Theory of Elasticity
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Lecture - 15
Constitutive Relation – I (Contd.)

Welcome so, this is the lecture number 15 of module 3, where we are discussing the anisotropic elasticity. In the last class we have discussed the strain energy function and the for a general anisotropic material what are the number of independent components. So in this lecture, we will use certain special type of anisotropic material where actually we will use the symmetric conditions, to derive certain class of materials, for instance monoclinic material, where we will observe a one plane of symmetry.

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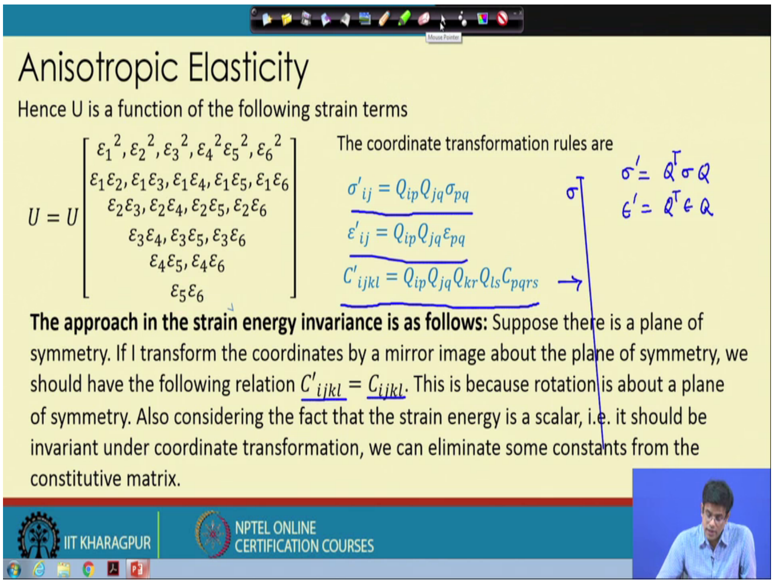

So, now we have seen in the last class that the strain energy function of the all these strains right epsilon 1 square epsilon 2 square and so, on.

So, now what this symmetry means actually will understand it first. Now, suppose we suppose there is a plane of symmetry in a material, so, if we rotate this material about that plane, then my material constant is essentially will change. So, which is C i j k l and C i j the C dash i j k l will be C i j k l. Now, but even if it changes, if I rotate this material, if I rotate this material the material constant will change and it will in a different axis the material constant will be different. But it is important to know that the internal energy of the strain energy will change right, because if I rotate simply a rotate a material its energy will not change.

So, this has a physical meaning and the mathematical meaning of this since strain energy is a scalar. So, it is invariant under any rotation or any reflection of a plane. So, that means, even though my material constants looks to be changed, my strain energy would not change.

So, this actually poses a condition that what should be the material constant for a particular symmetry of a given material. So, this actually results in a different type of different class of materials, again for instance the monoclinic material, orthotropic material, transverse isotropic material and isotropic material also.

So, essentially suppose there is a plane of symmetry, if I transform coordinate by mirror image about the plane of symmetry, we should have the relation like this. Now, this is because rotation about rotation is about to plane is a plane of symmetry. So, which is a natural suppose we rotate a vector, we get a different magnitude of different magnitude of different components of the vector. Similarly, if we rotate matrix we will get different transform matrix and similarly this is a fourth order tensor, so, if we rotate about a plane, so, it you will get a different components.

So, different material constants; so, which is represented by this, but actually this is this may not be true, because the material constant is inherent property of the body. So, if I rotate this and my strain energy function will not change and so, since my strain energy function will not change, my this constants will pose a restriction, what should be and what should not be in the constants.

So, let us see how we can impose these restrictions of elastic constants further for a different special materials. So, that is this is because also considering the fact that strain energy is a scalar that is it should be invariant under coordinate transformation. So, as I told scalar is invariant under coordinate transformation or rotation or deflation, so, we can eliminate some of the constants from the constitutive matrix and the if I can use certain symmetry is certain type of symmetries, we can actually get the different materials. Now, coordinate transformation has already been talk to you. So, this coordinate transformation, if I write it in a indicial form its looks like this for stress strain and, then if I write it in a for a fourth order tensor it looks like this.

Now, this is impossible to write it in a matrix form, but these can be written in the matrix form. So, if I write it sigma dash is essentially Q transpose sigma Q and similarly epsilon dash is Q transpose epsilon Q. So, Q is an orthogonal matrix you know and so, there that of Q is plus or minus 1, because it depends on the reflection.

