Theory of Elasticity

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Lecture - 14
Constitutive Relation – I (Contd.)

Welcome this is the 14th, lecture number 14th for the Theory of Elasticity course. So, we are actually in this module we have already discussed the stress strain relation and the constitutive matrix. For instance we have also discuss the anisotropic elasticity, matrix and all those things. But and in the second lecture number previous lecture we have also discuss the physical meaning of isotropicelastic material constants. So, here in this lecture, we will study deeply the anisotropic elasticity.

Our approach would be will start with a general anisotropic system and or anisotropic body or anisotropic material, we then impose different conditions and different especially symmetric conditions to arrive the finally, isotropic material. And in between from general anisotropic or allotropic material to isotropic will also discuss several cases of anisotropic material for instance orthotropic material, transverse isotropic material, monoclinic material all those things will discuss here.

So, let us see why anisotropic is important ah. In the beginning of the course we pointed out that micro structural effect on the material property or the homogenized material property or the macroscopic material property have the micro structural effect, has a great influence.

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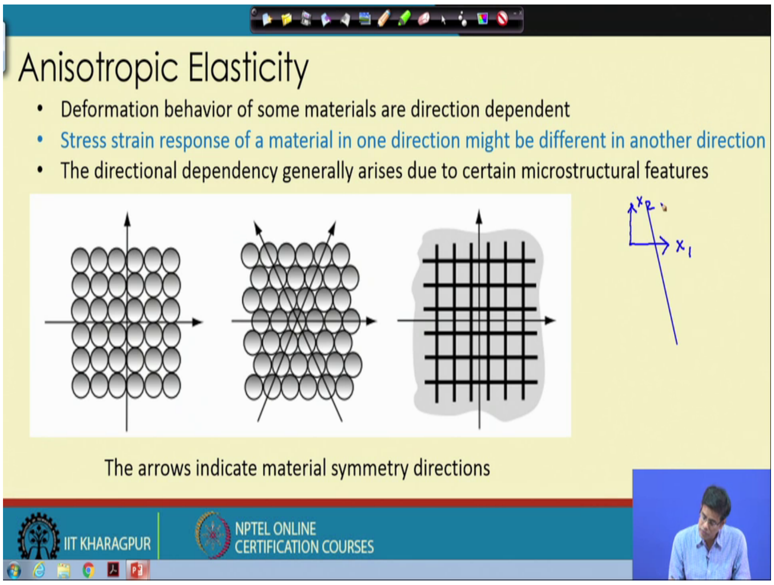

So, these essentially we discussed there that when isotropic body is when you have the material property in all direction are same. So, anisotropic means the material property at other directions may not be same.

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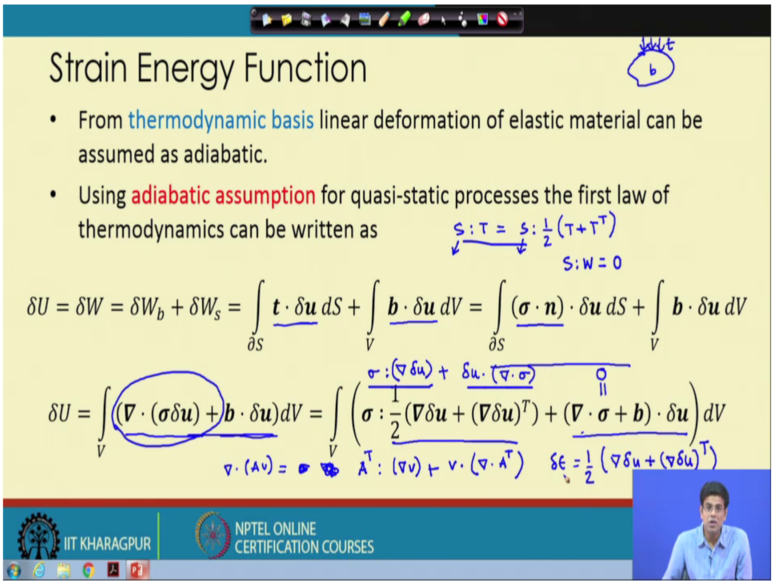

For instance, if I have this x 1 direction will have different material properties x 2 direction will have different material properties. So, deformation behavior is essentially some materials are directional depend direction dependent. Now, this directional dependency maybe arising due to the micro structural feature or the packing of material or the arrangement of different material in a sense and then finding out the homogenized property. So, for instance there are 3 materials.

