Theory of Elasticity
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Lecture -21
Constitute Relation – II (Contd.)

Welcome this is last lecture for Constitutive Relation, actually in this lecture before doing the summary I thought will discuss about the lamina constitutive relation.

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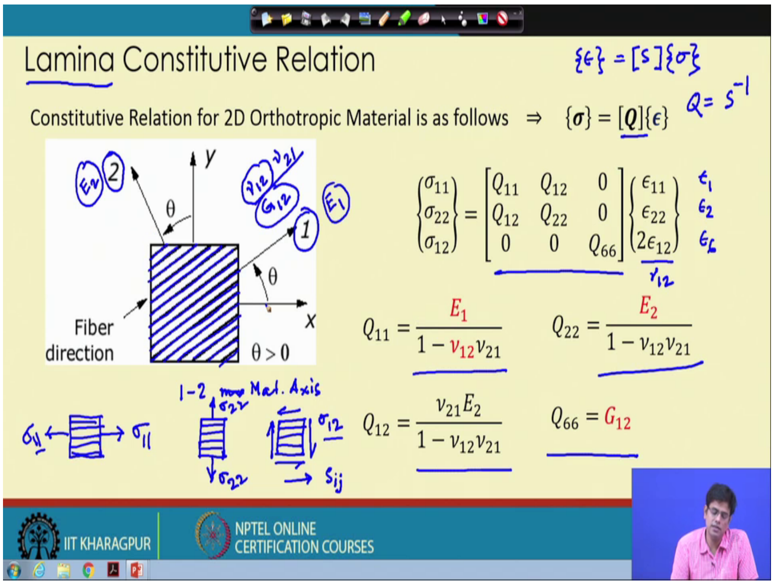

So, basically this word the lamina what does this means let us first give a definition what is essentially a lamina is. For instance we are in today's age is age of composite materials; so, in a composite material what you see is essentially a series of layers of material for instance it is called ply or oven fabric or essentially a plane. So, those planes I have a negligible thickness or very small thickness and compared to other 2-dimension and those planes are actually known as the lamina.

A lamina is a essentially a part of laminate, where a series a laminate is essentially a series of lamina is arranged in a suitable order as per the material behavior desired. So, the essentially the lamina is a part of a or a is a plane which is a of very small thickness such which constitutes the laminate basically.

So, the more detail of this lamination laminate theory or how the constitutive equation for laminate will be derived, it will be discussed in composite technology course or any other course possibly. Here what I thought I will describe that if there is a fiber orientation in this direction of lamina is, essentially a lamina is a 2D body of very in small thickness.

So, this laminate lamina is essentially a 2D body and it is made of suppose oven fabric or any made of unidirectional fiber. So, this fiber directions are along this now these direction what I say is the 1 2 direction and it is the material axis, so 1 2 direction is essentially the material axis. So, we write material axis, so now our loading axis or the structural axis is essentially x y direction. So, when we solve this problem with a given load or stress, then we need to use the constitutive equation in terms of structural axis or the x y system.

Now, in this x y system we already know that how to transfer this material properties from one axis to another axis. So, here I thought I will give a special case of a 2D orthotropic material or 2 dimensional orthotropic lamina where the fiber axis and the structural axis is essentially different. So, now in that case in 2D orthotropic material we know: what is the constitutive equation and the constitutive equation looks like this. So, here you see that the there are 4 independent constants, I have already discuss this things in a previous classes.

So, these are the stress components and strain components it is, it can be also written is in this form that epsilon 1 in voigt notation epsilon 2 and epsilon 6 this format. So, remember this is essentially the sheared strain gamma 1 2 and the similarly stresses can be also be written as sigma 1 sigma 2 and sigma 6. Now, the relation between from the previous to last to last class, we know how these engineering constant and these mathematical constants Q is here. I have denoted in terms of Q because this Q is coming from the essentially the plane stress constitutive equation, this is body under plane stress which we will learn in the next classes what is plane stress and plane strain condition.