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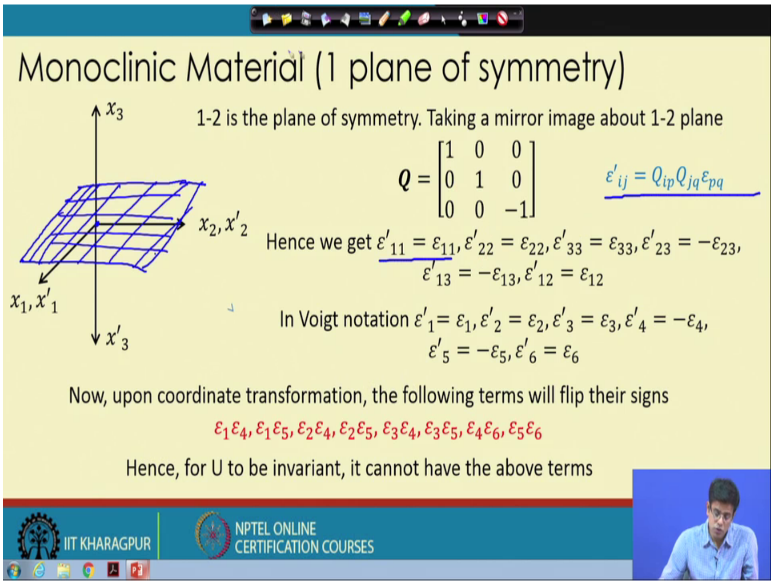

So, now let us see a how we can explode these conditions to define a different material. So, first we will start with the monoclinic material, what do you mean by monoclinic material? The monoclinic material, if we have one plane of symmetry. For instance, here if it is x 1 x 2 x 3 x 1 x 2 x 3 are the coordinate axis. And if this material is having plane of symmetry is 1 2; that means, if I can draw a plane from here to here with 1 2 system. So, if this is my the symmetry plane; so, here, now if I take the mirror image of the about this plane, then I get a x 3 will be opposite side naturally. So, this will give me a rotation matrix of this form right. So, which will be minus 1 in the third component 3D component of the rotation matrix; so, this type of material is known as the orthogonal monoclinic material.

Now, if I use my strain transformation formula, we can just simply a write it epsilon dash 1 1 equals to epsilon 1 1. So, we can do this; for instance let us see how we can do this.

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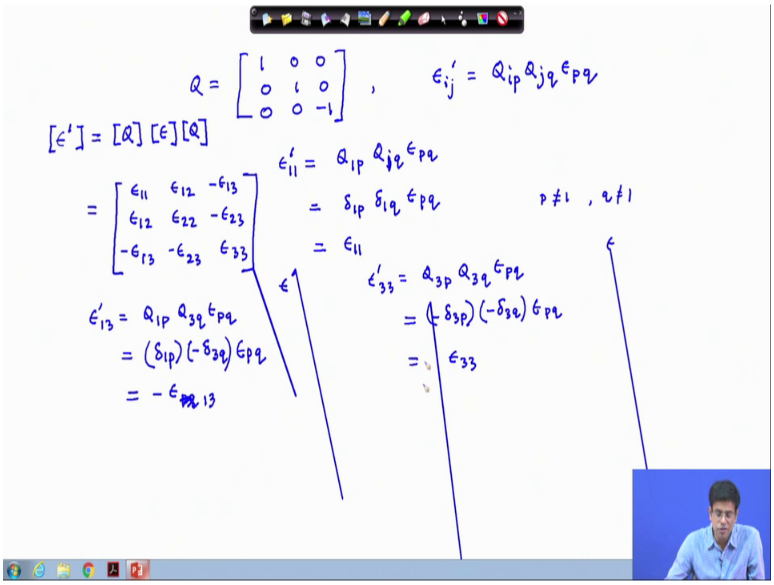

So, our rotation matrix is this so, which is 1 0 0 0 1 0 0 0 minus 1. Now, here my epsilon i j dash is Q ip Q jq epsilon pq. Now how we arrive this formula? So, essentially epsilon suppose epsilon 1 1. So, epsilon 1 1 dash will be the another rotated coordinate system is Q 1 p Q 1 p Q j q epsilon p q sorry Q 1 q.

