement from that does not matter, but this is the form this is a mathematical representation of the idealized system that we have we have derived so far. And what are the major assumption we took? We took we assume that that material is linear elastic material.

So, therefore, we have essentially 2 constants linear elastic and isotropic material. And then we assume that the it is a small deformation theory, strains a small anyway and that and that is the reason why this relation between the strain and displacement and the relation between strain and stress can be linear. So, these are the major assumption that we did that we had while deriving these equations.

But again these assumptions are limitations of these; equation as well because there are problems where this linearity assumption cannot be may not hold. And in some cases or there are problems where the elasticity assumption itself does not a material no longer remain elastic. So, in such problems these equations cannot be these equations we have to enrich this equation a in order to input in order to incorporate all these features that the material exhibit when it undergoes deformation ok.

Now, once we have this mathematical representation; next is a solution of this equation and let us see what are the techniques that we have for this solution of these equations.

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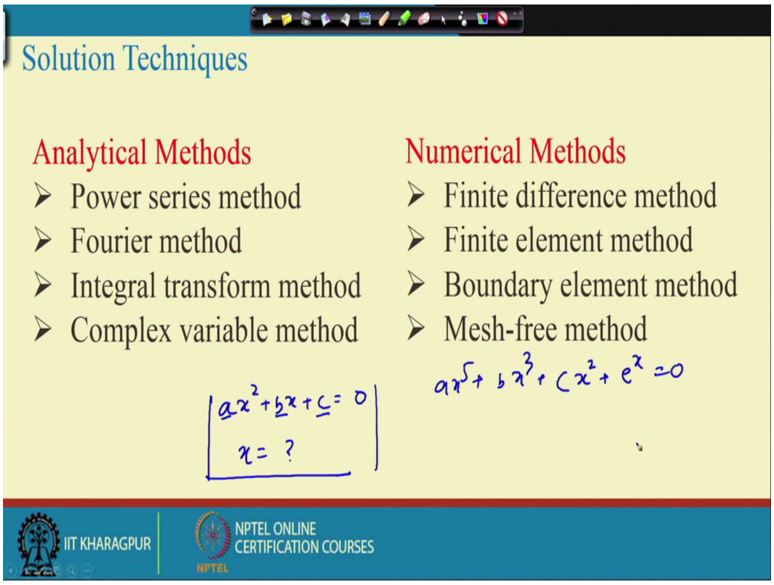

Now broadly these techniques can be classified majorly classified as a analytical methods and numerical methods. There is another method called approximate method, but I am not putting it in a separate category. So, analytical methods is something where a close function solution is available ok. For instance if I have to if I give you any quadratic equation; for instance if I give you a x square plus b x plus c is equal to 0 ok.

Now, we know that the solution of these equations is equal to x we can have the solution that is or we can have as a function of if you know abc and then we can write the what is the value of x from this expression. Even if you have some other expression then from which we can; we can solve the close form. So, this solutions are obtained using is an analytically solved.

Now, but there are problems suppose a same equation which has if I give you say a x to the power 5 plus b b x cube plus c x to the power x square plus e to the power x equal to 0; if I give you some equation like this which you cannot you cannot have a closed form solution. In that case what you have to do is we have to look for some numerical methods; numerical methods one very popular numerical method is for solving these non-linear equations is Newton Rapshon method.

So, similarly when we when we solve elasticity problem; so depending on there are some problems where your boundary conditions are very simple. The domain itself is very regular in those cases the problem can be the solution can be obtained through analytical method in a closed form way, but in but several problems rather most of the problems exist that we encountered in real life, the analytical methods are not sufficient and in that cases we have to go for numerical methods.

Now, the some of the analytical methods that can be used are power series method, then fourier method, integral transform method, complex variable method. Now in this class or in this course; we will spend next 3 weeks we will spend mostly on power series method; how different how the power series method is used to solve these equations in different Cartesian coordinates. And then polar coordinate system and then one more we can spend on complex variable method, this one and this one that would be our emphasis on ; in this course.

The numerical methods for instance these are the some popular methods like finite difference method, finite element method; you must be having course on finite element method and finite difference method in your curriculum. Then we have boundary element method, then there is another class of method which is different than finite element method; which is which do not rely on element unlike finite element method, which is sometime called as mesh free method or elementary free method or particle methods.

So, these are the some methods available numerical methods; so, we do not we will not discuss in detail all these methods. Now here one point please note you see the most of the problem or rather; I would say the all the problems that we really encounter in real life; that we when you go to industry and then you even if it is elastic problem and you encounter a problem.