So, that is not required right now, but here I have depicted the constitutive matrix in terms of Q. So, instead of C essentially C we derived for the 3D constitutive matrix and 3D from there, if we impose the plane strain plane stress condition when you get the Q and Qij x so that we will learn later. Now, this Q matrix if I write it in terms of engineering constant. So, there are in this fiber direction there are Young's modulus in E 1 will be Young's modulus and E 2 will be Young's modulus in these direction and then the Poisson's ratios will be nu 1 2 and nu 2 1 and then the Shear modulus between the in plane Shear modulus which is G 1 2. So, now among them this nu 1 2 and nu 2 1 both are both of them are not independent as we know, because we know the reciprocal relations which can also be proved from the bect is reciprocal theorem.

So, this nu 1 2 and nu 2 1 how it is related we know that. So, now finally the number of independent constants will be E 1 E 2 and 1 of them say nu nu 1 2 and G 1 2. So, these fours are my independent constant which is consistent with the 2D orthotropic material. Now with this we can write the Q i j S how it will. So, these are the forms and these forms also we learned it in our previous classes how to obtain this forms because, we essentially do 3 test here 1 is for this lamina what we do is essentially we do a axial test. Where, I put only sigma 1 1 in this direction and then use the compliance relation which will give me the Q S 1 1 the compliance relation. Also we know that is epsilon is essentially related to s and strain sigma.

So, these for each sigma how each epsilon will come out we know we have seen it for the 3D case. Similarly, another relation another loading case will consider, where will give only in this direction which is sigma to here and by the way this is the fiber axis here the fiber axis is here also, will give the sigma 2 and 2 2 in this direction and another is essentially the sheared pure shear stress where will give this. So, this gives us which is essentially sigma 1 2.

So, this gives us the 4 independent constants which is E 1 E 2 nu 1 2 and G 1 2. So, essentially we find out Sij from that Sij from that and once I know the Sij if I invert it I get essentially the Q matrix which is S inverse. So, from there we get the lamina constitutive equation and how it we; how the engineering constant is related with the lamina constitutive equation. So, now it is important thing is here that whatever we are doing it is in this constitutive equation all these 1 1 2 2 and 1 2 these are the fiber direction and loading direction or the structural axis remain same.

Now, when this structural axis is essentially rotated; that means, the angle between the material axis and structural axis is in a theta, here for instance here then how we can transferred this that we will see.

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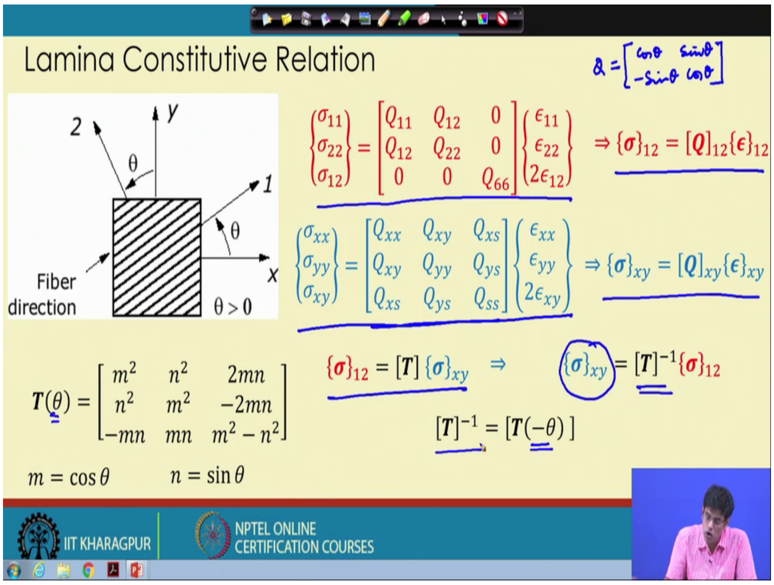

So, essentially this is a lamina constitutive equation which is this, so this is the my short form where in 1 2 axis I am writing the lamina constitutive equation. Now, similarly in the x y our objective is find out these constitutive matrix how it will look. So, we know if we from the strain transformation relation which we have learned also in this course in the previous class how to transform strains. So, in the strain transformation case or stress transformation case, we know what is a rotation matrix. So, essentially the vector we know if you write it the vector or displacement we know which is cos theta sin theta and minus sin theta cos theta whatever is the theta. So, this is a 2D case, so this is essentially rotation matrix.

