Theory of Elasticity
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Lecture - 24
Formulation of Boundary Value Problems (Contd.)

Hello everyone this is the third lecture of this week, we have been discussing formulation of boundary value problems in elasticity. And, if you remember we discussed there are 3 important parts for this: one is equilibrium, which tells you how the stresses a different components of stresses are related to each other and then we have compatibility and then we have another important thing which is the constitutive relation.

Now, in the last couple of weeks we discussed compatibly we discussed a constitutive relation in detail, in the last class we discussed what is equilibrium equation; today we are going to start compatibility equation. So, today's topic is compatibility equation, but let us before jumping into the compatibility equation directly let us understand what is compatibility physically.

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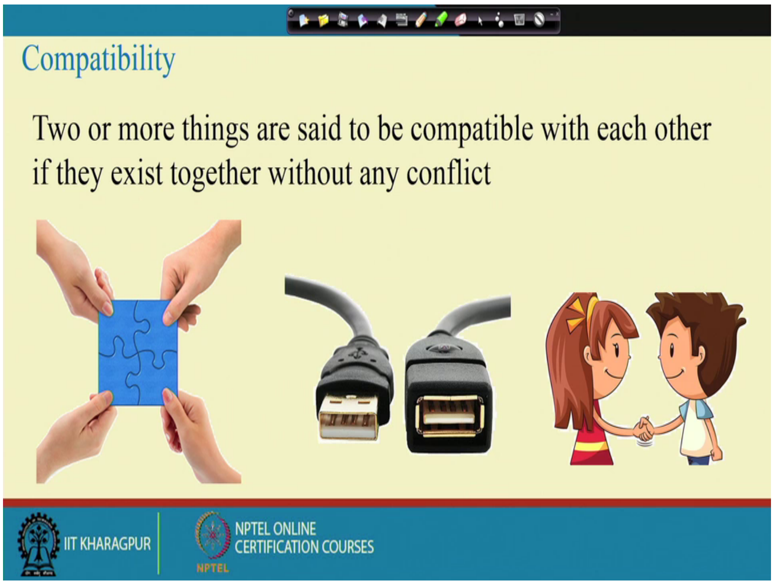

You see the compatibility means and we all are familiar with this term compatibility, compatibility means that if few things they coexists without any conflict then we say that these two are these things are compatible with each other, if not then we say that there in compatible. For instance if I take this small box now, this is the lid of this box and if I press it then it can, we can shut it. So, this is compatible with each other, but again then if I take another lid like this, but I cannot use this for this box ok. So, we say that these are these two are compatible with each other.

So, the this is the meaning of compatibility in a general sense that we all know and when we apply this compatibility the notion of compatibility. When we apply in the context of elasticity the remain exactly same, but the context change; context change in the sense that now you see here it is written two or more things are said to be compatible. Now, what are the things that we are dealing with in elasticity that is different. So, now what we are going to do today is we are going to see that what is the requirement of compatibility and what needs to be compatible in the context of elasticity and then we translate that compatibility in the form of some equations which are essentially called compatibility equation.

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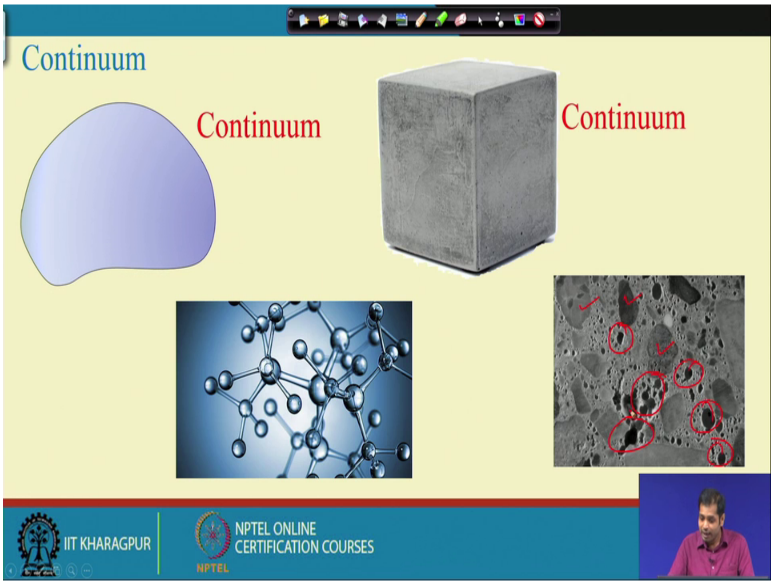

Now, another important aspect which we will not discuss in detail, that is the concept of continuum. You see for instance suppose take a chalk, now this is a continuum now if I break into pieces, let us before formally introduce continuum let us understand through some example. This chalk is a continuum now, if I break it and make it 2 this entire system, if we take the entire system then this no longer remain a continuum. Individually this is a continuum this is a continuum, but if I put together and then say this is my problem domain the entire thing is not continuum. Now, what has changed when we break this chalk, initially the chalk was like this where if we the, if we there is no physical separation in this chalk.

