Theory of Elasticity
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Lecture – 23
Formulation of Boundary Value Problems (Contd.)

Hello everyone this is the second class of this week, as you remember last class we discussed there are three important ingredients of equations for elasticity. One is equilibrium which tells us how the stresses are related different component of stresses are related to each other. And, then we have compatibility or strain compatibility which tells us how the different components of strains are related to each other and then third one which is the relation between stress and strain which the constitute relation. Now today we will start equilibrium equation ok.

Now, so before you do that let us since equilibrium equation is essentially tells us the summation of forces is equal to 0, in a particular direction one important thing whenever you say equilibrium by implicitly it is implied that we are talking about static equilibrium.

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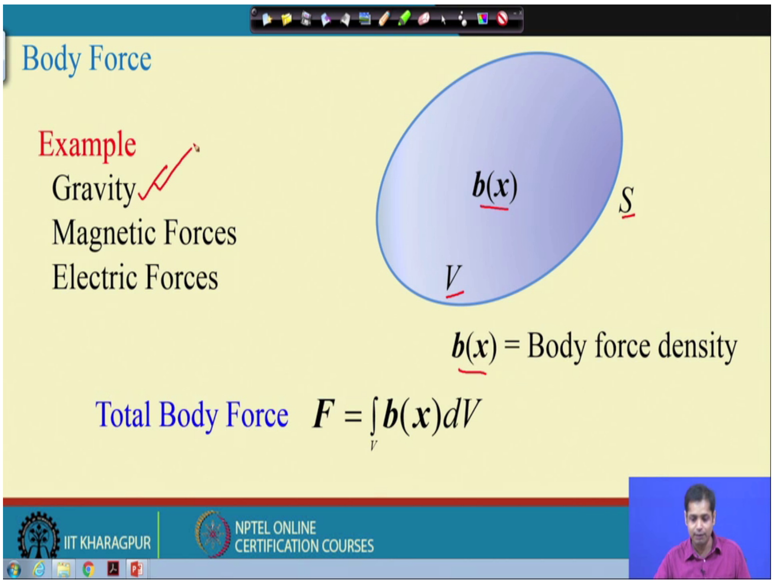

Now so, static equilibrium is what the summation of forces in all direction is 0, summation of moment in all direction summation of moments about any point is also 0. But the symmetry any Cauchy stress tensor automatically satisfy that moment balance equation. Now, since we have to at the end we have to write the equation in terms of stresses, but the equilibrium is always equilibrium of forces equilibrium there is nothing like equilibrium of stresses ok. So, essentially you have to start deriving the equilibrium equation in terms of forces, let us understand what are the different kind of forces that a body can have.

Now the first thing is the body force body force is for instance gravity is a body force, then the magnetic forces electric forces they are the body forces. What is that the body force is defined it is it acts over the entire body and it is defined by suppose the function say b x, for instance if it is gravity then it becomes the density how the density is distributed among the body. So, if the body force density is b x for a given arbitrary 3 dimensional object with volume V and the surface is S volume V and the surface is S the boundary of this object is represented by S and b x is the body force density it could be the it could be density it could be the body force density for magnetic field electric forces.

So, it depends on which kind of analysis you are doing depending on that the b x will be different. So, if b x is the density of the body force then it has to be integrated over the entire volume to get the total body force on the system. Now in this course possibly in this course we will be only dealing with this body force gravity the self weight of a any object. Now so, once so this in addition to that there is one more force that we can have is the surface force.

