Theory of Elasticity
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Lecture – 22
Formulation of Boundary Value Problems

Hello everyone. This is the 5th week of this course. This week we will be discussing formulation of boundary value problems, but being the first lecture of this week, today we will try to understand what is boundary value problem. And in the subsequent weeks we will see what are the associated formulation for a boundary value problem.

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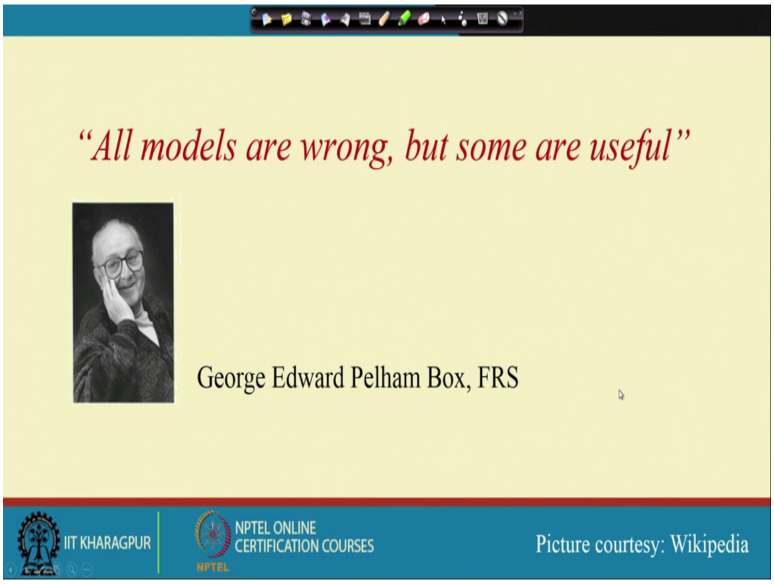

Let us start with a very famous quote, which goes like this: all methods are wrong, but some are useful. You see when we study any physical process, there are some steps we follow. And what are the steps? In general the steps we follow in any engineering analysis.

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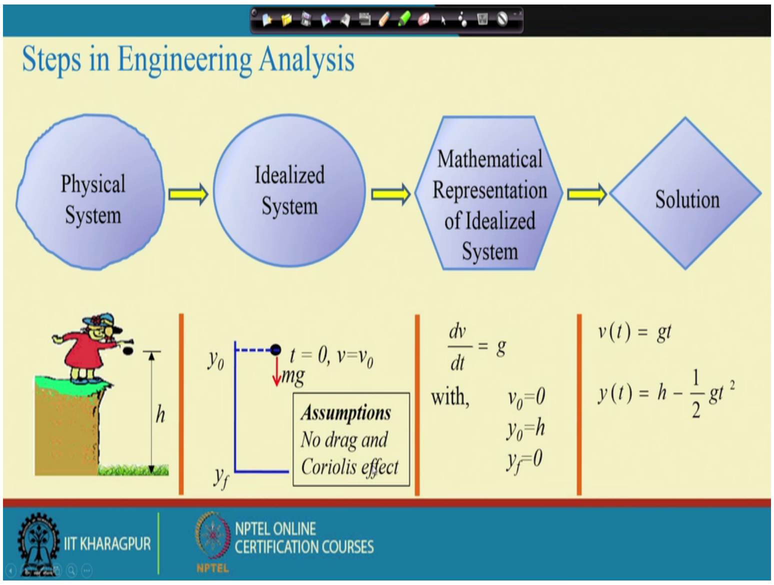

Suppose you have a physical system, that physical system could be anything, it could be it could be the deformation of a structure, it could be say vibration of an oscillator it could be say motion of a of a projectile. It could be anything any physical process we want to study.

Now, but because of the complexity associated with the physical process, and the many uncertainties we do not analyze the physical process as it is. Instead what we do? We first idealize the physical system. Now this idealization of the actual physical system is based on certain assumptions. Now once we idealize the physical system, then we represent the physical system through some mathematical model, which is the that model could be the form say differential equation, an integral equation. We represent that idealized system through some mathematical model.