So, if you look carefully that the arrows in this figure indicates the material direction. Now,a symmetry material, symmetry direction; so, these symmetry directions we need to understand carefully, and what will be the material properties, what will be the constants or the material constants here will discuss in detail.

Now, to start with before we enter into the details of these material constants and generalized Hooke's law for 3 dimensional body. So, we also want to introduce the concept of strain energy density. So, strain energy density all of you have learned it earlier ah, but let us see little bit more here.

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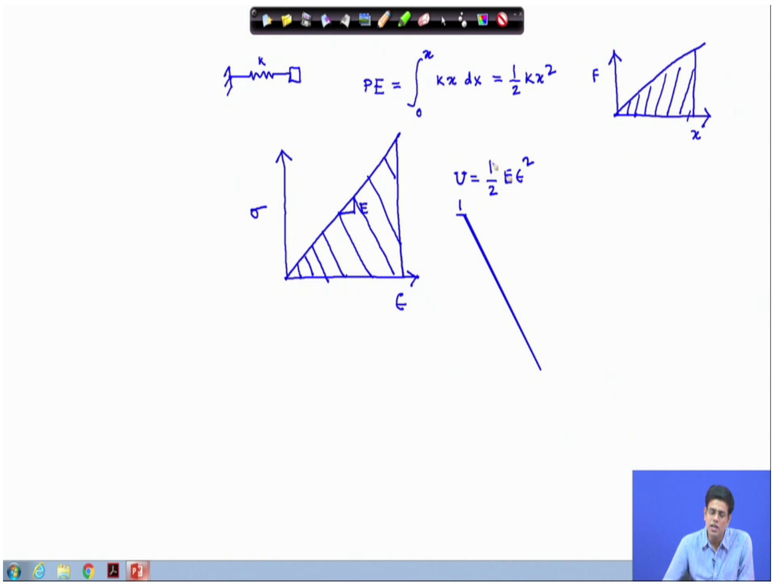

For instance, if you come from a spring system spring mass system; so, strain energy density of the potential energy, there we can equate with the work done. So which is essentially PE is essentially integral of 0 to x, k x, k x is the force and d x; so, which comes out to be the half of k x square. So, now, if you plot it clearly the force displacement curve or x here is a force then this becomes the straight line and half of k x is essentially representing the area of this curve.

Now, similar for the one dimensional stress strain body we have also discussed earlier; so which is essentially a one dimensional body. So, if you plot stress strain curve, so sigma epsilon and if it is a linear; so, we know that this is the slope and this is known as the Young's modulus and this is your strain energy. So, this becomes your strain energy which is essentially half of half of E into epsilon square. So, this obviously, is actually not related to the Poisson set. So, this strain is not the normal strain including Poisson set. So, this is purely hypothetical one dimensional body. So, this one dimensional body have only axial strain which is the uniaxial strain which is epsilon. So, this is the stress strain curve.

Now, the question is if for a 3 dimensional body what will be the strain energy for us. So, let us see how we can derive it carefully. Now, before doing that let us also considered a body which is something like this a arbitrary body which is something like this and there is a boundary condition implied on this body, and there is a traction force or the surface force and then there is a body force inside the body.

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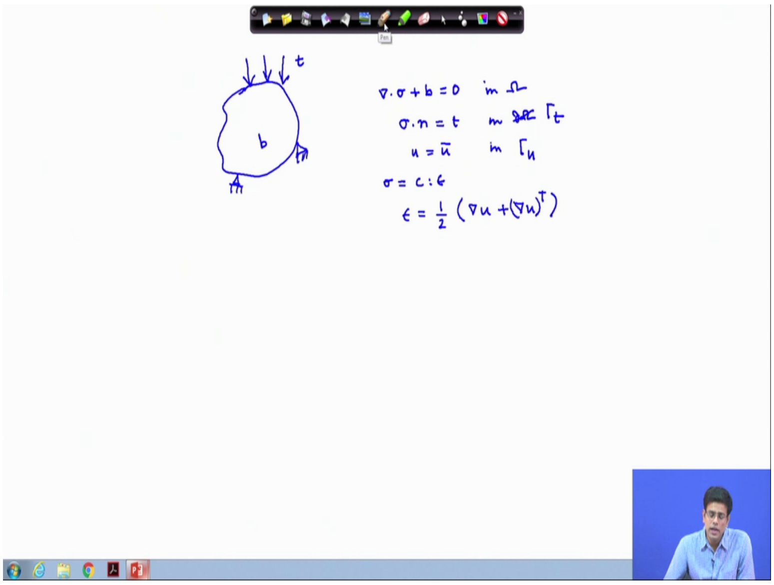

So, this body force is over the body. So, we know the differential equation for this body is del dot sigma plus b equals to 0. So, this is over the body or in the omega and then sigma dot n is equals to t the applied traction right. And this is the famous Cauchy's law or Cauchy's principle and then u equals to u bar or the displacement boundary condition this is also over lambda u. So, this we can write it in lambda t, right.