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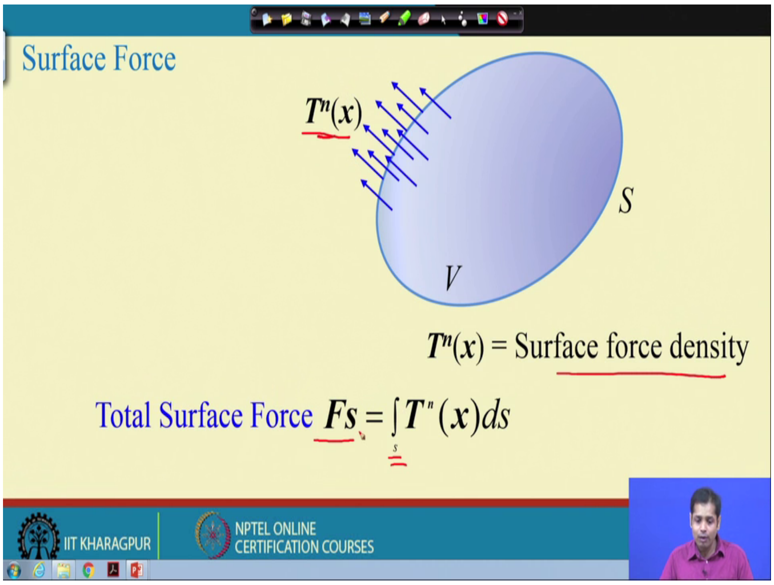

Surface force is if you recall in the in the in the previous class when we talk about what are the conditions boundary conditions specified at the surfaces, we discussed at the surface and those are the surface forces that is represented by a anys on any surface the stresses represented by traction at any point. Now, suppose T n x is the T n x is a function is a function which gives you how the traction how this how the it is essentially the surface force density, how it is distributed over the entire surface. Now if we integrate it over the entire surface S then we get the total surface force.

Now, this is the two important forces that a body can have we have the body force which is the by gravity is the body force and then we have a surface force the forces acting on the surface of any arbitrary volume. Now, once we have once we have these two kind of forces let us see how to derive the equilibrium equation for elasticity.

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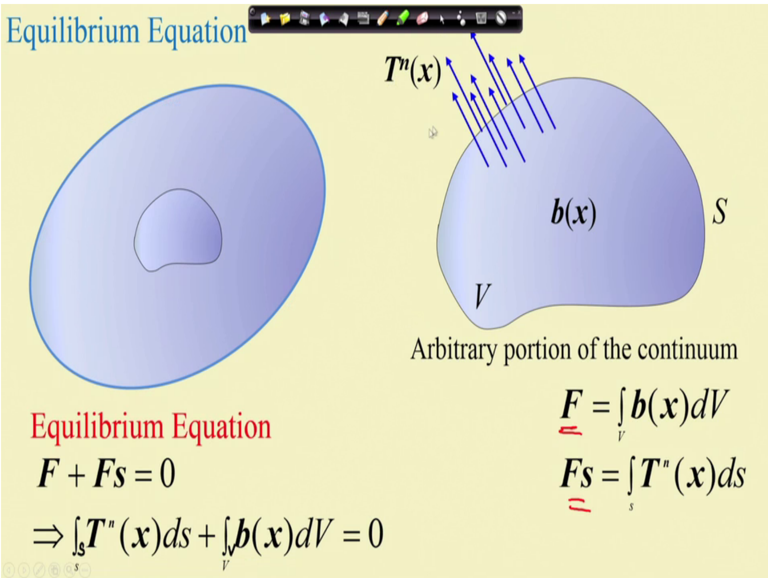

Now, take a arbitrary there are two approaches, one approach that we are going to do here which is very general and another approach is that is also available in any book, where you take an infinity symbol a volume element as we discussed by representing the representing the stresses. Represent the write the stresses on different planes and then convert the stresses onto into forces and then write the equilibrium of that forces that is another approach. This approach is again similar to that, but it have in a written in a very general form.

Now, suppose we have a 3 dimensional object like this, then we take a very small volume element into this object. An infinity symbol volume element into this in this in this object and let us draw the free body diagram of this object ok. Now, what are the things we can have on this object, we can if the body if this 3 dimensional body has a body force distributed in over the entire body, then this small chunk of continuum this is suppose this volume is V and again the surface is defined by S then this small chunk of continuum the one force is the body force. And, then since it is taken out from the taken out from this entire 3 dimension entire object, then if I take if I call that this is part number one and this is part number 2.