So, these rotation matrix you can transform vectors and from that vectors we compute the strain and stresses similarly then we get the transformation matrix. So, this is a transformation matrix form x y to 1 2, so which is T and T looks like this. So, it is in terms of m n where m and n are cos theta sin theta. So, now here if you want to we know the sigma 1 2 essentially the constitutive relation in 1 2 direction we know and in the most case when a material will be tested essentially along the fiber direction the properties will be given. That means, E 1 E 2 will be along 1 2 direction nu 1 2 G 1 2 will be given.

So, when a however in whatever way you want to align it or the loading axis maybe different with the material axis; so, in that case actually you have to find out the constitutive relation or the constitutive matrix along the structural axis. So, that is the objective here which we are deducing it here. So, now this Q x x Q x y Q x s Q x y and Q y y and so on, here you see that material is not essentially purely orthotropic material and which is natural because, the strength along these lines along these fibers and this direction will be different because it will be rotated.

So, essentially this is the x y system where this is the constitutive axis and the constitutive relation and this is the short form of that. Now here essentially if you find out if you want to find out stress attic x y component you have to invert it. Now this inverse is very simple if you substitute minus theta it will be T inverse is T of minus theta, essentially theta is a positive here theta is greater than 0. So, minus will be minus sin theta cos theta cos of minus theta will be cos theta, so this will be inverse so essentially T inverse we know. Now once I know this our problem is to find out the next this in the x y system.

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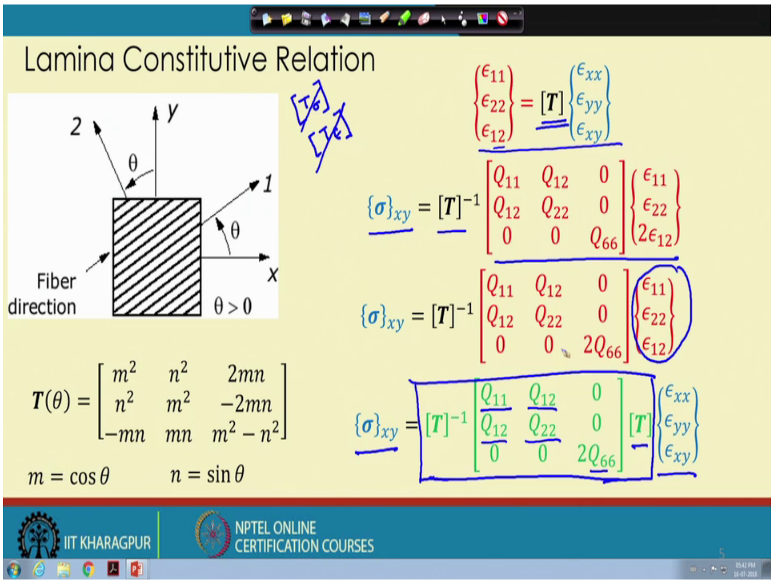

Now, it can be done in a little simple with a simple trick if you now transform the strain. So, if you remember that in the previous lecture I have used two different transformation matrix for T sigma and T epsilon. So, T sigma is for stress transformation matrix and T epsilon is the strain transformation matrix, but and this T sigma and T epsilon are related. So, we know that relation, but in spite of using those relations will not use this relations here, instead of that will just check whether if I write the this quantity this quantity instead of 2 epsilon 1 2. If I write epsilon 1 2 I can represent it in terms of T. So, similarly it will be instead of shear strain it will be engineering strain which is epsilon or tensorial strain whatever epsilon x x y. Now so if you do this now this will be so sigma x y sigma x y we know which is C inverse of T inverse of this quantity.

So, I now substitute the constitutive matrix for the 1 to coordinate system which is essentially this. Now to remove T this 2 I just multiplied with this 2, so this 2 can be removed. So, here now this quantity again I substitute with this relation which is essentially T epsilon x x x epsilon y y and epsilon x y. So, these quantity becomes now my the transformed constitutive matrix in the direction x y. So, this is the x y constitutive relation for the material in the, this fiber this unidirectional lambda. Now this if we multiply so we know: what is that component of T inverse, so which is essentially T of minus theta.