Now, once we have represented the idealized system, with the mathematical model, then next we try to solve this mathematical model. Now just to give you an example, the example that you studied in your school days and you that time also when you try to solve that problem you followed the similar approach. For instance, if I want to study suppose I if I have a ball and it is dropped from a certain height and then I want to study the motion of this motion of this object.

So, this is the physical system and then that physical system is idealized as this, with a suppose the height is y 0 y f and the m g is the mass, m is the mass g is the gravity. And that idealization from this the physical system to idealize to idealized system is based on these assumption, there is no drag force and the no Coriolis effect ok. Now once we have that idealization the next thing is the mathematical representation of this physical system which is the just the equation of motion.

Now equation of motion with some initial conditions. Now, once we have the equation of motion then we have the solution of this equation motion and like this. Now and if you see it is not just in engineering, it is in many places we follow the similar approach. Actual physical problem idealize it and then write the equation associated to the to the physics of the problem and then solve this solve this mathematical representation. Now every steps, at every steps we assume something every step is based on some assumptions. For instance, in the example that is given here, the one of the very important assumption is there is no drag force and no Coriolis effect.

The assumptions are very, very important because assumptions are limitations. Whenever we study a theory, it is important to know the formulation it is important to know the know the derivations, but more than that it is important to understand, what are the assumption the assumption which constitute the basis of that formulation. Because assumptions are the limitations the places where these assumptions are not applicable you of our formulation will not be applicable. Now, with this now at every step is based on certain assumption and with every assumption we slightly deviate from the actual system.

So, the solution that we get in through this process, the solution is not the not the solution of the exact physical system. Because there is nothing like; there is nothing like right model wrong model, whenever you have idealize the system or the idealize that idealization is at every step, we are essentially dealing with the wrong system with quote under quote. But what is important is the solution at the end, of the entire process the solution we get whether the solution is useful or not whether the solution give us useful information about the system or not.

Now, when I say useful, useful is a very subjective word. Because you see it depends on what is the purpose of our analysis. If you remember one of the very important assumptions that that constitute the entire this elasticity theory linear elasticity theory is your displacement as small this, small deformation theory. Now so, the we can only apply all this theory that we have been developing all this theory we can apply only those problem where the small deformation assumption is valid. And the solution we get from all these derivation all these formulation the solution gives us useful information about the system as long as the small deformation assumption is valid.

But if I have to apply the same formulation to a problem or to a system where the small deformation principal is not you not valid assumption, then we may not get the useful information about the physical system. We get some solution get some value but that value may not represent the physical system closely; now let us try to understand the step through an example from mechanics.

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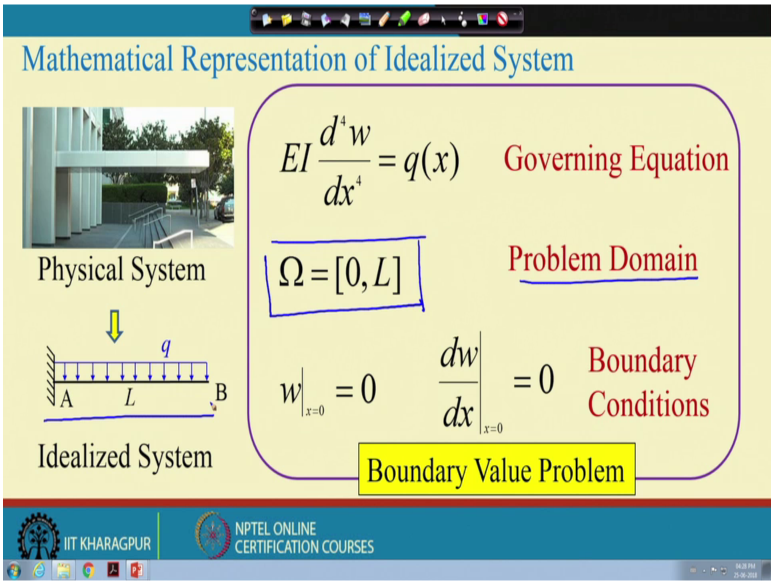

For instance, you would see take, this is a physical system we have it is a cantilever shade in front of a building. And suppose I want to analyze this cantilever shade and this cantilever shade is subjected to only self which. So, this is the physical system right.