If we have if we define any function, say for instance displacement, if we define displacement or stress any field variable; then that definition that can be defined through a continuous function it is, it can be continuously defined, but whereas if you break it and make it like this then this definition cannot be continuous because, the field variable is now no longer remains continuous. So, there is a discontinuity at this point so that is why this is no longer remain continuum. Now again continuum will we say something is a continuum it is also an idealization, we idolize this as a continuum.

For instance if we for instance it also depends on a what scale we are dealing with, for instance if we take the molecular structure of any arbitrary, any matter you take the molecule structure is this we have molecules and then we have space in between. So, in that scale there is no continuity between these two. So, what happens in that case at that scale it cannot be coined as continuum. But if you look at larger scale macro scale we assume that there is no space in between all the spacer, there is no void, continuously matter is continuously distributed like this. So, this is the continuum idealization; for instance probably this example will give you better understanding we take a concrete block.

Now, if you cut this concrete block see this is the this is the image of one plane of this block, we can see the aggregate these are the aggregates, these are aggregates, these are the aggregates and then we have cement matrix and then in between we have some voids, you see these are some voids, these are some voids. So, like this if you take any material we have some internal flows we have voids, but we assume there is no voids, there is no internal crack, no flow, we assume the matter is continuously distributed it is in so and then we can call them as continuum ok.

Now, but again at this point 1 at what we have we have there are separate branches of mechanics depending on at what scale you are dealing with, if you are dealing with that continuum scale you have continuum mechanics, if you are you are dealing with the say micro scale you have micromechanics.

So, depending on the that what scale you are dealing with what are the things that you consider your mechanics become different, the governing equations are different the associated variables are different. But here we assume it is a continuum and therefore if I have to formally define continuum. Continuum is what we assume the matter is continuously distributed in the body and then any field variable can be continuously defined over that body.

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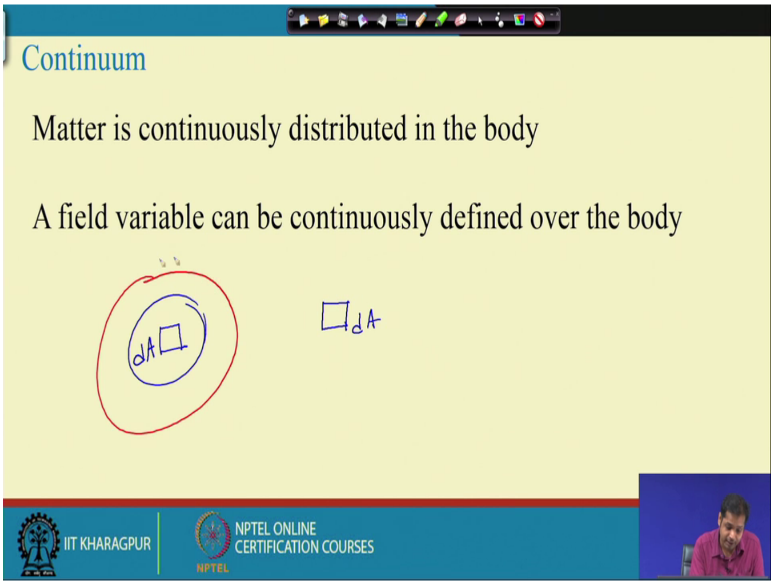

You recall we and the very when we when we discuss the concept was stress and strain, we introduce a term called infinitesimal area and infinitesimal volume. We had we have a domain like this and then we took an infinitesimal area like this or volume and we say that suppose this area is dA.

Now this continuum hypothesis the continuum assumption will also says that irrespective you taken an infinity, you can assume an infinitesimal area this area you can assume a infinitesimal area. And, if you take this area the property of this area this dA the property of this area will be exactly same as the entire domain. So, therefore the you can take if you are in the continuum hypothesis then you can use the notion of infinitesimal area and infinitesimal volume that we discussed some time back ok.

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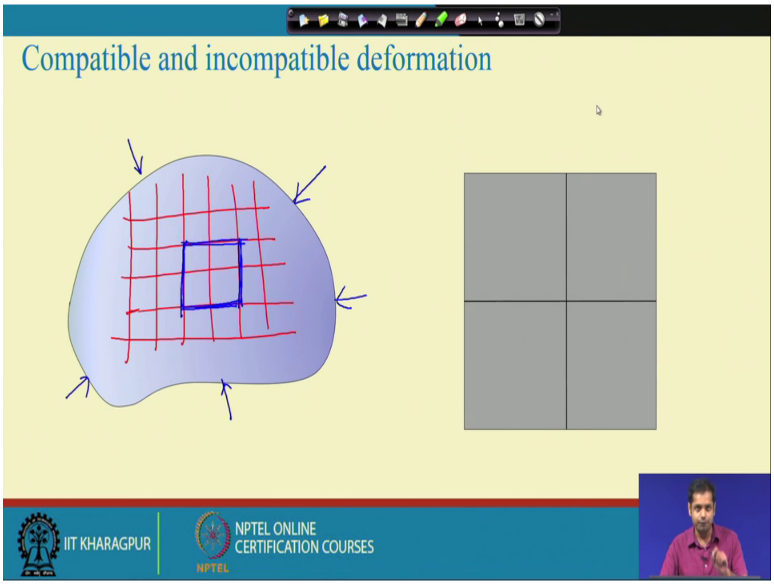