So, now, this body if I want to apply some force in terms of traction and then there is a body force included in that body, then what will be the strain energy. So, if this is a general body and the constitutive relation since we are talking about linear elasticity here only. So, constitutive relation follows the Hooke's law, where C is the fourth order elasticity tensor and epsilon is a strain. Now, strain is also represented in terms of half of del u plus del u transpose. So, this is we know from our previous lecture.

Now, to derive the strain energy of a body let us see how we can efficiently derive it. Now, suppose I have such body and then I consider the basic thermo dynamical consideration, where we can model linear elastic deformation or a linear deformation as a adiabatic process. So, adiabatic process means from our elementary knowledge we know adiabatic assumptions those are there is no heat and mass transfer between the systems. So, these systems have only internal energy or internalforces.

So, these within the adiabatic assumption the if I assume also that gets there is no dynamical effect that quasi static process it isno mass inertial effect is not there then first law of thermodynamics can be written that variation of the internal energy, or the change of internal energy, or increase of internal energy can be equated the work done. Now, as I told earlier the work done is there are two kind of force one is surface force one is surface force another is body force, body force. The surface force is t and body force is b.

Surface force in terms of traction; so we considered a general body here, so surface force in terms of traction and b is the body force. Now, if due to this applied forces there is a variation of the displacement which I see here in terms of del u, then work done can be the surface work done, work done due to the surface 4 is force is t dot del u. And similarly for the body force which acts over the body volume of the body which is b dot del u. Now, these should be equals to the del u that means, the change in the internal energy; so, within this adiabatic assumption. Now, t dot del u we can substitute t with the sigma dot n which is the Cauchy's principle. So, we can just substitute sigma dot n dot del u dS plus b dot del u dV.

Now, here you see we have also learned the divergence theorem. So, we can apply divergence theorem and convert this surface integral to the volume integral. So, if you remember the divergence theorem then this sigma dot n d u a can be converted as sigma del u into del dot sigma del u. So, this is from the divergence theorem. So, this quantity comes after applying the divergence theorem on these quantities. So, this quantity converts to the volume integral and then b dot del u as usual. So, this is my final thing.

Now, from here to here if we just use one of the identity or the tensor identity which is essentially del dot AV, A is any matrix then we can write it see V is any vector, A is any matrix. So, we can write it del dot del colon sorry; this is a transpose A is general matrix, A transpose inner product with del V that means, del V plus V dot del dot A transpose, right. If you use this identity then these equation becomes very simple.

So, here A is our sigma, A is our sigma. So, we are actually transforming this equation this quantity this quantity we are actually transforming. So, del our A is sigma and del u is V. So, if you and sigma we know it is a symmetric tensor. So, you can substitute, so sigma double colon del V, del V means here our del, so these will be finally, if I write it here sigma double colon del of delta u right then plus of plus delta u dot del dot sigma.

So, now this quantity this quantity from here to here how we arrived will discuss later and then this quantity is entered into the body forces trans. So, this quantity del dot sigma plus b dot del u since it is a scalar quantity, so we can flip sides. So, del dot sigma plus b dot del u. So, this becomes our, these quantity goes if the body force term.

Now, this del u the sigma inner product del of del u. So, this from here to here we use another identity which is if you know that is if there is a symmetric tensor S, if there is a symmetric tensor S and if there is a arbitrary tensor T, then S inner product T can be written as the symmetric part of the T that is S colon or S inner product half of T plus T transpose. So, I think we have also discuss this any second order tensor can be represented as a symmetric part and anti symmetric part.

So, symmetric part is half of T plus T transpose anti symmetric pass part is half of T minus T transpose. So, any if S is a symmetric tensor here and T is any arbitrary tensor then this identity follows. Now, corollary to these identity if W is a anti symmetric tensor then S inner product W is essentially 0. So, if W is as anti symmetric tensor, so in this is I can write it 0. So, now, these if we use here, so this is t del of del u which is any a tensor and sigma is a symmetric tensor. So, I can write this in place of this. So, which is what you I have written here, so half of del of del u plus del of del u transpose.