Now, this physical system can be idealize as a cantilever beam, which is subjected to uniformly distributed load say the uniformly distributed load is q. Now you go back to your strength of material quotes and what we have studied instead the material is now, we can write the entire we can write the equation of elastic line this is also called Euler Bernoulli beam theory, which says that if w is the deformer transverse displacement, w is the transverse displacement then this is how the this is how the displacement is related to q, through this differential equation so, the ordinary differential equation.

Now, once we have this equation, then what other information we have now, but you remember we cannot solve the we cannot have an unique solution from this. Because along with this we need some information about the system what are the constraint, what are the boundary conditions given to the system, what are the support condition. Now then what we have along with this we have the domain definition of the domain. In this case the domain is a line which is 0 to l and 0 and l are the 2 boundaries of this domain. And then in addition to that still it is we are not in a position to solve this example, we need one more example, one more information and that information is the condition at the boundary.

So, one condition is displacement it is a it is a fix support. So, displacement at this point is 0 and the slope at this point is 0. So, deflection w is 0 at x is equal to 0, and in addition to that one more condition is the slope at x is equal to 0 is 0 ok. So, we have 3 important information about this idealized system. One is the governing equation which tells you how this at least in this is how the deform load deformation are related to each other. And then we have the definition of the domain, the problem domain in this case it is just line 0 to L. And then we have the boundary condition ok.

Now, together these information, we can have a solution of this equation. This is called boundary value problem. Ok now because some conditions are specified we have a governing equation and then. So, in the in boundary value problem we have very 3 important part one is the governing equation, and then is the problem domain and then the boundary condition. The condition specified at the boundary. We can also have a problem like initial value problem, where we can have governing equation and the domain and the problem the initial condition of the system is known and based on the initial condition you need to you are studying the how the system evolves over now that is called initial value problem.

Similarly, we can have initial boundary value problem, where the initial condition and the boundary conditions both are both are specified and then you need to solve based on that information, but in this is the boundary value problem. Now, you see when you say the idealization of a system, it is just not the idealization of the geometry idealization of everything; idealization of the governing equation, when you derive the governing equation that is also based on certain assumption. When you derive when you take the problem domain, that is also an assumption that is also based on certain idealization. And then the boundary condition that you provide that is also based on certain assumption some idealization.

For instance, if you remember in the Euler Bernoulli beam theory one of the very important assumption was you studied in strength of material, that section plane, a plane section normal to the neutral axis in a undeformed configuration remain plane, and normal to the neutral axis even in deformed configuration. This is one of the very, very important assumption in Euler Bernoulli beam theory. And that assumption gives us this equation. So, this is equation is also based on certain idealization. And then these equation can be applicable only those class of problem, well the plane section remain plane thus assumption is valid. There are many problem where those assumption that assumption is not valid and where we cannot apply this equation ok.

Now, once we have understood this, what is boundary value problem, and there are 3 important part in boundary value problem. Definition of the domain what is the problem domain and then the governing equation and what are the conditions specified at the boundary. Now what we do is, today we will not start deriving the governing equation. We will start deriving the governing equation next class onwards, together we will try to understand what is the idealization of problem domain and what is the boundary condition.