Having said that now consider any arbitrary body, let us now introduce understand what is compatible and incompatible deformation. Now, suppose to take an arbitrary body any 3 dimensional object and suppose this object is you takes smalls you take some grid like this ok, just to identify you can assume that this is our infinitesimal area and suppose take a small grid like this ok.

Now suppose this grid is this one is this, now suppose this body is under some loading and then it is subjected to this undergoes deformation it is different kinds of loading and this undergoes deformation. Then what we are interested now is the, what is the deformation of this small square, which consists of 4 square 1 2 3 4. Now suppose let us see two kinds of deformation.

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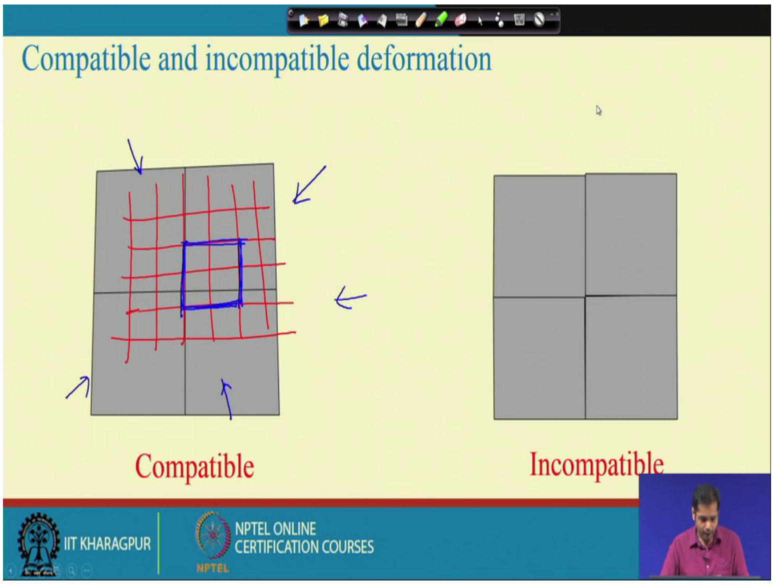

Now, you try to understand what is the difference between these two deformation, in the first deformation in the first deformation and you see the and the second deformation is incompatible deformation. Why you see that initial there is no space there is no gap in between all these small small element there was no gap, but when it deform every square deforms differently there is a small gap when the points the junction points are not also maintained properly.

Whereas, in the first case everything is if the deformation of 1 element and the deformation of another element they are consistent with each other ok, they do not contradict 1 do not contradict other. So, this case this is called compatible deformation and this is called incompatible deformation ok, just now the assumption of continuum we said in this case you see it was continuum and even the object undergoes deformation.

The continuum nature is preserved it remains as continuum, whereas in this case in this case what happens that the it may actually these block may actually come out if it is not incompatible it is not compatible, then the continuum nature may not get may not may not get satisfied. Again go back to the example of chalk this is the this is the continuum and then if I apply some if I apply some load on this object then what will happen this chalk will deform, but in the process of deformation this chalk will not be like this that is our assumption ok.

When it happens then all the equations that you we going to derive everything that are no longer valid, so we always assume that if it is a continuum this nature is preserved ok. Now and on top of that if there is no if there is no voids and no physical separation in between 2 material points exists before the deformation and after the deformation there is no physical separation or physical discontinuity arises in the material. So, therefore any variable you take it can be continuously defined over the entire domain. Now, once we have understood what is compatible and what is incompatible deformation, then let us try to understand that what is the what is the what is the use of this compatible.

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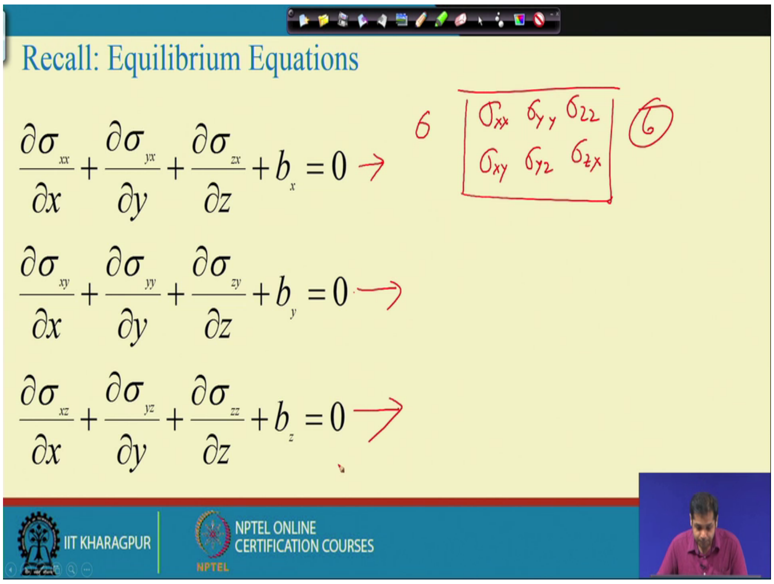

