Theory of Elasticity

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Lecture – 10
Concept of Stress and Strain (Contd.)

Hello everyone welcome to the fourth class of this week. In the first 3 classes, what we discussed is when a when a material is subjected to certain kind of threat, then as a response to the threat stress generates in the material and then, also the material undergoes deformation right. So, in the first 3 weeks we try to understand what is the definition of stress; how the stress is represented and what are the different properties of stress components.

Now, we will do the similar exercise for strength today. So, today's topic is Concept of Strain. You see in any engineering analysis one of the very important step that we do at the beginning of any engineering analysis is Idealization. We always perform analysis on the idealized system not on the actual system. For instance, if you recall your mechanics course or first year or structure analysis course, you studied how to analyze truss, you studied how to analyze beams frames and so on.

So, whenever we call there is no in nature there is nothing like, there is nothing like beam plates and so on; but when we when we say that something is truss, it means that we have idealized that particular structure as truss. Similarly, when we say that something is beam, means we have idealized the structure as beams. Now, every idealization is based on certain assumptions.

Now, these assumptions are very important because these assumptions are essentially the limitations of any theory; wherever those assumptions are not applicable, we cannot apply that theory. So, it is whenever we do any study any formulation, whenever you do learn any theorem, it is very important for us to understand what are the assumptions which constitute the basis of that formulation ok.

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You see suppose we one thing we have understood that deformation is a response of a material body of a of an object to the external threat. Now for instance, suppose we have this is suppose a plate ok. Now, if I apply some torque here, at this point and also at this point; then it undergoes deformation like this right.

Now, if I keep on increasing the torque or the moment here, not the torque; if we increasing if we keep on increasing the moment, the deformation increases deformation increases and you see we may have some point of time or we may have a deformation like this. Now, why I am showing this example because you see when the deformation is like this, this was a very small deformation right. But if I if I apply if I keep on increasing the load, the deformation increases, deformation increases; the deformation is so large or the deformation is such that it cannot be it cannot be coined as it cannot be treated as small.

Now, all the theories, all the definition of stresses definition of strain that we are going to do today, everything the relation between stress and strain that will be studying in subsequent weeks; everything that we have been discussed and everything that we will continue to discuss most of the time of this of this course, one of the important assumption is the Deformation is very small ok.

So, the if a Deformation is very small; so this is an idealization, we have we only can apply all these theory, all these definition of stresses, this strain and the relation only to those problems where the small deformation is a valid assumption. But there are many cases where the small deformation cannot be is not a valid assumption, you have to deal with large deformation, finite deformation and towards the end of this course will spend one week on how to deal with those finite deformation. But, till then, everything that we discussed with an assumption is a small deformation. Now, what is the consequence of this small deformation?

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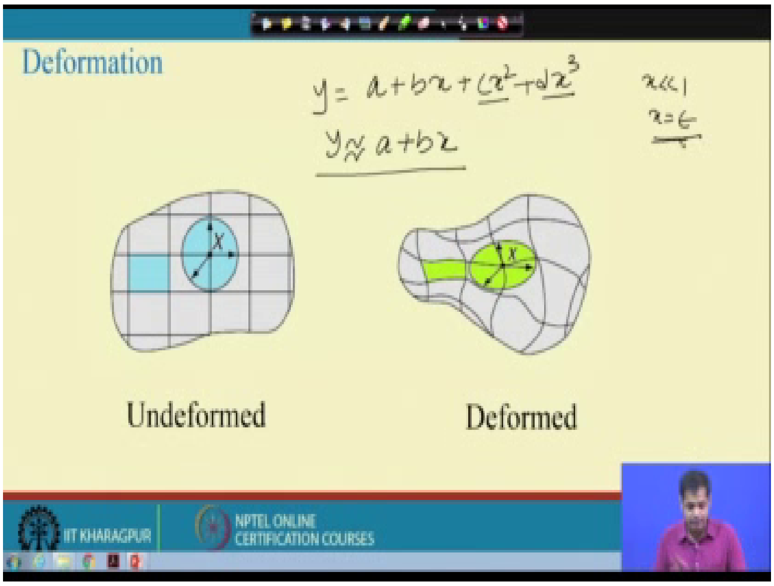

For instance, so suppose a material body like this and it is Undeformed configuration and this is the Deformed configuration. If you see these cells here this small rectangle and this is in deform configuration, these become this ok.