So, this is if you look carefully this is actually the del epsilon because we know strain expression which is del epsilon is half of del of del u plus del of del u transpose. So, and this is actually our governing equation, so this becomes 0. So, now, if you look carefully this del u becomes only sigma inner product del epsilon. Remember this del epsilon is a second order tensor.

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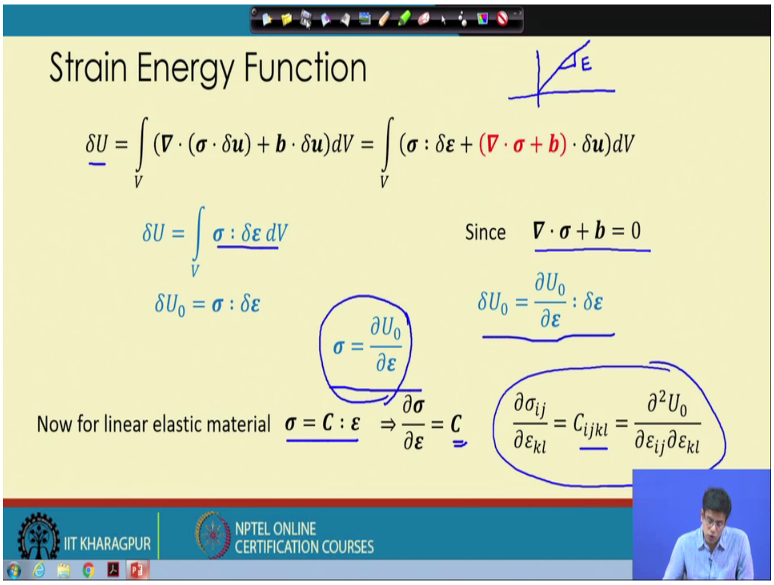

So, here if you now write it properly so which is this quantity turns out to be this. So, this becomes a 0 we know this becomes a 0. So, this is essentially 0 because it is a governing equation. So, and sigma inner product del epsilon will remain in this expression. So, I can write del u equals to sigma inner product del epsilon.

Now, if I take this quantity as the internal energy, so del of u 0 as this quantity and then if I assume from this that these strain energy is a function of strain then we can write this sigma as del of u 0 right del epsilon. So, this is the existence of the strain energy function.

Now, if we use now here that this is a linear elastic material. So, which follows from the Hooke's law and then again if I take the derivative of sigma with respect to epsilon then it will it will give in the constant. Now, if you look carefully that for a one dimensional material it is essentially that, so this Young's modulus as actually d sigma by d epsilon. So, here it is sigma is d sigma by d epsilon is actually the constant elastic constant and since this is this is a full strain tensor and stress tensor. So, this is a fourth order tensor.

Now, if I write it in ij or indicial notation. So, del of sigma ij by del of epsilon kl equals to C of ij kl and this is equals to del square u by del epsilon ij del epsilon kl. So, this is the, so the main thing from here is that there exist a strain energy density function, and strain energy density function is essentially represented in this form. So, and then if I take the double derivative of a strain energy function with this strains and that will give me the elastic constant matrix or the elasticity matrix popularly known as a elasticity matrix. So, we will use these relations in our formulation.

So, now once we agree that there linear for a linear elastic material that exists a strain energy density function and this function can be represented in this form then we can proceed for the anisotropic material.

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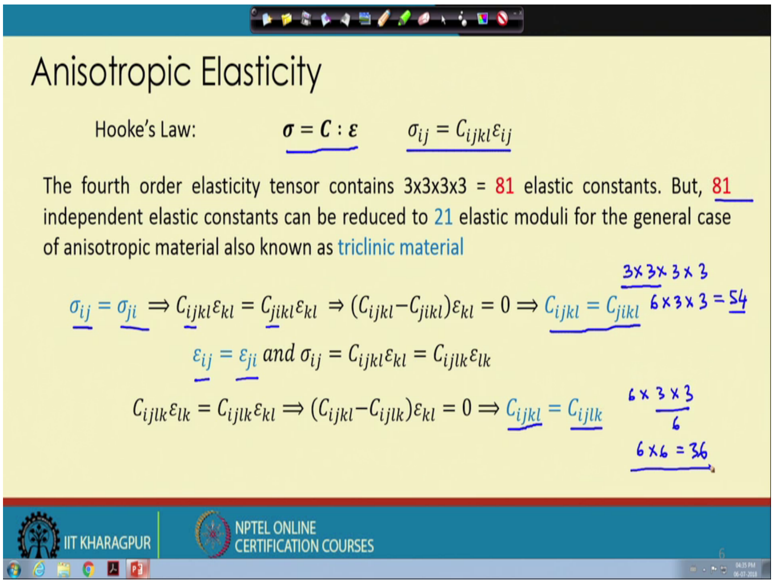