If you recall when we when we discussed traction, then we discussed we defined traction is a field through which these two different parts will interact with each other. So, if we take this thing out from this object then we will have traction on this surface and this through this traction the small part the part 2 will interact with part 1 ok. Now then we have suppose the body force density is b x and then in addition to that we have a traction T n x, it is a function of x means it is the traction may be different at different points ok. Because the state of stress is different at different points and the plane also the norm or the tangential plane is also different at different points. But for clarity it is not shown over the entire surface it is just shown it is a representative traction shown on the surface. Now, so if we have that so this is essentially the free body diagram of an arbitrary portion of the continuum ok.

Now once we you recall how you wrote the equilibrium equation in your in strength of material, when you draw the free body diagram in any object then the equilibrium equation free body diagram is essentially what we take the object from the entire system and write the forces acting on the object and then write the equilibrium equation on that free body diagram. Writing equilibrium equation means writing the forces in different adding the forces in different directions in a particular directions and then equating them to 0.

So, what are the forces then we have here, we have if we integrate these body force over the entire volume then we have body force. This is the total body force on the small chunk of continuum whose free body diagram is shown here and then we have surface force which is the integration of T n, T n is the surface force density so these 2 forces we have. Now, please note these forces are written in vector form. Now the equilibrium says then these 2 vector is the forces written in vector form this the addition of these 2 forces the total forces has to be equal to 0.

So, the equilibrium equation says that that summation of F the summation of body force and the summation of surface force that has to be equal to 0, there is no inertia here because we are talking about only static problem. So, this is the equilibrium equation now let us see what this equilibrium equation, let us now this equilibrium equation is written in terms of forces.

Now, let us write let us see how to write the same equilibrium equation in terms of stresses. Now, so once we have this then you see, so F is directly from here we can substitute F and F is here so we have this expression. It is not shown here properly, but you please note that this is in ts integration is over surface over surface and this integration is over volume this is very important. Now so let us see next, so this is the equation we have so this is total equation we have.

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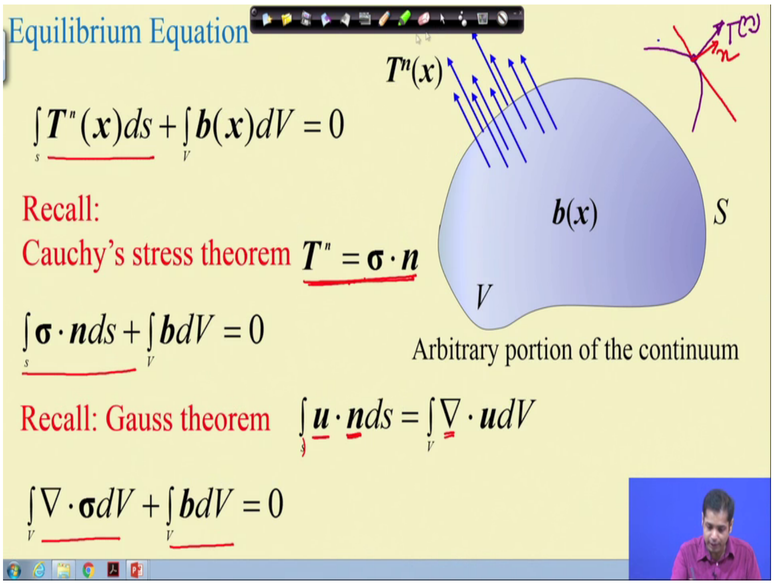

Now, you recall what we have you recall Cauchy's stress tensor, if we have any arbitrary plane if any arbitrary any arbitrary any arbitrary any arbitrary point in a any 3 dimensional object and on that point if we have if we construct a plane. And, suppose this is your actual object this is the actual object, then we have traction on this plane this is the traction on this plane and these traction can be represented and the sigma is the state of stress at that point and n is the normal to this n is the normal direction n is the normal, which defines the plane.

So, traction vector can be obtained as sigma dot n this is projection of sigma the stress. So, on this surface we have several points so at every points we project that stress on to a plane which is tangential to that surface or the plane which is defined by the normal vector n then we get traction, so essentially T n gives you this. Now, if we have this then substitute this T n substitute this T n here and we have this now.