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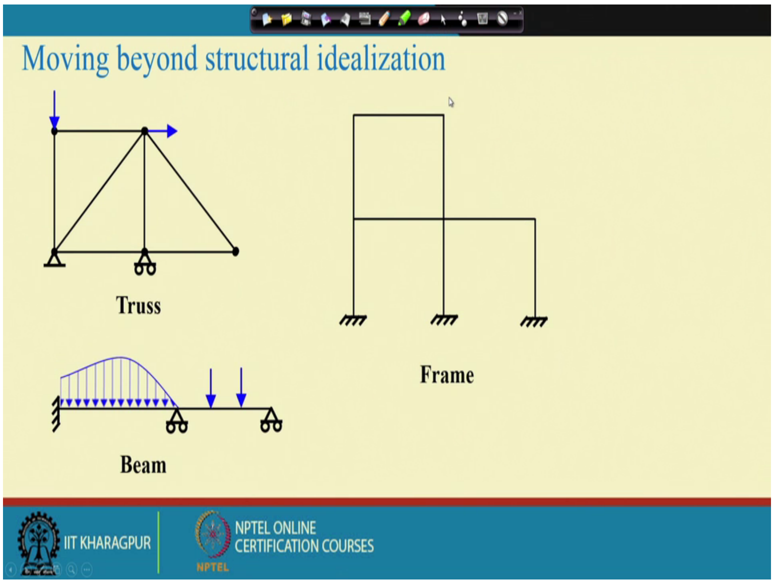

You see, for instance let me show you some structure. So, you I am I believe you all are familiar with these kind of structure. You studied in mechanics you studied in strength of material, and those who have taken structural analysis one, they have studied that in structure analysis as well.

The first one is truss, this is truss. One of the important characteristic of truss is all members are 2 force member and the members are connected to each other at the end. And load can be applied only at the joints not at the middle of the member. Then we have beams where that restriction is not there, a member can take both shear force and bending moment. And then we have frame where the actual force sometime you can consider we may not consider.

You see there is nothing like truss, beam and frame in nature. In nature anything you take any structure you take whether it is a building bridges or say any vehicles any ship or anything it is essentially a 3 dimensional object in space right. When you say something is truss something is beam or something is frame, it means that we are idealizing that particular object as truss as beam or as frame. Or if we say something as plate, something as shell this is essentially a structural assumption structural definition given to those 2 dimensional objects.

Now, so, these are some structural idealization that we have study. Now go back to the previous example, you see it was a beam, now beam is essentially what beam has a finite length it has it has cross section. So, essentially it is a 3 dimensional object isn't it. So, beam, but when we say that the cross section is very small as compared to the length of the beam. So, therefore, the entire object can be idealize as a line element, means we are giving a structural definition to this object and that definition of beam ok. Along with that how that definition how that object my behave along with that we are giving a structural definition to the beam.

The first thing, we have to do is we have to move beyond the structural definition. Now onwards there is nothing like beam truss, shell or frame in our formulation. Everything is a 3 dimensional object at the most what we can have is 3 dimensional object taking idealizing at 2 dimensional object, but still there is no structural assumption, structural definition given to this object. So, we have to move beyond this idealization. Now, once we understand then we have to move beyond this idealization, then beam cannot be represented as just a line. Beam has be represented as a 3 dimensional objects.

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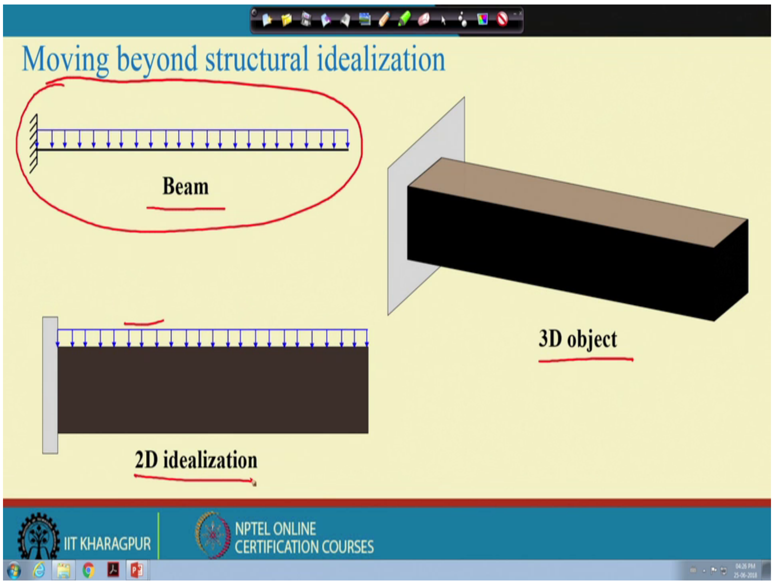

For instance, now if it is a beam same beam consider it is a beam simply supported cantilever beam, subjected to uniformly distributed load. But this is a structural definition, but we have to move beyond this definition. Now, we do not we our formulation is not based on the structural definition. Now the beam is actually a 3D object, this is a 3D object. Ok 3D object subjected to some load and it is deforming as a response to that load internal forces stresses generated in that object as a response to that that external loading.