We have to translate this understanding into a set of equations, but before that let us also understand through the through mathematics point of view, why this equations are why this compatibility are required. You see recall the equilibrium these are the equilibrium equations we have 3 equilibrium equations and if you look at the number of unknowns these equilibrium look at the number of unknowns, we have the 6 total 6 unknown. And, the 6 unknowns are sigma x x sigma y y sigma z z and then sigma x y sigma y z and sigma z x, these are 3 total 6 unknowns we have. Where as we have number of equations 3 so we cannot solve them all these 3 unknown, we need more equation we need another 3 equations ok.

So, we need another 3 independent equations then only we can these all these equations are independent equation, they all correspond to different axis if we recall when we derive these equations. Similarly just not 3 equations we need we need 3 independent equations.

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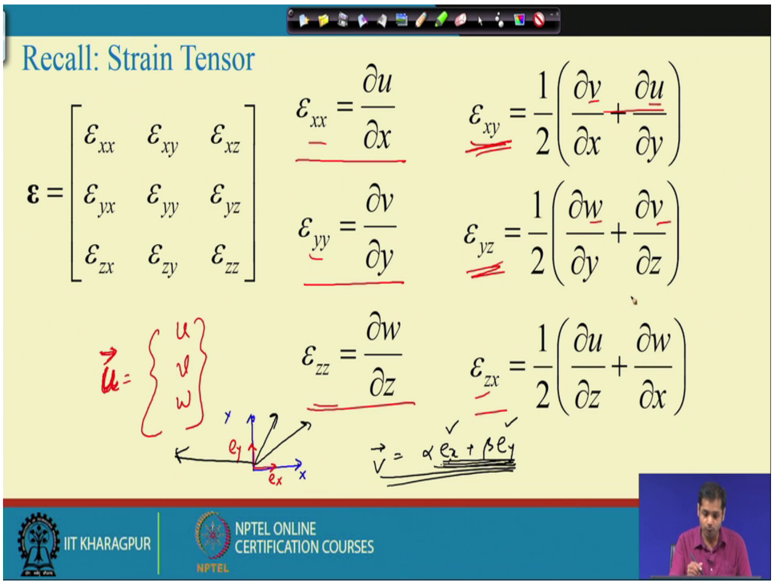

Now again see if we if you look at the strains we have suppose we have deformation field u is like this u v w, this is bold u in a vector form to. Now, so this u v they are the deformation ok, so this is essentially definition of strain displacement relation how the strains are related to displacement. Now, suppose you know the strain at a given point and you want to find out what is the displacement, then what we can do is we can integrate this strain and find out the displacement.

Now, you see we have 6 strain components right this 6 can components we have, but when we integrate them and these 6 when we define this 6 6 strain component the strain components are defined independently they are the function of displacement, but these strain components are defined independently. Now, so we really as at this point we do not know is there any relation between the strain components, but at when we derived it we derived it just based on the deformation of a representative element deformation in u v and w and then we define the strain components.

Now, when we integrate them and get the displacement what is the guarantee that the displacement will be unique, because if you if you integrate this then you get u if you integrate this then we get v, if you integrate this we get w even if we if we integrate this we get some relation between u v w if we integrate these 3 equation also we get u v w.

So, what is the guarantee that we get an unique displacement field and therefore if the displacement field is unique, then these all these strain components cannot be independent. They are some there is some relation between all these strain components it is as simple as that if you look at in this way that suppose you have in 2 dimensional in 2 dimensional in 2 dimensional space, suppose you have a 2 vectors like this is x coordinate this is y coordinate and in here your unit vector is say Ei unit vector is E x and here unit vector is E y or the basis vector is E x and basis vector is E y.

Then you take any vector any vector you take on this space any vector you take then this vector these vector can be defined as alpha E x plus beta E y the linear combination of these 2 vector. So, the only 2 independent basis vector E x and E y, if you take any other vectors those 2 vector those vectors cannot be independent there can be they can be represented through this E x and E y. So, similarly we have a 3 strain components 3 directions and we have 6 strain components and out of this 6 strain components there has to be some relations. So, that we can have we can have they are not independent ok. So, what we need to do is now we need to find out what is the relation between all these strain components and that relation is the compatibility relation ok.