Another important theorem that you has you have we have discussed in the first week that is gauss theorem, divergence theorem and what it says it essentially gives you tells you how the surface integral of an any object surface integral can be can be projected on to volume integral or the volume integral can be projected on to the surface integral like this is ok. So, now if we have if for any variable any field variable u and if at any point if there is if n is the outward is a normal vector, then the integration over the surface can be converted to volume integration like this when del is the differential operator, del you remember del is del del x del del y del del z.

Now, if you substitute this if you substitute this from T n from this and then again this surface integral if we convert it to volume integral, then what we get is we eventually get it this. So, now this is also integration over volume integration over volume this is also integration over volume, now let us combine them together. If we combine them together then what we have is this if we combine them together what we have is this.

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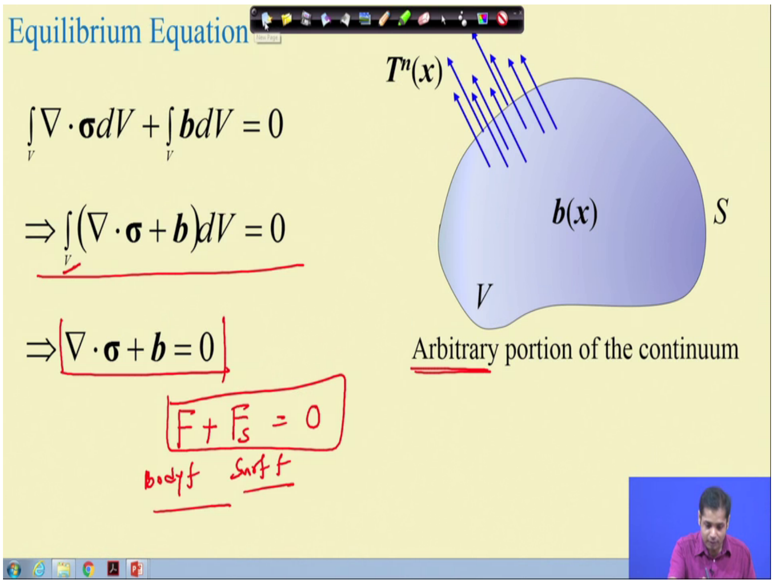

Now this is important now you see this is true for when we when we took this chunk of continuum from the entire domain, we did not have any specific assumption we have just taken arbitrary manner ok. This is an arbitrary portion this is very important what this arbitrary portion. So, this is arbitrary so this equation is valid for any for any part for if any part you take any chunk of any portion you take from the entire domain this equation is valid.

Then if it is so then this is only possible if this integration of something over d v and that d v that over this domain is arbitrary and that is taken any that that can be taken anything, then we can have is this integrand has to be equal to 0 the integrand has to be equal to 0. This is only possible this integral has 2 equal to 0. Now you see you recall our equation was F plus F S is equal to 0 it was body force body force and this was surface force surface force and this equilibrium equation is now written in terms of stress.

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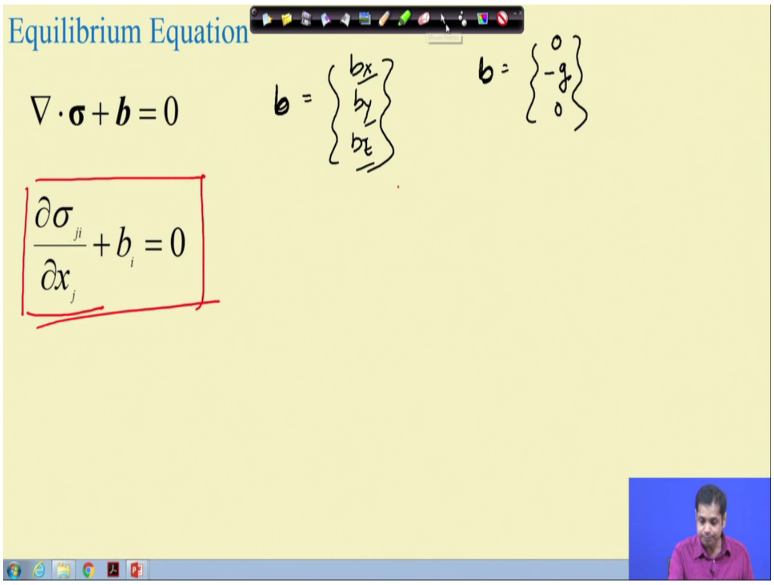