Now, Q 1 p will be delta 1 p right, because this is one. So, now Q 1 q will be also delta 1 q so, it will be epsilon p q. Now, if you look this delta 1 p 1 p and q will be 1, then only these are the 1, otherwise all those quantity the 0. So, in these quantities if p equals to 1 and q equals to 1 then only this quantity in means if p not equals to 1 and q not equals to 1, then quantities these quantities 0.

So, finally, I can write only this is epsilon 1 1 right. Similarly epsilon 2 3 and epsilon 3 3, let us see epsilon 3 3. So, epsilon dash epsilon dash 3 3 equals to delta Q 3 p Q 3 q epsilon p q right. Now, Q 3 p is essentially minus delta 3 p right, because this is minus. So, similarly 3 q is minus delta 3 q epsilon p q right.

So, if I do this now the again with this similar condition only at q 3 3 p and q are three it is valid if it is p not equal to 3 and q not equal to 3, then this quantity becomes 0. So, this becomes 2 minus becomes epsilon 3 3. So, this can be achieved in a different way also for instance, if I use this matrix formula for instance this epsilon transpose a epsilon dash, this is my strain matrix right, or strain tensor, then I can multiply Q transpose which is Q transpose is Q essentially here so, and this is my epsilon and this is my Q.

Now, if I multiply this quantity I will get like this epsilon 1 1 epsilon 1 2 minus epsilon 1 3 epsilon 1 2 epsilon 2 2 minus epsilon 2 3 epsilon minus epsilon 1 3 epsilon minus epsilon 2 3 epsilon 3 3 so, if I multiply this. So, it is essentially epsilon dash 1 1 epsilon dash 2 2.

So, let us see epsilon 1 3 so, epsilon dash one three epsilon dash 1 3 is essentially your Q 1 p Q 3 q epsilon p q right. So, here Q 1 p is actually delta 1 p right and epsilon Q 3 q is actually minus delta 3 q and this quantity is will be valid if only p equals to 1 and q equals to 3, otherwise all components will be 0. So, this becomes a minus this minus will come out. So, this is essentially p q. So, delta is essentially the chronicle delta. So, this epsilon 3 3 becomes minus epsilon 1 3 epsilon 1 3.

Now, so if I do it for all component of strains I will get like this. So, this gives me the this formula, where this epsilon 2 3 epsilon 2 1 3 and how it will look, if I rotate this strain components and so, in Voigt notation similarly epsilon 1 3 and epsilon. So, epsilon 1 3 and epsilon dash 1 epsilon 2 dash epsilon 3 dash and so, on right.

So, now if you look carefully these systems will these systems will change only here. So, these systems upon coordinate transformation only following terms will flip. because epsilon 1 epsilon 4 epsilon 4 is actually minus right. So, epsilon 1 epsilon 4 will change epsilon 1 epsilon 5 will change, epsilon 2 epsilon 4 change and so, on.

So, these quantities will change right, but as we have discussed that in U is essentially the invariant, because U cannot change even though these signs of these quantities are changing. So, U cannot be U cannot change, so we can just simply write the strain energy function.

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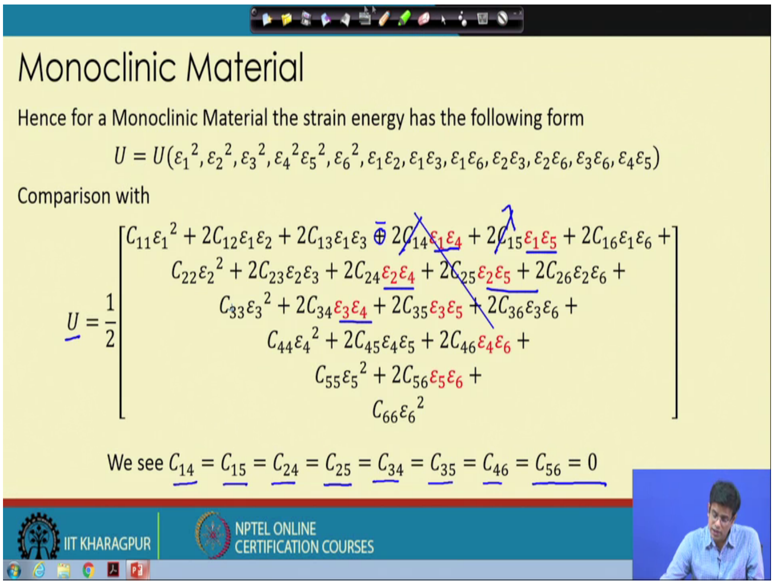

Simply this form and write this whole strain energy function for the anisotropic material and, then say these quantities will be changing the sign. So, these quantities will be changing the sign, accept these quantities other quantity design perfectly fine.