So, before we write this anisotropic elasticity. So, we again write this hooks law is in this form and then hooks law in initial form in this form. So, we have also mention that fourth order elasticity tensor C ij kl have the 81 component 81 elastic constant. So, this is very simple 3 cross 3 cross 3. So, I run from 1 to 3, j runs from 1 to 3, and k and l runs from 1 to 3. So, there are 81 elastic constants, but these 81 elastic independent constants can be reduced to 21 elastic module for the general anisotropic case and also known as triclinic material.

So, the general anisotropic material elastic constant is not 81 it is 21. And how it is? So, this is comes from the first it will come from the stress strain symmetry which is also known as the minor symmetry. So, we know that symmetricity of the stress that means, sigma ij is sigma ji. So, I can write C ijkl epsilon kl equals to C jikl is epsilon kl. So, you see that I just change these indices. Now, if I take inside take this quantity this side and then C ijkl minus C jikl epsilon kl equals to 0. So, if we take epsilon kl is nonzero, then it represents this, so sigma ijkl, jikl.

Now, if you do this, so if we assume that sigma ji ij and sigma ji sigma ij and equals to sigma ji, then obviously, ji and ij can permitative in 9 components. So, from here actually we are able to reduce 3 cross 3 cross 3 to this can be 9. So, 9 cannot be there since ij and ji are same. So, I can reduce it to 9, so 6 independent components. So, 6 cross 3 cross 3 so it becomes 54 constants. So, already you have reduced 81 to 54 constants.

Similarly if you take epsilon ij and epsilon ji are equal that means, strain tensor is a symmetric tensor then also we can prove this that sigma ij kl and sigma ij is equals to sigma ij lk. So, now from here actually we can reduce this 6 cross 3 cross 3 to this will be again 6 independent component, so 6 cross 6 so 36 component. So, you see from 81 components we have already reduced to 36 components by using the stress strain symmetry.

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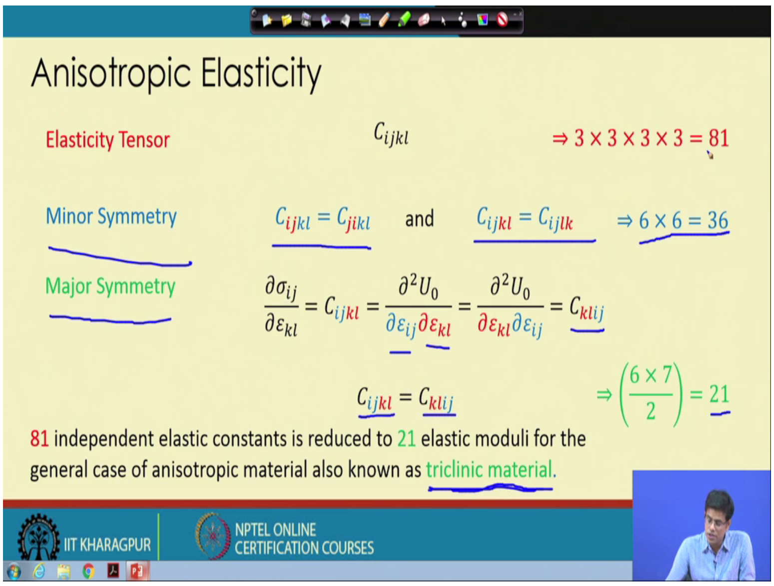

Now, if you if you use now the existence of the symmetric condition existence of the strain energy function. So, here then we can further reduce it. For instance here the elasticity tensor this is C ijkl is a 81 component and if you use this stress symmetry and strain symmetry. So, this is known as the minor symmetry. So, we can reduce it to 36 component.

Now, another symmetry is known as the major symmetry where will prove that C ijkl equals to C klij that means, these 2 indices are flipped. So, we know that C ijkl can be written as a second derivative of the strain energy function with respect to strains. So, now if we change so this is the partial derivative, so partial derivative we can flip this sides and then we can write it C klij. So, now this is known as the major symmetry. So, finally, C ijkl and C klij these are the major symmetry.