Now so, if I write this equilibrium equation in initial form, this is the divergence of sigma and you recall the very first week first week we discussed how to write the divergence what is divergence of divergence of a of a of a tensor, first order tensor or second order tensor divergence of a second order tensor can be written as in this way and then the b is the b is a body force and the body force can have density say b is b as a vector can have say b x b y and b z direction ok.

So, b x is the body force density in x direction b y is the body force density y and it is z direction suppose if you are taking gravity and which is acting only downward (Refer Time: 14:18) b can be b we can have something like this which is 0 minus g and 0 and so on. So, this is the equilibrium equation written in terms of stresses and written in indicial form.

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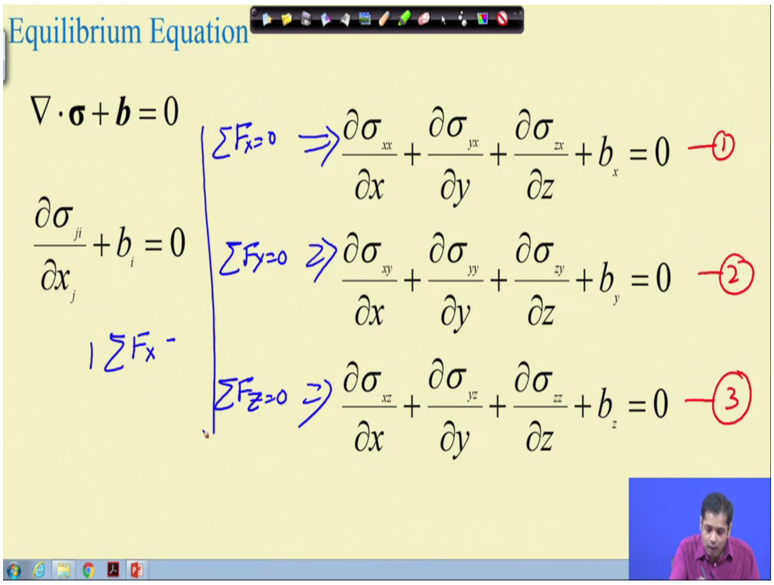

Now next is so if I expand this equilibrium equation, it is written in indicial form if I expand it if I write I j in terms of x y z and so on. So, these are the 3 equations we get this is equation number 1 equation number 2 and equation number 3. So, these is equation essentially what equations we have the equilibrium equation equilibrium equation it is a summation of summation of force in x direction is equal to 0 x direction is equal to 0 and then summation of force in y direction is equal to 0 and summation of force in z direction is equal to 0.

So, essentially it is same but now in strength of material you use this equation all this equation and use equilibrium equation it is same equilibrium equation, but now written in terms of stresses. So, already you know this equilibrium equation, so this is the 3 equilibrium equations we have and these equilibrium equation is with respect to Cartesian coordinate system ok. Now let us write the same equation with respect to cylindrical coordinate system.