So, now to have this so, this quantity remains constant. So, you if you have to have this U to be constant and that this rotational change so, then this quantity has to be 0 right. So, this quantity has to be 0, now similarly these quantity has to go 0.

So, similarly we can see that C 1 4 C 1 5 C 2 4 C 2 5 C 3 4 C 3 5 C 4 6 and C 5 6 will be 0. So, the reason is actually that U cannot change even though strain component changes. So, it will be minus so, this case it will be minus, but original case before the rotation it was a plus, now it is minus.

So, it cannot have this quantity means it cannot; so, then the, if it is minus and the rotation. So, strain energy it changes which is not possible because strain energy is invariant under coordinate transformation. So, this to achieve this or the invariance property of the coordinate transformation of the strain energy, we have to have C 1 4 is 0 C 1 5 is 0 and the other constants will be 0. So, this is one way or one approach of finding out the material constant.

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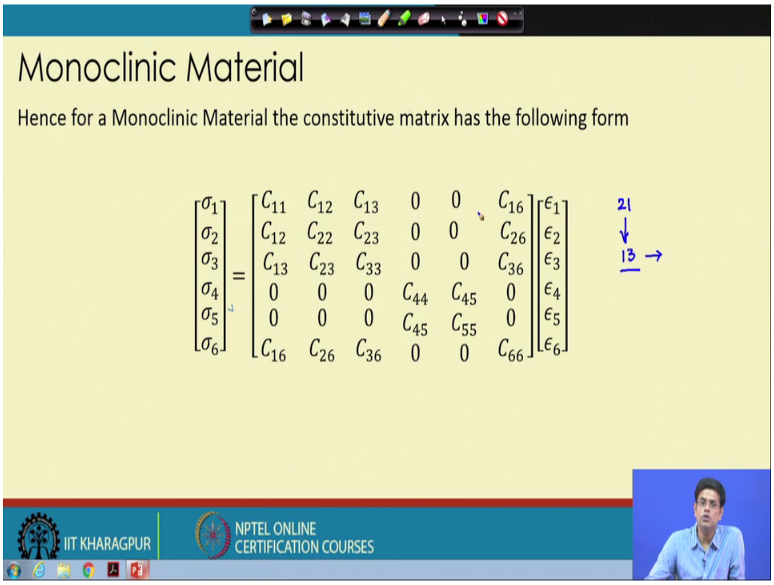

So, now, see if you look carefully so, under this coordinate transformation, or the one plane of symmetry the material matrix looks like this so, as you have seen so, these quantities these quantities goes to 0, these quantities goes to 0, but this quantities are not.

So, this is coming from the invariance of the strain energy, another or another way of doing it is to directly use this equation, for instance this equation. So, if I use this equation with the rotation matrix. So, we know the rotation matrix for the monoclinic material. So, which is suppose now we want to deriving this form.

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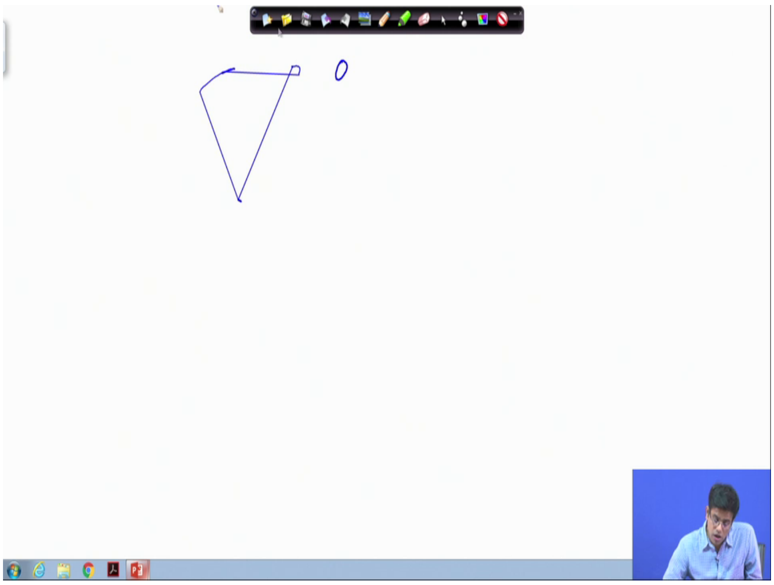

So, we know the rotation matrix of this form.