Now, before that suppose we have a function, suppose we have a function y is equal to y is equal to say a plus b x square b x plus c x square plus d x cube and so on. Suppose a function a non-linear function is a polynomial. This is how a and x are related to each other a, b, c, d are the constants. Now, if x is very small if is x is very small ok, then what happens then all these, the contribution of these terms are very small.

When x is very small, x is essentially tends to 0. This small x is equal to epsilon which is very small number. Then, the contribution of all these term because they are the higher order term, the contribution of all this term becomes 0. So, in that case your y can be written as a plus b x ok. So, then in that case y and x can be idealized as if x is very small can be idealized as the relation between y and x can be idealized as linear right.

Now, y and x depending on the context, you substitute the parameter it y and x. Suppose if x is stress and y is strain, then the relation between stress and strain is linear. If the x is displacement and y strain, the relation between strain and displacement is linear and that linearity we can that linearity is valid assumption as long as this x is very small and that is the thing that we have been discussing and we will continue to discuss with a deformation is very small ok. Now; so, it is undeformed configuration; it is deformed configuration ok. What we are going to do today is if a material, if a sub if an object is, if our object undergoes deformation; then, the strain is essentially the measure of the deformation.

Then, how to define strain? The same way, we define stress where else, you will define how to define strain and what are the and how to represent strain in different way different form and what are the properties of the strain tensor like the properties of stress tensor that we studied in previous lectures ok. Now, before that let us see how a material in a three-dimensional space; if we have an object in three-dimensional space, what are the possible mode the that object can deform?

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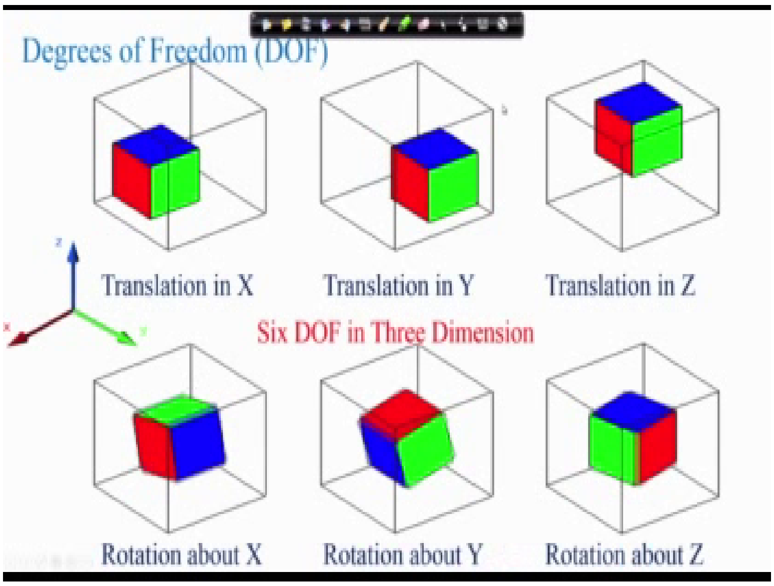

Here, you all might have you all know I believe the term called Degrees of Freedom. It is the freedom of an object that that can in three-dimensional space the number of freedoms that an object can the different way that an object in a three-dimensional space can deform.

So, in three-dimensional space, any object can have a 6 Degrees of Freedom. When I say any object without any constraint ok, any free object can have at most 6 degrees of freedom and then, and what are those 6 degrees of freedom? 3 Translation in 3 direction and then, 3 Rotations about 3 axis.