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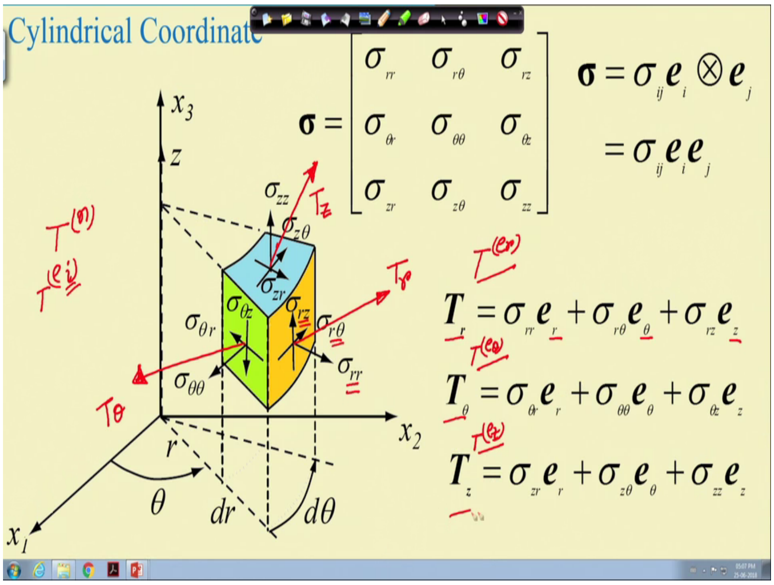


Now, cylindrical coordinate system the stress is defined as this we have 3 axes, one is one is r one is this axis r this r and then we have theta this is theta and then we have z and suppose e r e theta and e z are the corresponding basis vector in r theta and z direction ok. What are the stress component we have now suppose this plane if we take this plane is normal to the r direction ok.

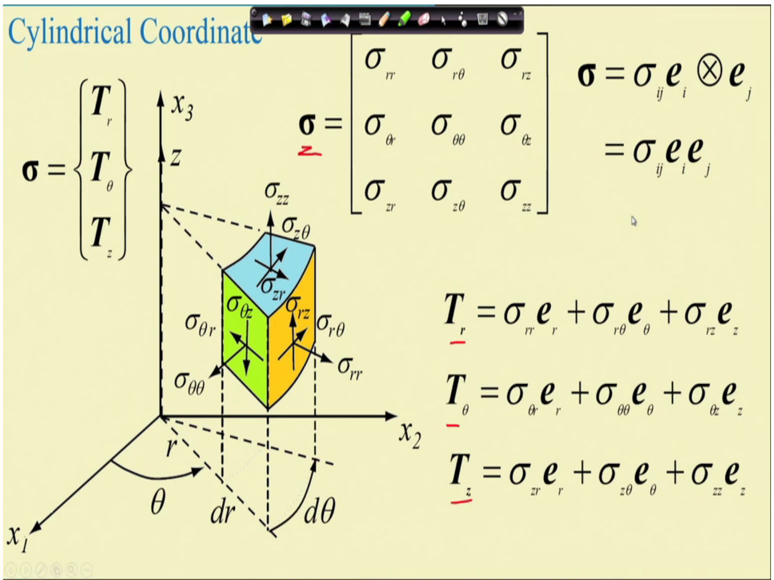
Now we have 3 stresses on this plane 1 is normal stress and 2 are the shear stresses, now normal stress is represented as sigma r r because it is acting on a plane normal to the r and the stress direction of this stress is also in r direction. This is sigma r theta and which is acting on a plane which is normal to r that is why the first r and the second is it is acting in the direction of theta that is why second theta and then this is sigma r Z it is acting on a plane normal to r and in the direction of Z that is why it is r Z.

Similarly if you look at the other plane also we have 3 stresses and then 3 stresses are the representation of this stresses we follow the same approach ok. Now if we have this now the same sigma if you recall that the stress any second order tensor can be represented by in indicial form as this. This e i and e z are the basis vector in r theta and Z direction and sigma i j is the component of this and if it is this is a tensor product. So, tensor product between 2 first order tensor i and j are the basis vector they are first order tensor the tensor product between 2 first order tensor gives you second order tensor ok.

So, sometime you do not have to write this explicitly the tensor product if you write e i is a it automatically means it is tensor product. So, essentially it is a same matrix same second order tensor at any indicial form. Now so another thing we know that suppose this is a plane, now this is a plane on this plane we have can have a traction like this suppose that traction is T r ok. Then similarly on this plane we can have traction like this is T z and on this plane we can have traction like this and suppose this is T theta ok.