At the most what we can do is, we can do this 3D object can be idealized sometime as plane problem 2D idealization, that towards the end of this, towards the end of this course. We will have one week on plane problem plane stress problem plane strain problem, there we will see how on under what circumstances a 3D object can be idealized as a 2D object, but still there is no specific structural definition given to this object ok.

Now, this was the first part of this definition. If you go back to the definition, one important part was this right the definition of the problem domain. In this case the problem domain is idealized as a line and therefore, the problem domain is this, but now onwards the problem domain is, if it is a 3D object it is a 3 dimensional domain. If it is a

2D object, the 2 dimensional domain. It can be idealize as a 2D object then a 2 dimensional domain ok. So, this is the problem domain. Now, let us see once we have understand that that now the domain is this, we have to give the boundary also accordingly. For instance, here when we say that it is idealize is a line then it has just a one point, but now the beam is not idealize as a line it is a 3 dimensional object then it has a several surfaces.

Then the boundary condition we have to give on the surfaces. Let us see what are the boundary conditions we can have on the surfaces of any 3 dimensional object. Ok, now. So, let us start with this let us take these example let us take this example first.

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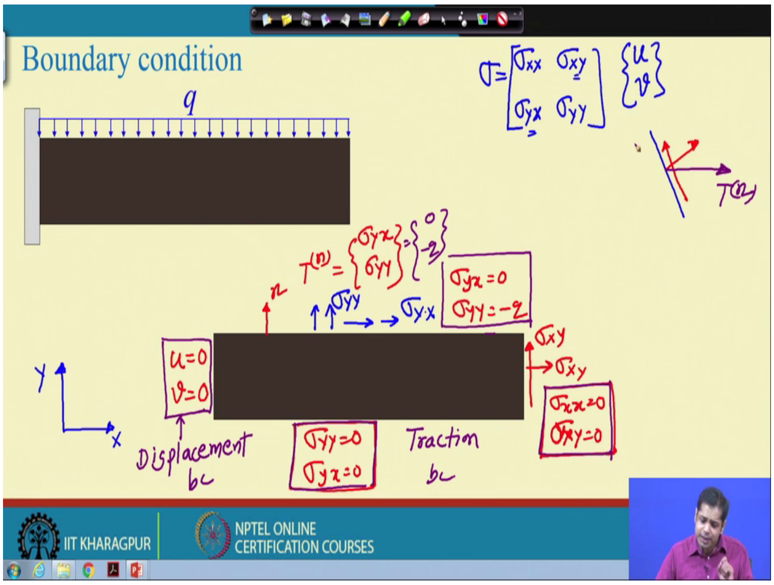

So, suppose this is the example. Now you recall we have 2 things. We have on the surfaces; we can have 2 kinds of boundary condition. One boundary condition is the displacement boundary condition, and another boundary condition is called traction boundary condition. Traction we already had, we already have idea about what is traction. It means that on the boundary for instance this is the beam and then this is the this the beam entire beam without any forces in the boundary.

Now, it has 4 sides right. Now you see what are the information we have at the 4 sides. First information we have at this side is, first information is this is a 2D problem then what is the what is the displacement we have we have deformation. This is if we take this as the this as the coordinate axis, this is x and this is y ok. Then we have say u x u and v ok. So, u is the displacement in x direction or y is the displacement in y y direction deformation in y direction.