Now, if we impose this condition then the number of independent components in C ijkl further reduce and it becomes 21. So, now, this 21 component even though it is a full anisotropic material, this 21 component is actually the independent components. So, 81 independent elastic constant is reduced to 21 elastic moduli for the general case of anisotropic material, and also known as the triclinic material or allotropic material. So, this material has no symmetry, no reflection symmetry or no deflection symmetry. Now, this material is known as the triclinic material it is the most general anisotropic material. Now, but contrary to 81 constants it has only 21 independent constants. So, let us see how it looks.

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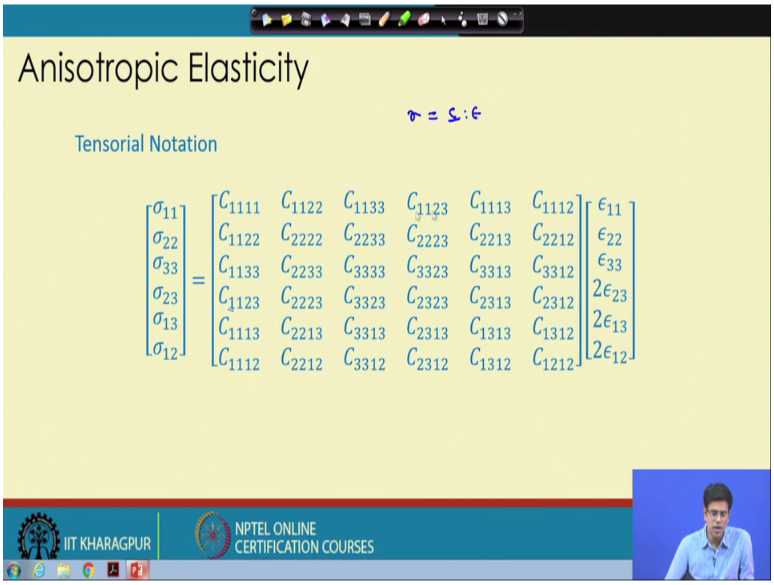

So, if we nowright it in tensorial notation. You see this is in a rectangular matrix this is this cannot be written in a matrix form, because as we know we have we observes sigma is essentially C colon epsilon. So, essentially its C is a fourth order tensor. So, this we can write it in this form.

Now, look carefully these are the symmetry lines. So, these components are only independent and these are the component of the stress tensor. So, this is possible for this case only, in general this is not for possible for the higher order tensor. So, now, ah, but we do not work in this form actually we work in a matrix form. So, we will now use our previous concepts of using Voigt notation to convert this stress strain relation to the matrix form.

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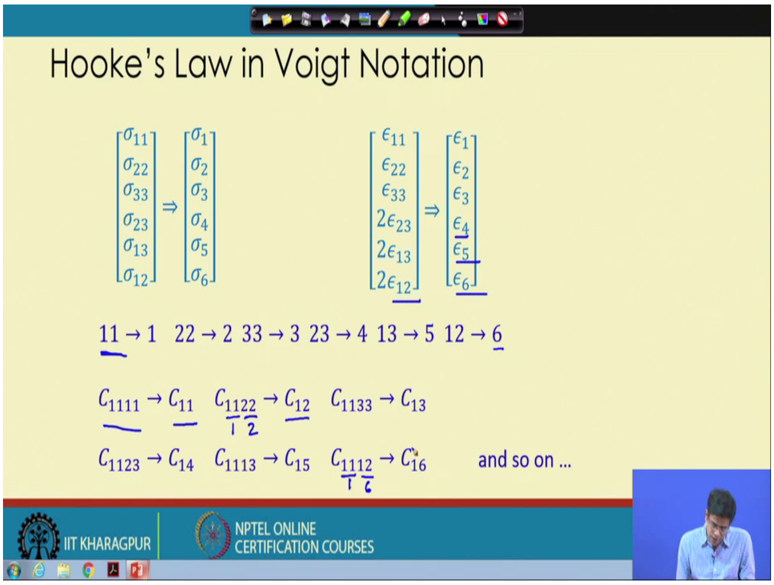

Now, this can be very easily done and I think in the previous class we have also discuss; so, Hooke's law in Voigt notation. So, basically what we do we transfer 11 to sigma 1, sigma 22 to sigma 2, sigma 33 to sigma 3, sigma 23 to sigma 4, sigma 13 to sigma 5, sigma 12 to sigma 6. Similarly strains is epsilon 11 to epsilon 1, epsilon 22 to epsilon 2, epsilon 33 to epsilon 3 and 2 epsilon 23 to epsilon 4. So, this is engineering strain. So, now, 2 epsilon 13 is epsilon 5, 2 epsilon 12 is epsilon 6.