Now T r what are the stresses we have on this plane we have sigma r r sigma r theta and sigma r z and e r e r e theta e z are the basis vectors in r theta z coordinate, then T r can be represented as this ok. Similarly T theta is also can be represented as this and T r z also represented as this, just one thing please note recall when we discussed traction when you define traction we used a notation something like this T n ok.

T n means this T this is a traction on a plane normal to normal to which is defined by a normal vector n, similarly T e i is a traction on a plane which is which is defined by normal which is defined by the basis vector e i, so T r is essentially T t is essentially T e r ok. This is essentially T e theta and this is essentially T e z just to avoid this again and again it is repeat written as T r T theta T z do not get confused with the notations.



Now so these are the traction on different planes that can be represented at this and if we have this then, then sigma this sigma this sigma can be represented as this sigma can be represented as this traction the traction.

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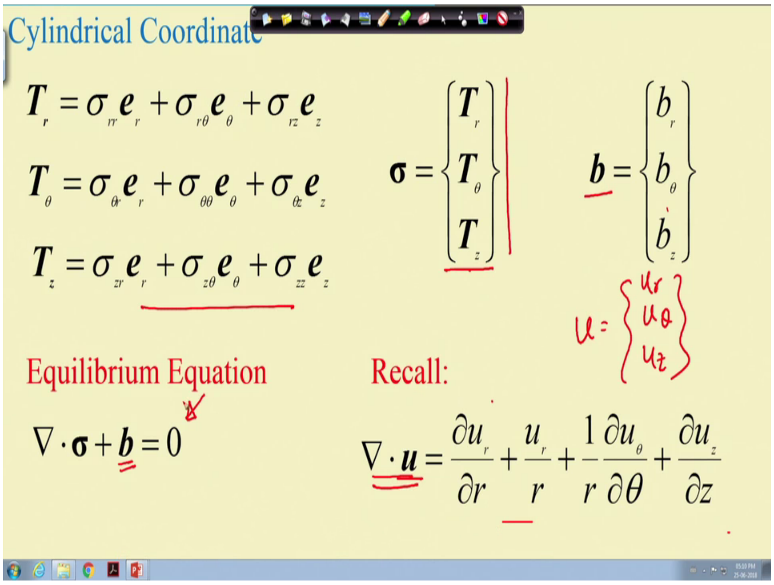

Now once we have that so we have sigma represented in terms of traction on different planes like this. Now each individual say T r T theta T z that each individual they are first order tensor and sigma is the second order tensor, so they are not scalars please remember it. Now next what next once we have that suppose b is written in terms of in r theta z direction is b r b r b theta and b z, the way we wrote b x b y b z in Cartesian coordinate system and then we have equilibrium equation is this which is a general when we derive this equilibrium equation we did not have any specific coordinate system.

This equation is whether you use Cartesian coordinate system polar coordinate systems spherical coordinate systems cylindrical coordinate system any coordinate system you write use your this equations will remain same. Now, but the representation of sigma and b and del operator the divergence operator they are different in different coordinate system and when we when we when we replace that representation in this equation we get the equation with respect to different coordinate system and that is what we are going to do right now ok.

Now so you recall in polar coordinate in cylindrical coordinate system, if you have any vector any first order tensor u which has component which has component u is equal to say u r u theta and u z. Then in cylindrical coordinate system the divergence operator takes this form in Cartesian coordinate system what you what you get it ok. Now, this is the operator, now next is next is we substitute we have in this expression what we have we have expression for b this is the expression of a b. We have expression for sigma this where T r T theta T z are expressed like this and then we also have we also know that if any first order tensor u the divergence of that in cylindrical coordinate system it can be written like this. Now, next what we do is we substitute this all this b del u and this thing this thing into this equation, if you substitute that then what you get is we get this equation.

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Cylindrical Coordinate

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + b_\theta = 0$$

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{\sigma_{zr}}{r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

So, this equation is the same equation that the equilibrium equation written in cylindrical co-ordinate system. Now if you are if you are polar coordinate system then the z axis we can z axis will vanish then we will have we will have we will have just 2 equations one is in n and one is theta direction and all the term associated with z they vanish. Similar way you can write the expression equilibrium expression with respect to in spherical coordinate system.