And then what we have is we have stresses. Then stresses we can have if you recall it is a 2 dimension. So, we have sigma x x and then sigma y sigma x y, and then sigma y x, and then sigma y y. Sigma x and sigma y x there are same because this is symmetric ok. Now this is sigma. Now, let us see what are the information we have at 4 sides. This side what information we have, this side is a fixed side. So, we have u is equal to 0 and v is equal to 0. This is the information that we have at this boundary whatever solution we can get that solution has to satisfy this condition. The condition specified at this boundary ok.

Now, what other boundary conditions we have? You see you recall that if we have a plane, then on the plane we can have a normal stress, and then we can have shear stress right. And on this plane the stress can be represented by traction vector, which is T n where n is the normal to this normal to this plane ok. Now, on this surface also we can have traction vector. There traction vector is essentially here if you take then, this is your this is n, then in this case traction vector will be what? What are the forces we have in this we have we can have normal force, we can have say normal stress sigma y y, and then we can have shear stress. We can have shear stress which is sigma y x y x, and what is this traction this traction will be sigma y x and sigma y y ok.

Now, on the what are the conditions we have at this boundary. At this boundary the condition we have is sigma y x, y x that is equal to 0 and sigma y y that is equal to minus q. That is the information we have at this boundary right. And that is the information we have at this boundary ok. Either we can write sigma y x is equal to 0 or sigma y is equal to minus q, or we can write the traction vector, traction is equal to this 0 minus q. Both are fine in terms of traction vector or in separately you can write. So, this is information we have at this boundary.

What information we have at this boundary at this boundary? What are the forces possible we can have sigma x sigma x x, and then we can have sigma x y. Now in this case sigma x x is equal to 0 and sigma x sigma y, x y is equal to 0. That is the information we have at this boundary. And similarly what information we have at this boundaries? Sigma y y is equal to 0 and sigma y x is equal to 0 y x is equal to 0. Ok now you see there are 4 sides and all 4 sides some information we have. At this side we have displacement specified. So, this is displacement boundary condition and at this. So, this is displacement boundary condition and at this surface, this surface and this surface tractions, has specified. By tractions has specified means this is outer surface there is no traction on this the that whatever solution you get that solution must satisfy this condition.

Whatever solution we get that solution must satisfy that on this surface, on the top surface the sigma y has to be the minus has to be minus q, because it is subject to uniformly distributed load. So, these are called traction boundary. Traction boundary condition, all these are called traction boundary condition. So, what information specified at the given boundary based on that we can have a displacement boundary condition, or we can have traction boundary condition. Now, you see boundary condition does not ok.

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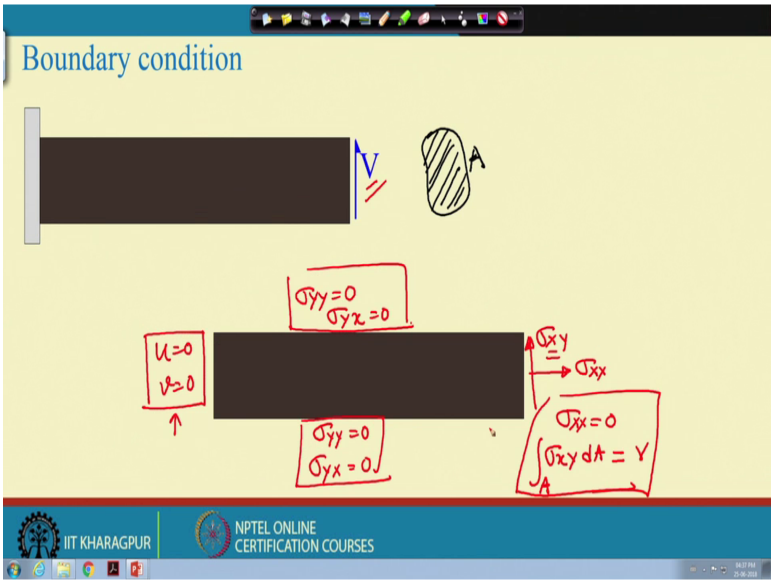

Let us move to the next example then probably take the same example, but now instead of a uniformly distributed load, we have a vertical, we have a shear force at this ok.