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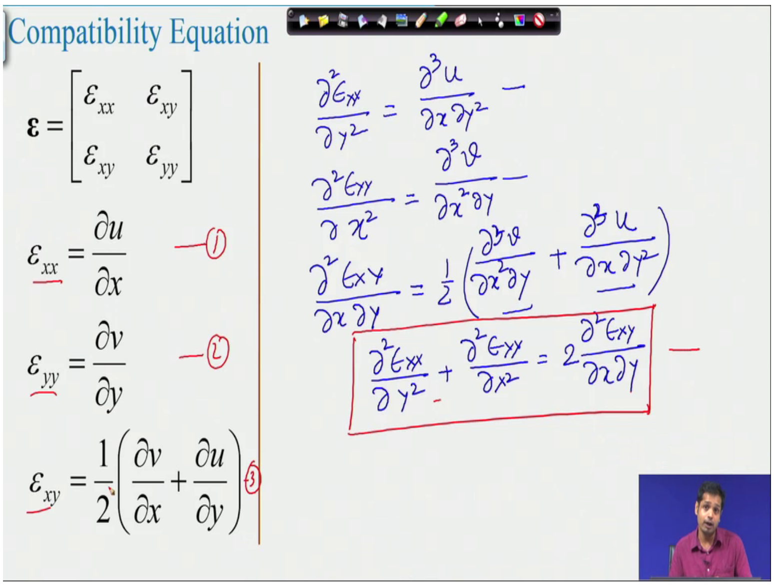

Now, what we do is we will we will we will derive in 2 way, 1 is let us start first we will start with 2 dimension case where 6 this is the strain tensor if you remember and then this is the definition of strain in 2 dimension and then another the definition of shear strain. So, what we are interested now is we are interested to find out what is the relation between these 3 strains these 3 strain ok. So, let us start first what we do is we differentiate this first one the first equation take this equation 1 this is 1 this is 2 and this is 3. The equation 1 the first equation we integrate with respect to say del epsilon del 2 epsilon x x del y 2 and this becomes del 3 u del x del y 2 and similarly if we differentiate the second one del y y and with respect to 2 we are doing it explicitly only for 2 dimension.

Next when it comes to come for 3 dimensions we won't do it like this we will be doing it using indicial notations ok. So, this become del cube v and then del del x 2 del x 2 del y and then similarly let us differentiate this one epsilon x y by x y del x del y and this becomes half remember half is require where it depends on which strain you are taking whether you are taking epsilon or gamma. In engineering strain or tensorial strain now this becomes if you del 2 v del x 2 and as long as this u v are continuously defined over the domain we can interchange this variable.

So, this is q so this is also q and this is u and this become del x del y 2 and then you see this you can substitute from this equation this and you can substitute from this equation and this you can substitute from this equation and if you do that then what we get is finally, that del 2 epsilon x x del y 2 plus del 2 epsilon y y del x 2 is equal to 2 del 2 epsilon x y del x del y right. So, this gives you how these 3 strains are related to each other. You see now in 2 dimension how many equations we have in 2 dimension we have equilibrium equations in 2 dimension, we have 2 equilibrium equations in 2 direction and then how many unknown we had unknown unknown stress components.

We had 3 stress components sigma x x sigma y y and sigma x y. So, we needed one equation and then this one equation is this one equation say together the equilibrium equation and this compatibility equation gives you the 3 equations and these 3 equation can be used to solve for all 3 unknown. So, this gives how these strain components are related to each other. So, they these strain components are not independent, if these strain components are if these relations are satisfied then what will happen that the every material point or the small element cannot be cannot and cannot under cannot undergo strain in different direction independently ok, so this is the compatibility relation.

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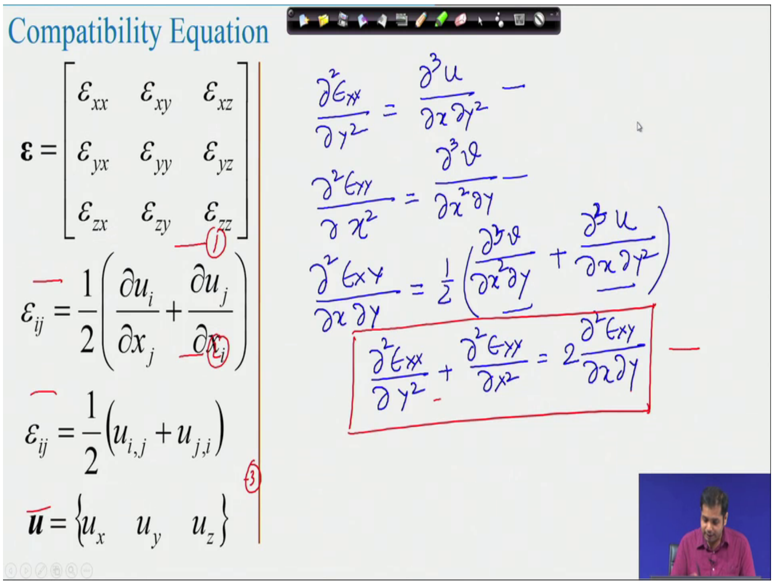

Now this relation is this relation is written for 2 dimensions ok, let us do this exercise in 3 dimensions. Now, so let us start 3 dimension case this is the this is the strain tensor and these are the corresponding stain component you see in 3 dimension what will happen, you have 6 strain components and then these strain components can be written like this either like this either like this in indicial notations and then u is essentially u x u y u y u I is essentially your xyz.