Then you will see that you have to use some numerical technique; there the analytical methods not feasible there. So, essentially for the practical purposes numerical methods we will have to use the end of the day. But still will be spending next weeks and another one week on complex variable approach. So, we will be spending 4 weeks on analytical methods there is a very important reason for that you see let me give you an example before I state what I what I intend to say.

Suppose you this is the problem encountered in defence ok; we have suppose you take any structure ok, either you take beam or plates for instance you take beam we have a beam like this; suppose you have a beam like this.

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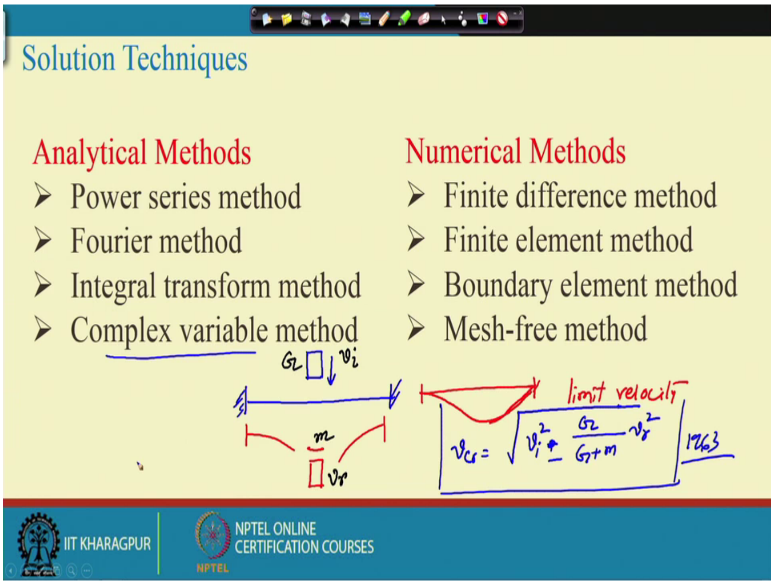

And then which is subjected to some impact ok; impact by projectile say the when you fire, this bullets come from the gun and that bullets hit any target, but that target could be a beam, that curve could be a plane or any structure or wall.

Suppose if it is a beam or a plate take a clamped beam now once this targets hits this structure, then what may happen is this structure may fail like this ; this structure may fail like this and this is pro this is the projectile. So, it breaks or it undergoes say undergoes huge deformation ok.

Now one important term that is used in defence; in defence is the limit velocity limit velocity or, ballistic limit. Now what does it means it means suppose in a very crud way if I have to tell you, that suppose you have a structure then you fire a bullet and then the bullet has some velocity for some kind of bullet the bullet has some velocity.

Now, if the velocity small then what may happen; this = target may not fail considerable deformation may take place, but it is not it may not get penetrated. But if you keep on increasing = the velocity of the bullet, what will happen? At some point of time this target may break ok; the bullet may penetrate the target ok. And that is the limiting velocity the velocity at which it happens that is the limit velocity ok, there are many other definitions of the ballistic limit.

But this is one way we can say; so that is the limit velocity suppose] this is a very important measure to quantify the safety of a structure or the strength of a structure. Now suppose you have to find out limit velocity of a structure ok; now any problem you solve there are 3 majorly 3 approaches. One is the experimental approach you do experiment in lab and then you find out the limit velocity; how do you have to do the experiment?

You make a sample and then fire projectile and you keep on firing projectile with different velocities and then see with your all this camera. And all these data acquisition system at what velocity the penetration takes place and you say that this may be a limit velocity. Another approach is the numerical simulation where you keep on simulating the same thing the same experiment, but you simulate in in computer.

You simulate this numerical this numerically simulate this entire event and keep on increasing the velocity and then see at what velocity this penetrates and this is your limit velocity. Now suppose, but this is time consuming task right; now suppose if you have to quickly access what is the limit velocity? Quickly suppose in some reason you have to understand you have plane and you want to quickly access that what is the limit velocity of the structure.

There is a model the analytical model which tells you that if it is I am just give you an example to tell you the importance of analytical model. Suppose this is the mass of this projectile is G and then which is limiting the initial velocity of the projectile is v i; v i is the at what velocity that impact takes place. And once the penetration is done then v r is the residual velocity of the projectile. And suppose the after the penetration what happens is some chunk of the object from the target comes out. And suppose that is suppose some chunk of the object comes out with the projectile ; some chunk of the target comes out with the projectile.

So, suppose this velocity this mass is m; small m then there is then it says there is analytical method which analytical formulation which says that that velocity the critical velocity will be square root of v i square plus G by G plus m v r square ok. This is the critical velocity this is minus this is the critical velocity ok; now this expression was derives in 1963 ok.