Now, take once again. So, this is again same at the this boundary, this is again same u is equal to 0, and v is equal to 0 this is the condition specified on this boundary, it is a stress free boundary. So, either you can write it an a traction form or you can write sigma y y is equal to 0, or an sigma y x is equal to 0. So, this is the condition specified at this boundary. At this boundary also sigma y y is equal to 0 and sigma y x is equal to 0. This is condition specified at this boundary. What happened to this boundary see there is on this boundary what are the forces we can have? We can have sigma x x and we can have sigma x y right. Now there is no what we know normal stress on this there is no stress in the in x direction. So, naturally sigma x x is equal to 0. So, that is one condition specified at the boundary

Now, what about sigma x y is it 0 or it is something else? What is information given here is that total shear force on this, on the surface total shear is v. If the cross section is if the cross section of this object is this, then this is say A cross section is a then what it gives you, this tell I that integration of sigma x integration of sigma x y dA over entire area that is equal to v ok. So, total shear stress integrated over the entire cross section, that gives me total shear force and that shear force is v. So, this becomes the another boundary condition ok.

So, all 4 side these are the different boundary conditions, that displacement boundary condition here and all other 3 sides there traction boundary condition. In this case traction boundary condition this is also sometime called as stress free boundary condition. Because this surfaces stress free these surface is also stress free. Let us see one more example.

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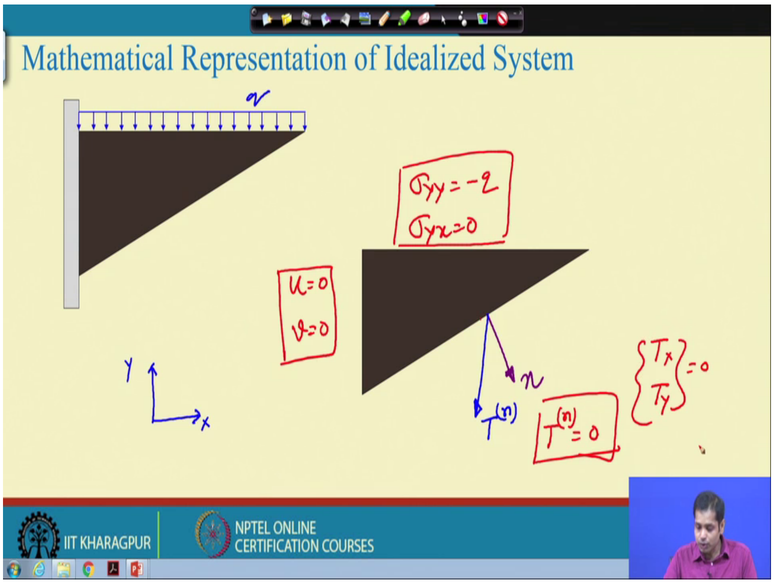

Suppose in this case what are the information we have in this case. Here also we have same thing u, is equal to 0 v is equal to 0 this is the displacement boundary condition. And then on this surface what we have if this is q, then we have here sigma x sigma y y, sigma y y is equal to minus q and sigma sigma y x is equal to 0. This is another boundary condition.

Now, what is boundary condition here is you see the this is also a stress free boundary. And these boundary can be represented by a normal vector n, and a traction say a traction on this which is T n ok. Now this traction since it is a stress free boundary. So, here boundary condition is the traction on this boundary is 0. Now this traction you can represent in a if this is an x y coordinate system, what coordinate system you take this is x and this is y coordinate system. Now, if the traction the projection of T n is T x in x direction and projection of traction is T y in y direction. So, these essentially tells you that this is equal to 0 ok. So, this is another boundary condition.

So, as we come across different problem, as we solve different problem, we will try to see different kinds of boundary condition not only in Cartesian coordinate system, we will see polar coordinates system cylindrical coordinate system we will be doing it as we come across the different problems ok. Now so, this is the boundary condition.