So, this will be minus these are the sin will be flipped essentially and then this we can evaluate T inverse and then T also we knows and Q 1 1 Q 1 2 Q 1 2 and Q 6 6 Q 2 2 we know. So, if we write it this constitutive matrix in a proper manner which is essentially this.

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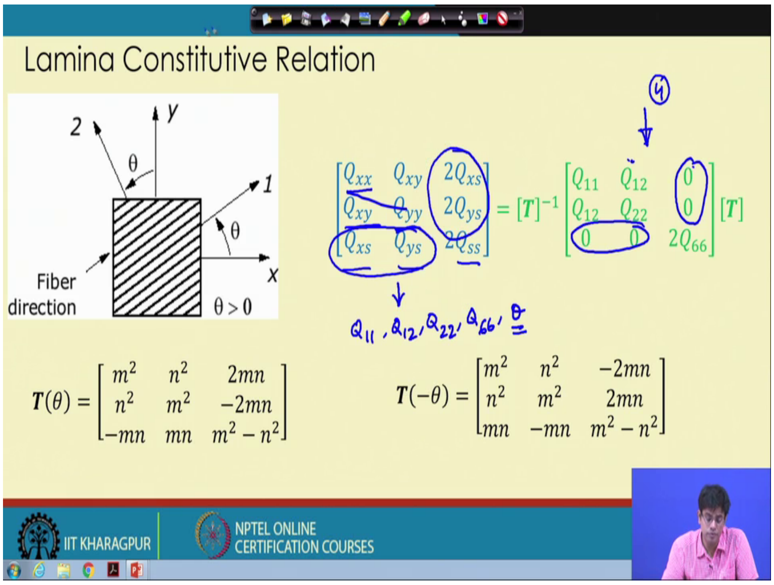

So, we have which is essentially this also we can compute the each component of this constitutive matrix, which is sigma Q x x Q x x Q x y Q x x and then Q s s Q y s Q y. So, it is it is important to know here that this is again 4 independent constant here and this matrix looks that these terms are the 0 terms. So, these are due to the Poisson's effect, but these are the coupling coefficient where shear strain and normal strains are shear stress and normal stress are coupled here these are the coupling coefficients. So, these coupling coefficients in 1 2 direction it is 0, but in case of a rotated direction this coupling coefficients are not 0.

So, this is important here to know so essentially even though these material is essentially originally the in 1 2 axis it is orthotropic. But in x y axis this coupling coefficient arise even though this coupling coefficients are not independent because, this coupling coefficients are functions of all Q 1 Q 1 2 Q 1 Q 2 2 and Q 6 6 and theta which is cos theta which is through m and n cos theta and sin theta. So, essentially the fiber parameter Q these are the number of independent components here essentially are Q 1 1 Q 1 2 Q 2 2 Q 6 6 and theta.

So, depending upon the theta this coupling coefficient and actually all Q x x components will be this components will be different. So, even though material so once again I am repeating that even though material is orthotropic originally in 1 2 axis, it is not necessary that x y axis it will be again the same structure of the constitutive matrix will be preserved because it is rotated and this rotation will also depend on how the loading axis of the structure is aligned.

So, this is I wanted to share with you for the lamina constitutive relation, which probably you will learn it in more detail when will learn composite technology or composite mechanics. Where how to find out a constitutive equation for a laminate will be also discussed is, a laminate is essentially symmetrically or anti symmetrically placed several lamina and through the thickness integration we can object the constitutive behavior of the laminate.

So, this actually completes our constitutive equation part. So, to summarize actually I summarize finally what we have learned here, what was objective and what essentially we have learned. So, first of all we started with very general elastic material where the with the stress strain curve and we decided to confined ourselves to a linear part of that elastic curve which is essentially the constitutive properties, essentially proportional with the means it is a constant essentially stress is proportional to strain. So, linear constitutive equation which is essentially comes from the which is known as the hooks law. So, we have derived the Hooke's law.