Now, this is derived with a very simplistic assumption, but what it is what is observed is this gives when your when your impact velocity is very close to the critical velocity; this gives a reasonable approximation of the limit velocity, which can be used not for the detailed study of the behavior of the structure, but for a quick estimation of the limit velocity.

Now, why I gave this example? That reason is the exercise that we will be doing in next 3 weeks and also another one week, if you look from industry perspective that exercise you may not do in industry on a regular basis. But that exercise will give you a platform; if you have to develop an analytical model for some problems, for some physical process these exercises will tell you, what are the possible ways that you can adapt to come up with an analytical model for physical process.

And also when we when we talk about complex variable and method; from complex variable method we will try to solve some of the example and that example will take us to a very fascinating branch of mechanics, which is called fracture mechanics. But that we will discuss you do not have to bother right now for that whenever we come to complex variable method we will discuss that there ok.

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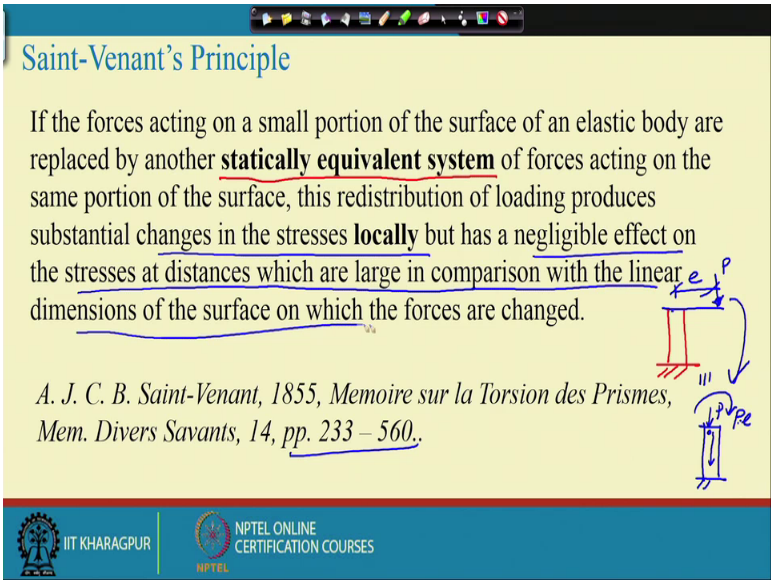

So, now these are the things that we will be doing in the next 3 weeks, but before that few things is very important,2 important things that is important that is called first is Saint Venant's principle; Saint Venant's principle; if I exactly state it what is written here in this what is written here.

Now, allow me to read it you can also read it; if the force acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system, that is very important. You remember in mechanics, you studied equivalent force couple system where the set of force is replaced by equivalent force coupled.

For instance, if I give you an example, suppose if I have to apply a load like this; column or a tower or a anything a post. And then if I have a load like this and if this distance is e and this load is P and this can be the equivalent system could be like this; this is P and then moment P into e ok.

So, this is called equivalent; equivalent system. Now what it is saying is that if that is statically equivalent system; if you replace the forces by a statically equivalent system of forces acting on the same portion of the surface, the redistribution of the loading produces substantial changes in the stress locally. When you come from this to this, what will happen? Because of this are statically equivalent systems.

But what do you what is what happens is that the because of this change, there will be a change in stresses and deformation locally. But you as you move from this place it will be this effect will be very small. So, negligible effect on the stresses at a distance which are large in comparison with the linear dimension of the surface on which, the forces are changed.

Just to give you an example so if I have to give you an example for instance quickly, if I take suppose again the same thing.

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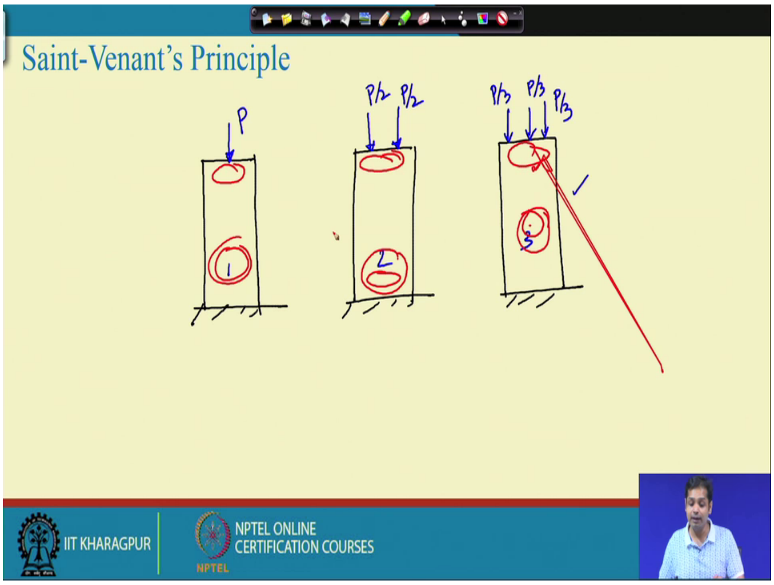