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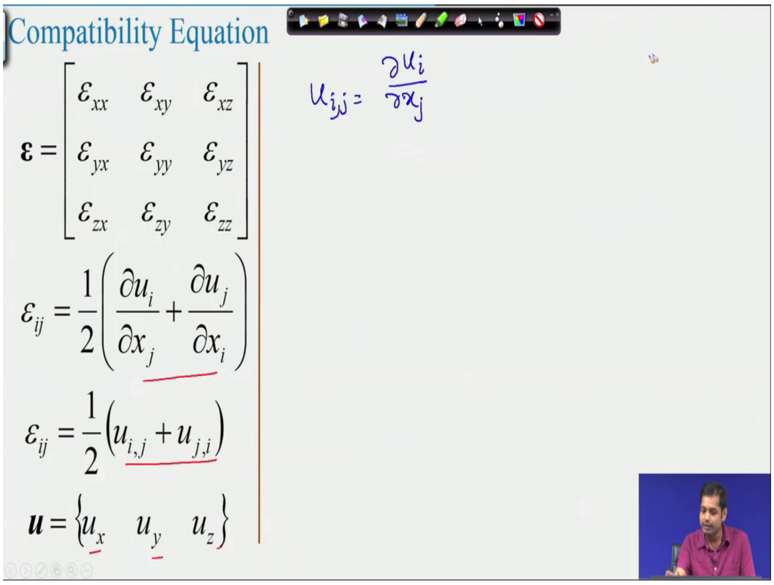

Now, what we do is we will write this equation in indicial notation and you when you do that this exercise you will realize that writing equations or deriving equations using indicial notation makes your derivations very very easy. Otherwise if you have to do this all these operator and all these derivatives it becomes very tedious you will see that in when in 3 dimension we can do it very very easily ok.

So, let us first we will do the similar thing that we that we did in the previous case, but since we have several strain components we have to do it several times. Now, first let us let us take another important thing you see that you please look at this thing this is very important, whenever we say that u I comma j this actually means del u i del j del x j i could be xyz and j could be xyz now.

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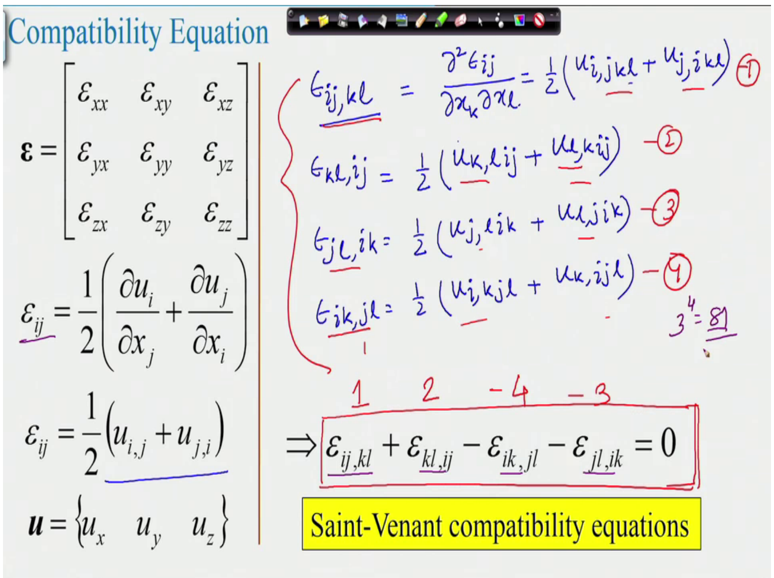

So, first take epsilon i j epsilon i j and differentiate with k l differentiate with this essentially means that del 2 epsilon i j del x k del x l this is written like this. Now so, if we do that what will be this will be if we look at from this expression this will be half of you can do this exercise u of i and then j k l j k l plus then u of u j comma i k l ok. Only thing is that it is now already this one is already differentiated with respect to x j, now additional derivative is done with respect to x k x l and similarly additional derivative is done with respect to x a x k a.

Now similarly if we take epsilon epsilon k l epsilon k l and this differentiate with respect to i j this become half of similar exercise u k. When we write epsilon k l epsilon k l means u k and it was already differentiated with respect to one. Now additional differentiation with respect to i j plus then u l which was already differentiated with respect to k additional differentiate is i j this is one and then epsilon j l j l i k this is half again the same thing this is u j comma l i k plus u l comma j i k the same thing and finally and why we are doing these exercise it will be clear shortly j l this is again same thing u i and then derivative with respect to k k j l k j l plus derivative with respect to k and then i j l ok.