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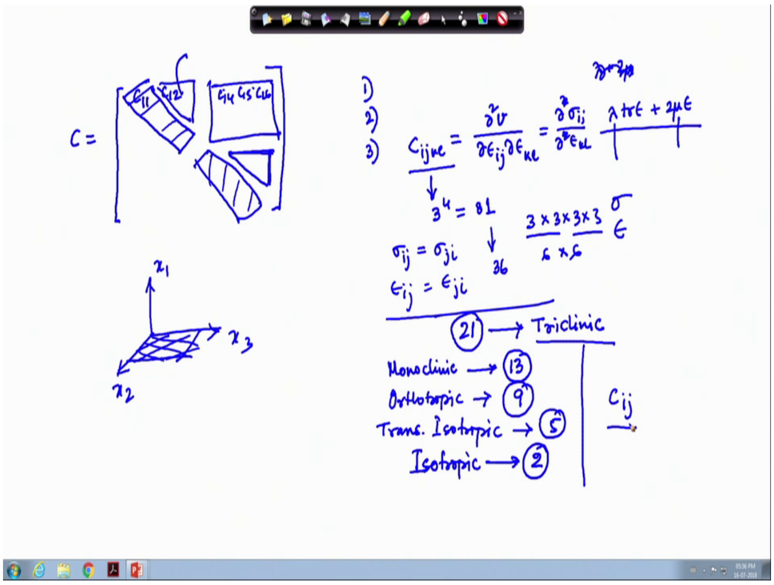

So, first part we have derive the hooks law and essentially we have derived it for the isotropic material, for which our basic knowledge from the strength of material or solid mechanics from which we know this. So, we have derived the hooks law for an isotropic material which is essentially um, if you remember which is lambda plus 2 mu in this form and if I write it in a tensorial form.

So, lambda trace of E plus 2 mu epsilon so this is the form. So, this is trace, so trace is epsilon k k. So, and then in a matrix form which is lambda plus 2 mu mu 0 and so on. So, this we know we have derived it for the isotropic material, we have also derived bulk modulus and first and second lame constant which is essentially a lambda and mu and how it is related with the Poisson’s ratios.

Then we found out the relation between actually what is the physical meaning of this constant for instance, if you if somebody wants to find out the material constant for a material how do you find out. Essentially material constants are not measurable it is a derived quantity, essentially if you measure stress you measure stress you measure strain and then from that you basically stress by strain you calculate young’s modulus.

So, for a torsion test for tension test for a sheared test do all those test and then found out different constants material constants for the isotropic material. Now once we have sufficient knowledge with the isotropic material then we went to the anisotropic material. So, to start with an anisotropic material we first derived the strain energy form and then finally with that strain energy density we are introduced a fourth order tensor which is Cijkl for a general anisotropic material, where there is no symmetry is assume which is essentially partial derivative of the strain energy function with respect to strain.

So, we have derived this thing and with the hooks law in implementing the hooks law. So, which is a again we can write this is del square sigma ij del square epsilon sorry del sigma ij del epsilon kl. So, in this form also we can write it. So, now here with this we have introduced this fourth order tensor and this fourth order tensor in general will have 3 to the power 4 in component which is 81 independent material constants.

So, these 81independent material constant is essentially not all are independent, if we use that stress and strain symmetry sigma ij equals to sigma ji and epsilon ij is epsilon ji that is the introducing the minor symmetry. So, then we can reduce it to from 81 do we can reduced it to 36 constant. So, this is very simple essentially if you write it 3 cross 3 cross 3 cross 3 and then this is essentially 6 this is essentially 6, so it is 36 component and then if we use major symmetry that is.

So, partial derivative which is Cijkl I can write it equals to Cklij, so which is just shifting the partial derivative. So, del square are by del x del y can be written as del square del y del x, that same logic is applied and so with that we can just arrive finally the 21 independent constant. So, with this 21 independent constant we say this is a triclinic material.

So, this is a triclinic material where there is no symmetry or reflection or even rotation symmetry is assumed. So, it is a essentially a triclinic material, now from there actually first we have derived the once we know the 21 independent constants, we know which coefficients are essentially responsible for it. So, we are derived the hooks the stiffness matrix or the material constitutive matrix which is C 1 1 C 1 2 and so on.