Now, so what we do you know what we did here is that contraction operation this is known as the contraction operation in this. So, 11 in this in place of 11 indices we write 1, 22 in place of 22 we write 2, 33 to 3, 23 to 4 and 13 to 5 and 12 to 6. So, if you use this now then C 1111 becomes C 11, C 1122 becomes C 12. Here for 11 I write 1, 22 I write 2, so becomes C 12 and then the other side as usual. For instance C 1112 which is C 11 I write 1, C 12 I write 6 because C 12 indices I am defining as 6 so and so on. So, I can now write it a matrix notation proper metric notation for the Hooke's law. So, let us see how it looks.

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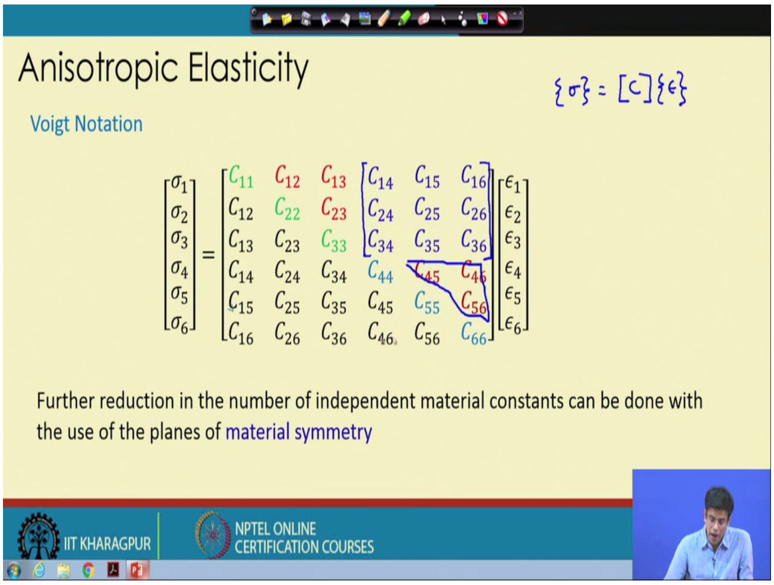

So, here now my Hooke's law looks like this. So, sigma 1, sigma 2, sigma 3 and sigma 6 to epsilon 1 to epsilon 6 and this matrix is the proper elasticity matrix. So, now, the in the matrix notation I can simply write sigma vector, C matrix or the elasticity matrix and this is my strain vector, right. Now, you see this elasticity matrix is symmetric and there are 21 independent components. So, this is for a general anisotropic material. So, there is no symmetry impose symmetry in it or anything ah. So, but if you look carefully this matrix I have written in terms of different colours except this portion because this is a symmetric matrix.

So, if you look green coefficients so these are responsible for the normal components or the normal strain normal stress, these are responsible for the normal stress. And these are the red components here represent for the Poisson's effect. If you remember the Poisson's effect that is if you pull a bar, then there is a contraction in the vertical direction also. So, this is this coefficients are responsible for the Poisson's effects on the normal stress. So, and these are the shear stress components. So, C 44, C 55, C 66 these are the responsible for the shear stress is component and these portion that is this portion these are the coupling coefficients.

So, these are the responsible coupling to strains that is the shear and normal strains now this portion also C 45, C 46, C 56 these are also known as the chains of effects, chains of effects. So, these are the different physical meaning of the anisotropic material.

Now, this is for a triclinic material we know. Now, again if we again assume material symmetry for instance we can assume the material is symmetric about one plane or reflection about one plane, then we can further reduce this components and how to do that will learn in the next class. So, further reduction of number of independent material constants can be done with the use of planes of material symmetry. So, this will do it in the next class. So, let us see in the next class how it looks.