Now let us call this equations the equations 1 equation 2 equation 3 and equation 4, what we have to do is see the we have to find out the relation between strains. Now, in the last when we derived it for 2 dimension you recall what exactly we did we eliminate the displacement term. So, therefore we are left with only strain term and that gives you the relation between strain, same thing here we have all these strain all these displacement term here what we have to is we have to eliminate this displacement term and all these exercise the entire thing all these 4 components we obtain just to eliminate the displacement term.

You see now there are some similar term you have, now if you take the plus minus though what you get is you get you get this like this. So, what you get what we do is if you take 1 plus 2, if we take 1 this gives this is 1 epsilon i j k l and this is k l i j this is 2 and this is j this is i k j l i k j l means this is your 4 this is 4 and this is j l i k and j l i k this is 3.

So, if you take 1 plus 2 minus 4 minus 3 then all these displacement term they cancel each other and we did all these just for that, so that the displacement terms get eliminated. Now once we have this then these is very important you see that this gives you how the strain components are related to different strain components are related to each other and this is called the Saint-Venant's compatibility equations written in terms of this equation this compatibility equation is written in terms of strain.

We will also see the next class the compatibility condition can also be written in terms of stresses, but let us let us wait for that. So, now you see what is i j k l what is this i j k l say use this i j k l you recall. At the first class first week we discussed the concept of tensor and how the tensor different order of tensor can be written in indicial form, what are the how many how many indices are there in this epsilon.

Epsilon i j when we write epsilon i j there are 2 indices i and j th epsilon is a second order tensor right and what is epsilon here how many indices we have we have i j k l so 4 indices. So, it is a fourth order tensor and similarly you say all these are fourth order tensor. So, when we have fourth order tensor then fourth order tensor how many components we have if we if we recall that 3 to the power 4 which is 81 components. So, essentially how many equations it gives you 81 equations.

If you compare term by term the first term in this first term in this first term in this and first term in this it gives you one equation. Similarly if we compare all 81 terms all 81 elements in every fourth order tensor and then we get an equations for each corresponding element so we have 81 elements 81 equations. Now, you see that is the thing, but 81 you recall you think yourself if we if we have to write all these expression using this derivative term all these 81 equations and everything it becomes so tedious right.

But in this case it is just writing 4 equation such that 4 equations are also very simple to write using indicial notation and that is the beauty of indicial notation and it is the very important that you write from the you write from the beginning, you start writing different equations deriving the equations through indicial notations ok. Now, here is one important before we close one important point, but you know this in 3 dimensions how many equations we have equilibrium equations in 3 dimensions we have 6 equilibrium equations and how many unknowns we have sorry in 3 dimensions we had 3 equilibrium equations and number of unknowns are 6.

So, essentially we need a 3 equations, now but we have 81 equations but these 81 equations are not independent equations, we can show that is out of this 81 equations only 3 equations are independent. The first thing what we can do is if we ok.

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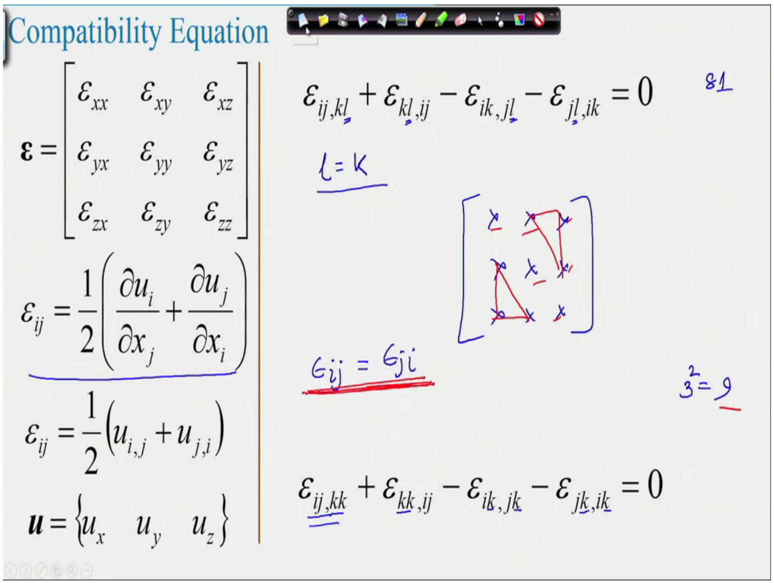

Now, this is the equation the strained compatibility equation, what you can do is the first thing is if we replace l because, of the symmetry you see if you replace l is equal to l with k l is equal to k, we will see that you will get the similar expression and if you do that you get an expression like this. So, l is replaced by now the l is replaced this exercise you please l is replace all this l term is replaced by k if you do that then you have this equation and now look at these equations this is a fourth order tensor 81 elements we had.