So, we know this first 3 components are responsible for the normal stress, then these are the coming from the sheared stress component and these are the Poisson's ratio components Poisson's ratio and these are the coupling coefficient, where the normal and the sheared stress are coupled. So, see from C 1 4 to C 1 6 these are the essentially the coupling coefficient, where normal strain and shear strains are or normal stress and shear stress are coupled so these are the coupling coefficients.

Now these are also the known as a chains of coefficients, so these are the coupling between two sheared strains essentially. So, these we know it from a general triclinic material. So, from that triclinic material we again try to reduce and define a new material which is first we define monoclinic material, where mono in the monoclinic material we have found out the constant is number of independent constant is 13, which is essentially a done through in variance of a strain energy function. So, we since strain energy is a scalar, so it will be scalar is a in variant.

So, we use that argument to derive this 13 independence constant and for a monoclinic material actually there is a 1 plane of reflection symmetry. So, where date of Q equals to essentially minus 1 if you write in terms of rotation. So, so date of Q is minus 1 or determinant of rotation matrix is essentially minus 1. So, 1 plane of symmetry if we have then we find out thirteen independent constants and how the constitutive tensor or this symmetric will looks like that also we have derived.

So, essentially then we defined an ortho material which is known as orthotropic orthotropic material, where instead of 13 instead of 1 plane of symmetry we assume another plane of symmetry or another symmetry plane. So, each since 2 if the 2 planes are mutually orthogonal then third plane is also essentially will have the symmetry. So, finally the orthotropic material is have 3 plane of reflection symmetry.

So, that gives us the number of material constant is further reduced which is 9 and then this orthotropic material we again consider a plane of isotropic, if you remember the figure this is how it is if it is x 1 then x 2 and x 3. So, what we assume that x 2 x 3 plane is essentially this plane is plane of isotropy and then we reduce the number of material constant is 5. So, this since this is an isotropic plane, so there will be 2 material constant because isotropic we know it for 2. Then from this thing we again if we assume all 3 planes are planes of isotropy, so then we get into this the number of constant is 2.

So, then what we did is essentially these are all in terms of C ij’s, so in a voigt notation C ij so this C ij’s are again related to the engineering constant; so, which we are more familiar with like young’s modulus Poisson’s ratios these constants. So, these constants we then again derived from a 3 different test tension and shear test for each cases for a 3D material and then for an orthotropic material how these mathematical engineering constants a mathematical constants C ij and engineering constant E 1 E 2 mu 1 2 and so on is related. Now then what we did is essentially how these engineering constants are bounded that we have also derived.

So, we found that for a general orthotropic material E 1 E 2 E 3 G 1 2 G 2 3 and G 1 3 should be greater than 0 and Poisson's ratios nu 1 2 nu 2 1 nu 1 2 nu 1 3 and nu 2 3 should be less than half. So, this actually states that the Poisson's ratio cannot be arbitrary large arbitrarily large and then if 1 of them is large, then 1 has to the other 1 cannot be arbitrary large and then some restriction with the Poisson's ratios with the values of the Poisson's ratio with the relative young's modulus.

So, which is nu ij mod of nu ij if you remember is less than equals to Ei by E j to the power half. So, square root of Ei by E j those things we have derived. So, this poses some restriction on the engineering constants which cannot be arbitrary. So, then once we done that we then derive some material derive some rotated, if the axis structural axis and the material axis are not same or material axis is not aligned with the structural axis, then how the constitutive matrix will look like that also we have derived.

So, the learning objective for this course is essentially even if a general anisotropic material is given. So, how to how it is constitutive relation or constitutive matrix looks like that is our main objective. And then, what are the physical meaning of this material parameter and how to obtain this material parameter at least for an orthotropic transverse isotropic anisotropic material, that we have learned in this module.

So, in this in the next module essentially will start with the balance law. So, till now what we have learned is the material what first we have learnt some elements of tensor algebra and tensor calculus. Then we have defined: what is stress strain, then so essentially we have defined strain displacement relation and then here module 3 and module 4 we have defined how stress and strains are related.

So, through a constitutive matrix and this constitutive relation is linear for this case and this is known as the hooks law. So, then once we once we complete third and fourth module, in the next module will learn how a physical problem has can be solved. So, basically balance laws or the conservation equations of the elastic solids. So, this completes this module.

Thank you.