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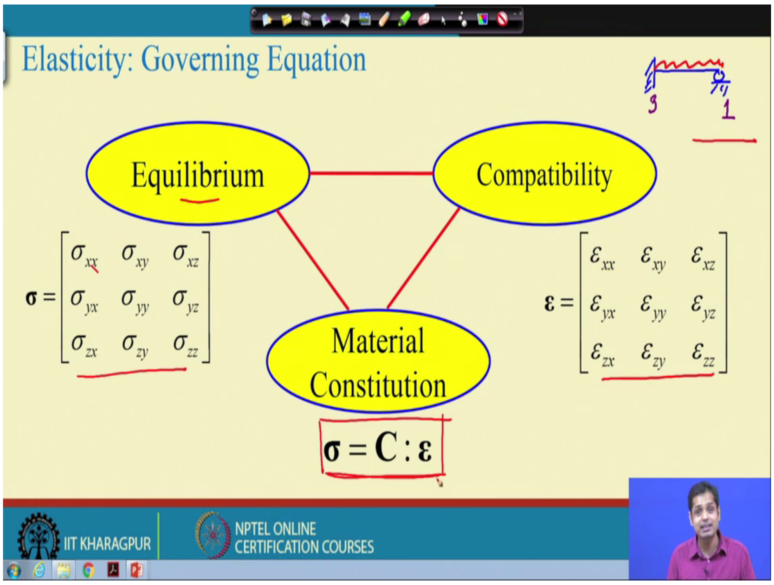

Now, before you close, you see when we derive governing equation for elasticity, from the next class onwards, there are 3 important pillars. There are 3 important ingredients of that that formulation. One is the equilibrium equation. What is equilibrium? If you recall your strength of material that if you have a beam, suppose like this take a propped cantilever beam which is subjected to uniformly distributed load and then it is a propped cantilever beam.

Then you try to solve these using static equilibrium equation and there you define 2 kinds of problem or other 3 kinds of problem depending on how many equations you have, and the how many unknown you have. For instance the equilibrium equations tells you the summation of force is equal to 0 summation of moments is equal to 0. In 2 dimension, it is essentially it gives you 3 equation summation of 2 forces and summation of moment. And then if your number of unknown the number of say in these case the number of unknown are number of unknown are 4, because here we have 3 support reaction here we have one support reaction total number of unknown 4.

So, if the number of unknown are more than the number of equation then we call the structure is statically indeterminate structure. And then we have if it is same, the number of unknowns and number of equations are same then we call it statically determinate structure, but if it is number of unknowns are less than the number of equation then essentially this structures become we cannot have an unique solution of the of this solution of the problem and physical manifestation of that is the structure is unstable.

Now, you recall. So, equilibrium equation essentially that, summation of forces is equal to 0 summation of moment is equal to 0, but here we write the equilibrium, there is now we are we are not dealing with forces and moments we are dealing with stresses. So, equilibrium equation, the same equilibrium equation summation of forces is equal to 0 that equilibrium equation, but now it will be written in terms of stresses ok. Moment equilibrium not explicitly required, because if recall in one of the lectures we discussed the symmetry of Cauchy's strain tensor gives you that equilibrium automatically ok.

So, this is the equilibrium equation. Again recall your strength of material when you studied in determinate structure, the equilibrium equation alone when not satisfied, not sufficient to find out the solution of the entire problems. So, in additional equations were required. And, how you constituted that additional equation using the compatibility; displacement compatibility. So now, we are not dealing with what directly displacement, but now instead of that what we do is the same compatibility relation, but will be written in terms of strain ok.

So, it has 2 important part, one is the equilibrium which tells you the how these stresses are related to each other. How the different components of stresses are related to each other? Then we have compatibility which tells us how the different components of strains are related to each other. And then we have the material constitution which relates the stress and the strain, which you already studied in the 3rd and 4th week, for different kinds of material, what is the material constitution and for linear elastic material what this constitution essentially can be reduced to. So, these are the 3 ingredients of a equation for elasticity. Next class we will start with equilibrium ok. So, we stop here today. So, next class will be next class we will discuss the equilibrium equation. See you in the next class.

Thank you.