Now how many indices we have here we have i j k k now k k is a repetition, so as per the convention repetition means summation so it is a dummy index. So, essentially the index indices we have here only 2 here also is a k k is a dummy index k k is a dummy index repetition k k is a dummy index. So, it has how many indices we have we have 2 so it is a second order equation second order second order tensor. Second order tensor means how many how many how many how many components we have we have 3 square means 9 components ok. Now out of this 9 components epsilon i j here you see epsilon i j is symmetric tensor, this is the symmetric tensor is not it so epsilon i j is equal to epsilon j i.

Now, if we if we have a symmetric suppose if we have a matrix if we have a matrix which as these 3 by 3 matrix, if you if you just to just to have a better understanding if the matrix we have 80 we have total 9 components. But if it is symmetric matrix then essentially what we have these all are these is a symmetric these this term and this term is essentially same.

So, how many components we have we have 1 2 3 4 5 6, similarly if this is symmetric then we have essentially 6 components. Now next what we do is you this is written in indicial form. Now, let us expand this form writing i j as x y and so on and if you do that we will get 6 such equations ok, because of the symmetry of i j these equation gives you 9. But because of the symmetry of i j you will get 6 such equation and these 6 equations will be these are the 6 equations.

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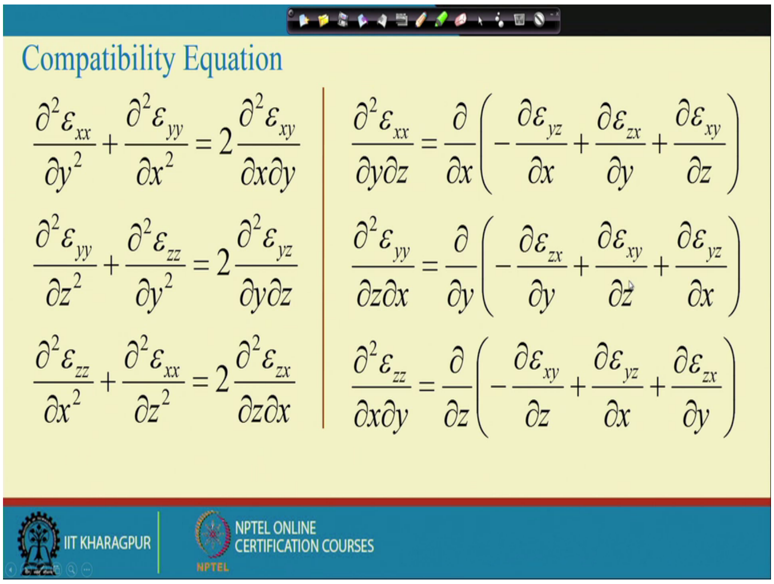

But again you can please try this, but again we needed only 3 equations but we have 6 equations. Again you can show that these 3 6 equations are not independent equations, out of these 3 equations only 3 are independent and what you do is I leave it to you try to find out the relation between these 3 equations and get the these equations are a second order equation you see. But when you when you find out try to find out a relation between these 2 combined this relation, what you get is you get a fourth order equation.

Very similar to you remember what we did in the case of this 2 dimensional problem epsilon x x was the first order equation and then when we then we had to differentiate it to get the relation between all the strain component. Similarly, here also you have to differentiate these equations further and then combine them to get the relation to get the

3 independent equation and that equation will be fourth order equation that I leave it you ok.

Now before I though this is the compatibility condition in 3 dimensions. Now here just 1 important point you see the compatibility equation is required, so that we can have we can define an unique displacement field continuous displacement field over the domain ok. Now, these if your if u is unique suppose if you have a continuous displacement field and these strain are these strains are defined with respect to u. So, the these equation is this equation is necessity to have this to guarantee the displacement field right to guarantee the continuous displacement field.

Now, the here question do not take it this is obtained from displacement field that is fine, but it is important to check whether these displacement the compatibility when we say when we have a strain components which satisfy compatibility ok. Then we have to integrate this strain component to get the displacement. Now, we have not yet proved we have derive the compatibility equation that is true, but we have not yet proved that this is the necessity necessary and sufficient condition for a displacement field to be continuous or an you can have a unique displacement unique continuous displacement field, these are the necessity and sufficient condition necessity and sufficient condition.

We have not proved it yet we have also not proved it that under what circumstances these may not be the sufficient condition for the uniqueness of the displacement field you put a star mark here. In some of the week in 1 of the weeks we will discuss the complex variable approach, we will come to this point once again and we will see that we will first try to understand try to prove whether, these conditions are necessary and sufficient condition for unique displacement field or not one thing. And, then also we will try to explore that different conditions where this may not be the sufficient condition for unique displacement field.

We will discuss that when we discuss the complex variable approach ok. So, once we have discussed the equilibrium equations then the compatibility equations today and then also the constitute relation last week, we are now ready to formulate the problem and that we will be doing in the next class. See you in the next class.

Thank you.