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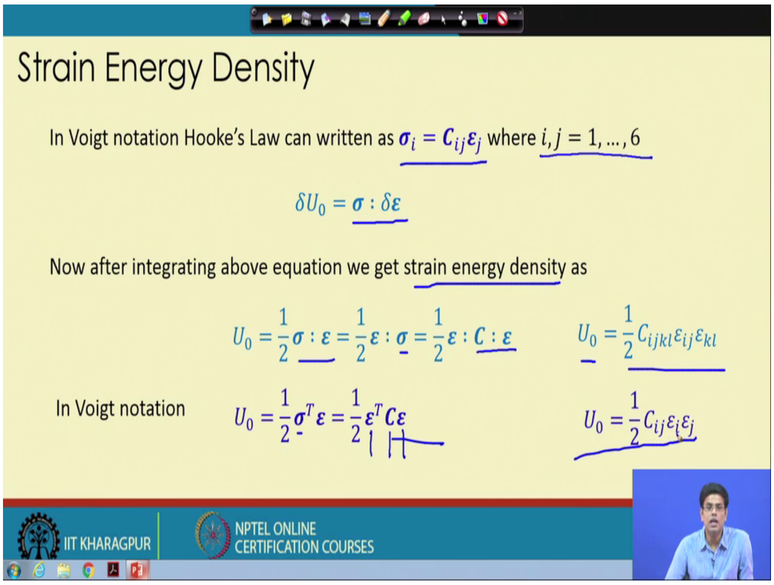

So, before that let us also derive the strain energy density here that in the Voigt notation we know what is sigma i equals to C ij epsilon j, where sigma ij runs from 1 to 6 this we have seen from the previous discussion that the Hooke's law in mark notation.

And now if you look the in our objective is how what is the form of the strain energy in this Voigt notation or the tensorial notation. So, if you now remember that our variation or the increase instance energy or the strain energy is sigma inner product del epsilon. So, now if we integrate this equation over the body then we get the strain energy density. So, the strain energy density is actually of this form, right, so which I represent in terms of U U 0 here, so half of sigmainner product E. Now, since this is a scalar we can flip it. Now, in place of sigma I substitute the Hooke's law which is the C colon epsilon in a tensorial notation. So, now, if I write this form U 0 in initial notation which is simply this so half of C ijkl epsilon ij epsilon kl.

So, this is a tensorial notation, but in Voigt notation also I can write it in a vector form these are vectors law, so which is half of sigma transpose epsilon, so half of epsilon transpose C E. So, C E comes from the substitution of sigma. So, this I can write it in a vector form. So, specifically the here epsilon is a vector, C is a matrix and epsilon is a vector strain vector which we have seen in a previous slide. Now, if we write it in the initial form, so which is essentially half of C ij epsilon i epsilon j. So, this is the form of the strain energy.

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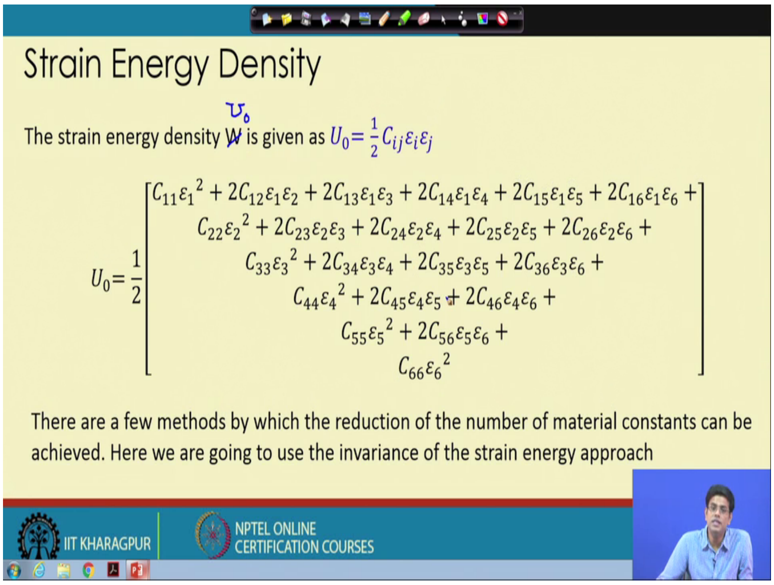

Now, if I write it if I expand it explain this strain energy, so the strain energy here it will be U. So, this is U strain energy U or U 0 here. So, U 0 is half of C ij epsilon I epsilon j. Now, if I write it for all components of strain and all modulus means ij runs from 1 to 6 this is the total form of the strain energy. So, C 11 epsilon 1 square C 12 epsilon 1 epsilon 2 and so on.

So, you see finally, this quantity is a scalar. So, even though this is a strains is vector, strains are vectors and Cis a matrix here. So, strain energy is a scalar. Now, so it is important to know that this strain energy is a internal energy or resistance of the material which it offers when we try to deform the body. So, now we can discuss how this strain energy will change if we assume any material symmetry or any material reflection symmetry or any other type of symmetry. So, we will discuss it in the next class.

Thank you.