Suppose you apply a load like this suppose this load is P ok. Now suppose this distributionnow what you do is suppose you take anotherproblem where same thing, but the load is P by 2 and then another is P by 2. Or you take another problem where here load is say 3 loads, you apply all are P by 3 P by 3. So, you see all this; all this systems system 1, system 2 and system 3 they are statically equivalent system ok.

Now what this Saint Venant's principle says is now because of this application of the load; you have some stresses generated throughout the body, you have displacement throughout the body. Similarly when you change this it will also give you in the stress distribution in the strain distribution throughout the body, this system is also give you some stress and strain distribution throughout the body.

But what this principle says is if you change this then locally if you take a point locally around this zone ok. This is just this zone I am showing just for the representation around this zone you may have some drastic change in the stress field or the strain field, but as you move further as you move further; suppose you consider another zone here, another zone here, another zone here which is far away from this change; your effect will be very negligible. So, the distribution of stresses here, distribution of stresses here and the distribution of stresses here they will all they will all be same the effect is very negligible.

Now, for the time being you take it for granted this the Saint Venant's principle; you take it for granted, we will be solving one example like this.

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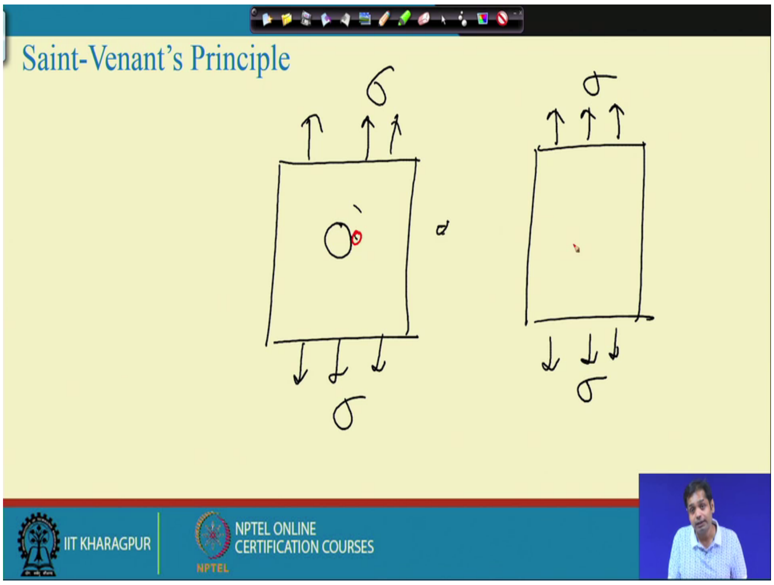

We have a plate with a central hole and which is subjected to in plane load say some say sigma; sigma. And then we will compare this we will compare this solution with a solution like this sigma and then and then sigma.

So, where there is no hole and there is a hole. So, it is essentially some small changes you have in the problem. And then we will plot this distribution of stresses and then see if we move far away from this effect from this; this change well the distribution of this stress in both the problems almost same. Whereas if you consider the distribution of stress and the displacement and the strain very near to this change; your these 2 are drastically different.

That is what essentially Saint Venant's principle said we will try to demonstrate this we will see whether it is actually happening or not, but for the time being you can take it for granted. Another important thing is that we will be using very often is the principle of superposition. The principle of superposition is something that you all know and you probably; that time you did not know this is a name called principle of superposition, you you learnt it in your in your 10th standard or the or the or the 7th standard when you actually first you are introduced to algebra