Only thing what we have to do is we have to start with this definition we have to start with this definition and then write del sigma and b in terms of particular coordinate system and then when we replace that coordinate system representation in these equation we get the equilibrium equation with respect to that coordinate system. Now, you do not take this the final expression for granted, you have to you must derive this equation and the derivation is very straight forward you have to just replace them and then you see what is the what is the what becomes dot product of r theta e r e theta dot product of e theta e theta and so on and you get the final expression all this.

Not only in this class in many classes we will see the (Refer Time: 24:17) many things we may not be derive explicitly, we discuss how to derive something and then we give you the final derivation. So, please do not take that final derivation for granted, you must derive all this everything that we discussed here you must derive on your own then only things will become clear ok.

Now, you see so this is equilibrium equation as I as we discussed as we started our lecture there are 3 important ingredients equilibrium equation compatibility and the stress strain relation the constitutive relation. Today we discussed equilibrium equation next class we will see what is will derive compatibility equation, but before that there is one very important very important very important observation.

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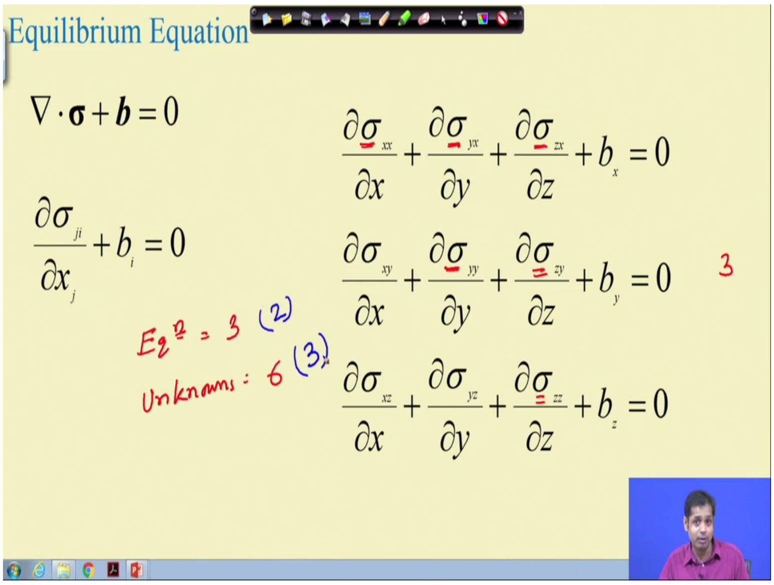

You see this is the equation that we have in the cylindrical in Cartesian coordinate system and we have only 3 equations right, we have 3 equilibrium equations. Now let us see how many unknowns we have we have sigma x is unknown sigma y x unknown and then sigma z x unknown sigma y y unknown sigma z y unknown and sigma z z unknown, these are not unknown because sigma y x and sigma x y are same.

So, essentially we have equations we have 3 and unknown we have unknown we have 6 right. If I have if we need 2 dimension then also you will see that your equation will be having you have 2 equations, you will be having 2 equations in 2 dimension 2 dimension and number of unknown will be 3 sigma x x sigma x sigma y y and sigma x y.

Now, so we have less number of unknown recall in strength of material when you analyze some structure, you applied the equilibrium condition and you found that you have the equilibrium equations equilibrium equation not enough to analyze the problem completely. Then we had another set of equation we derived another set of equation based on displacement compatibility, that since the that is the compatibility equation here the same compatibility equation, but written in terms of strain that is why it is strain compatibility relation.

Even in polar coordinate system if you check there your unknowns are you have 6 unknown and then your number of equation are 3. So, next class we will discuss what is compatibility what the compatibility physically mean and then how that compatibility between different parts of this different parts of a continuum can be ensured and when we translate them when we write that compatibility equation in terms of mathematical equation. What is that equations what form that equation takes then we will stop here today see you in the next class.

Thank you.