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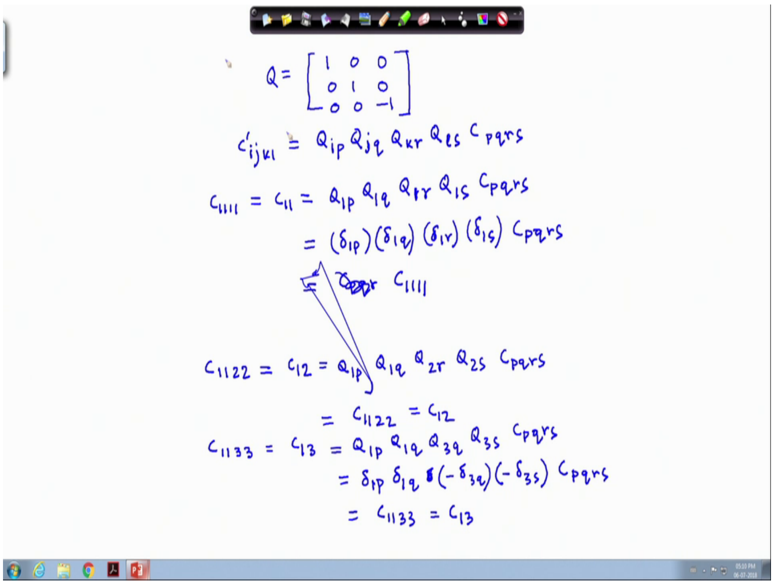

Q is essentially your 1 0 0 0 1 0 0 0 minus 1 right. So, now our C dash C dash i j k l is essentially Q i p Q i j Q q k r Q l s C p q r s. Now, similar to the strain transformation you can directly for instance C 1 1 1 1 right, which is essentially C 1 1 right for our contracted notation in the Voigt notation. So, if I write Q i p so, q i essentially it will be Q 1 p Q 1 q Q 1 r Q 1 s and C p q r s. So, if you look carefully this is again this will be delta function so, conical delta.

So, these so delta 1 q delta 1 r delta 1 s C p q p q r s. Now, if you see carefully that this is your these quantity will be non zero only when this p q r s are all 1 right.

So, this I can directly write that this is C p q r s this C 1 1 1 1, because this will be non zero only when p equals to 1 q equals to 1 r equals to 1 and s equals to 1. Similarly we can find out for instance the C 1 1 2 2 for instance say. So, C this is C 1 2 right so, which is Q 1 p Q 1 q Q 2 r Q 2 s and C p q r s. So, which will again with this logic it will come as C 1 1 2 2 which is essentially C 1 2.

Now, if I write C 1 1 3 3 how it looks again this will be C 1 3 right. So, Q 1 p Q one q Q 3 q Q 3 s C p q r s so, again with this logic these will be the delta 1 p delta, 1 q delta, this will be minus since 3 q. So, 3 third component is negative so, minus delta 3 q minus delta 3 s. So, C p q r s it will 2 minus will be there so, it will be C 1 1 3 3 which will be C 1 3 right.

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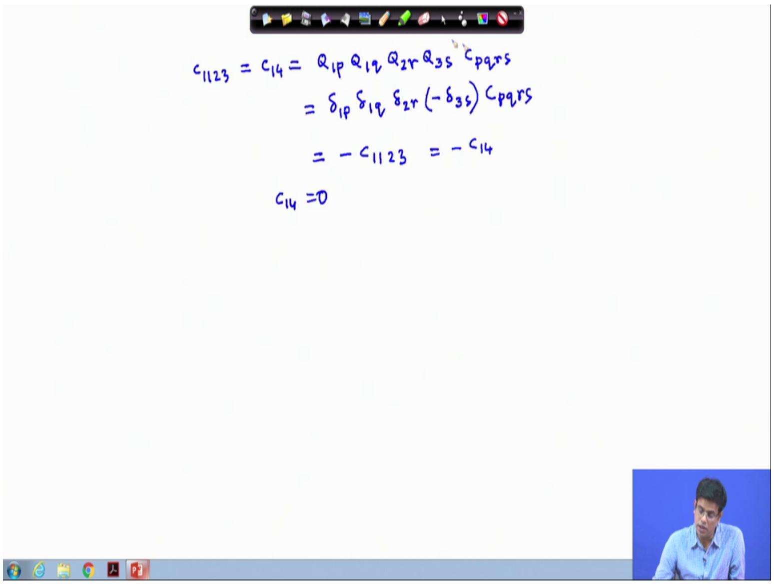

But in case of a, suppose I want to have C 1 1 2 3 right. So, which is essentially C 1 4 right.