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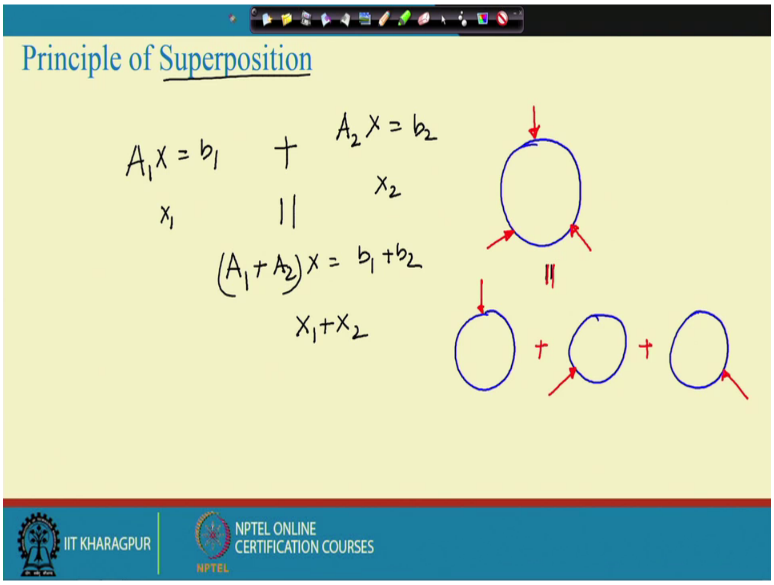

Now, what it says that suppose you have a you have you have 2; 2; 2 you have say A 1, A 1 X is equal to b 1, you have a set of linear equation like this and then you have A 2 X is equal to b 2 and then you have a linear equation set of linear equation like this. Suppose here the solution is X 1 and here the solution is X 2 ok.

Now, what linear; what linear principle of superposition says that now if you have a system called A 1 plus A 2; X is equal to b 1 plus b 2, then the solution will be X 1 plus X 2. And you see you all know this, so this is equal to this will be is equal to this plus this that is what linear superposition ok. Now this is valid if your equations are linear; if the equations are non-linear you cannot say that ok.

Now since the linearity is one of the very important assumption in our formulations. So, this superposition is applicable what it tells you in the context of if I have to tell you this same thing in terms of some example. Suppose you have a system like this, you have a system like any arbitrary system any arbitrary system we have a load like this like this and like this ok.

Now, you want to suppose you want to analyze it ok; now this can be equal to equal to a system with a same system which is subjected to we have 3 system. And then this system is subjected to load; this subject is subjected to load this and this system is subjected to load this. This plus this plus this is equal to this that is what linear superposition says ok. And we will be using this principle very often whenever we have different kinds of load acting on a acting on a acting on acting on an object, we try to split into pieces and then then combine them.

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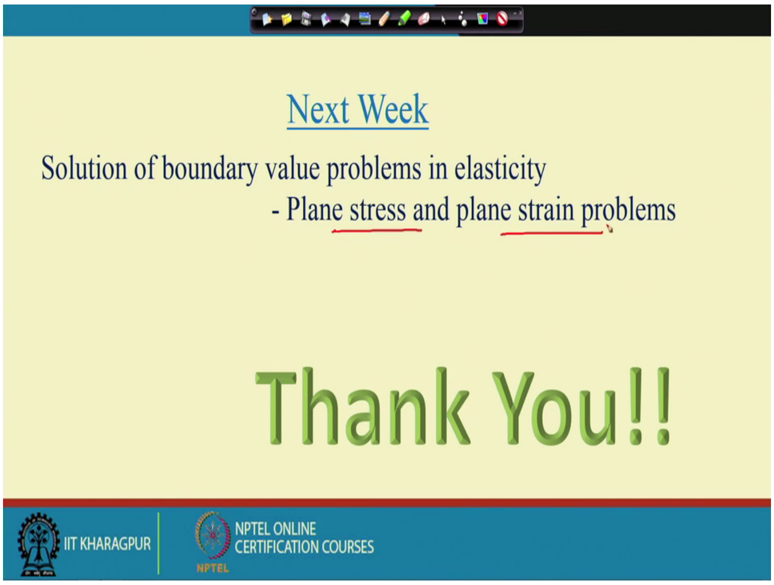

Now so with this we are almost ready to ready we have already set the platform all the necessary information that we required to go further and apply to solve some problems in elasticity; we are almost ready for that. So, next week what we do is we will the we will see how this the boundary value problems that we have formulated how to solve them. And then next class we introduce one of the very important concept; next week that is plane stress and plane strain problems.

You remember I we discussed that there is nothing like 2 dimensional problem in real life; all problems are 3 dimensional problem. Any object you take this is essentially a 3 dimensional object right, but under certain circumstances; we can idealize this object as 2 dimensional object. So, that we can our analysis become simpler we have to we have to deal with the less number of unknowns, we have to deal with less number of equations.

We will see that what are the circumstances, what the situation where the 3 dimensional problem can be projected; can be idealized as 2 dimensional problem and what are the different kinds of such idealization 2 idealization; one is plane stress and plane strain, we will discuss in detail in the next week then I stop here today see you in the next class.

Thank you.