So, if I write it so, delta Q 1 p Q 1 q Q 2 r Q 3 s C p q r s. Now, this is delta 1 p this is delta 1 q this is 2 r so, delta 2 r but this is 3 s. So, it will be negative so, delta 3 s C p q r s. Now, again with this logic when p equals to 1 q equals to 1 r equals 2 and s equals to 3, then it will be non zero otherwise it will be 0. So, if I write. So, it will be minus C 1 1 2 3, Now, which is essentially minus C 1 4.

Now, if you look if C 1 4 equals to minus C 1 4, then C 1 4 has to be 0. So, this way also we can derived for the monoclinic material. So, now, if you look carefully for the monoclinic material, this is C 1 4 is essentially 0. So, how many constant this monoclinic material is have so, 13 constants, because 1 2 3 4 5 6 7 8 9 and then this 4 constants 1 2 3 4.

So, if you look carefully that initially we have started with the 81 constant and, hence 81constant we proved, in the last class that it is essentially the 21 constant for a triclinic material. So, there is no plane of symmetry. Now, if there is a one plane of symmetry if there is a one plane of symmetry, then this triclinic material the independent number of constant will be 13. So, this is for the one plane of symmetry and this type of material is known as the monoclinic material.

So, this type of material exist a also and, if we if we directly say that material is monoclinic, we can directly write it constitutive equation that it is of this form. And it is coming from the invariance of the strain energy, and also there is another approach that by which we can directly compare the stress strain relation to find out which constant will be 0, in that case essentially stresses and strains are equated in two different coordinate system.

Now, the Hook's law we know. So, with this Hook's law essentially I briefly discuss this thing so, in the stress strain approach what essentially is done.

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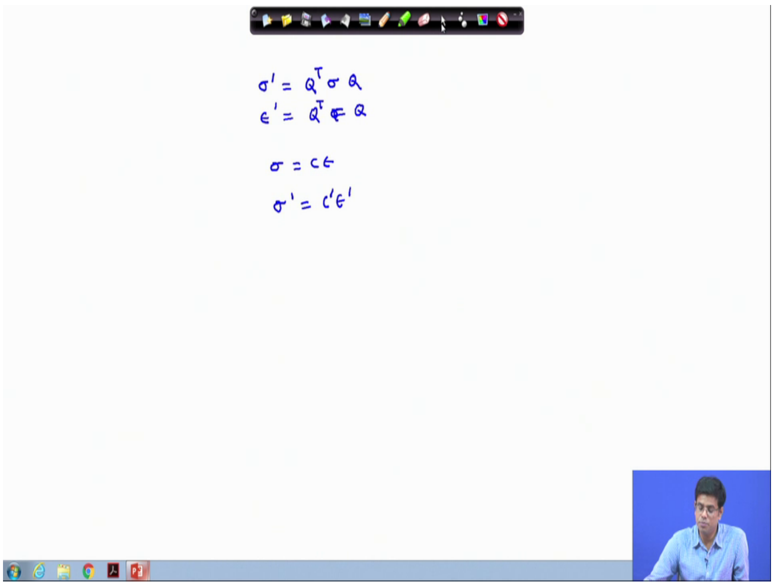

Is sigma dash is essentially Q transpose sigma Q and epsilon dash is essentially, Q transpose sigma epsilon Q. Now, Hook's law we know sigma is essentially C epsilon and, then in this format also in the transform coordinate sigma dash equals to C epsilon dash.

So, these stresses comparing the stresses comparing those stresses we can also find out which component will be 0 and which component will be non zero, in the two coordinate system. So, this is over all the process of finding out material constants, if I have a special case of symmetry and if I have a special case of symmetry, then what is the procedure for the finding of the material constant?

So, this is for the monoclinic material. Now monoclinic material is having one plane of symmetry. Another type of material where two plane of symmetry, or essentially three plane of mutually orthogonal three plane of symmetry, that may type of material is known as the orthogonal orthotropic material. So, we will discuss this in the next class.

Thank you.