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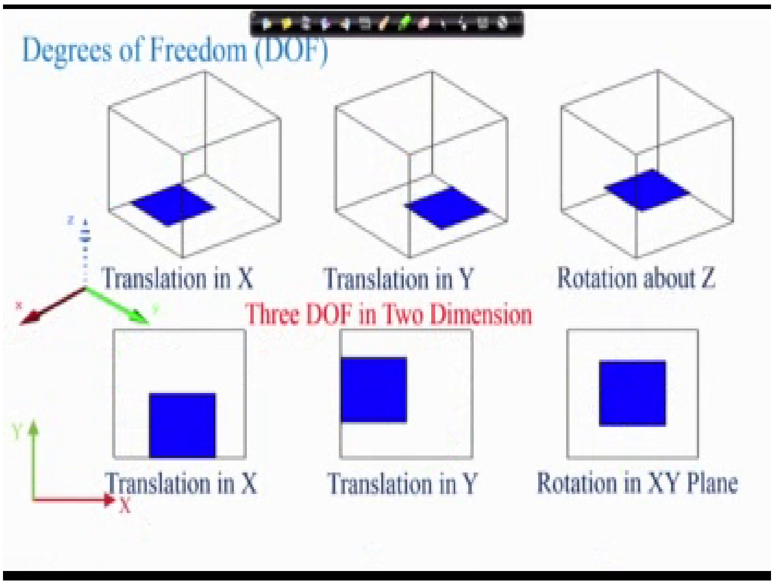

Now, if we project them into onto the two-dimensional space, then on two-dimension we have only 3 degrees of freedom to translation in the direction of 2 axis and then, one rotation in plane rotation about the normal axis.

So, this is the three Degrees of Freedom that an object can have in Two-dimension. What we do will try to understand or define stress in two-dimension and the same thing can be extended to three-dimension ok so now, ok. So, you see, so this is the deform confidence the undeformed configuration of a beam, not a beam is an configuration of any object and the corresponding deform configuration is this ok. Now, if we take any rectangular any cell here, this cell maps to this.

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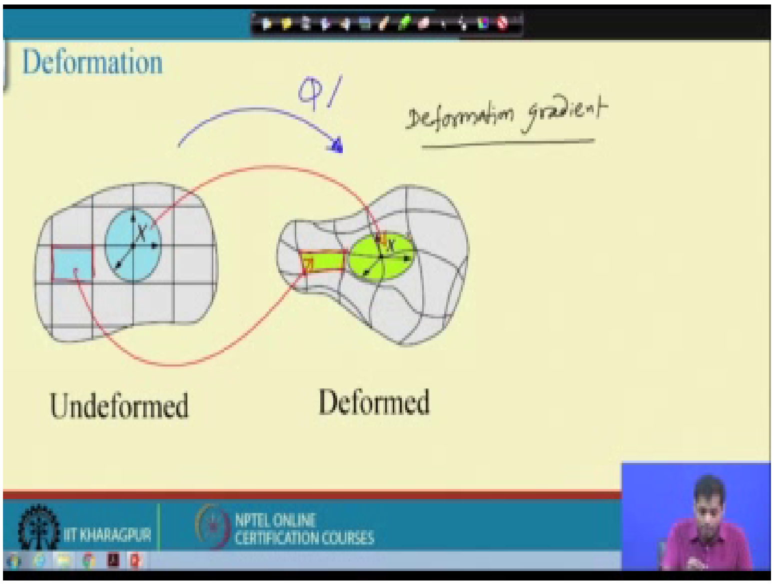

This cell actually goes here or if you take any circular small domain here, this is transfer to this like this ok. Now, this transformation, this map from undeformed configuration to deform configuration, suppose this map is this map is represented by phi ok.

We will not discuss what is this phi and how to construct this phi? But this map, this projection must satisfy certain properties ok. Now, in a very general way a general continuum when we apply when we define strain, ideally we should we should take care of this map and then, from that we define what are the different components of strain and what are the different measure of strain.

Now, but here we would not do that. Here, what we do is we take a small element for instance we take a small element, this element and which is un in deformed configuration become like this; will take a small element the same way we took a very small element infinitesimal element in the case of stress when we define stress, similar way we take a small element and then see what are the possible deformation mode that small element can have and then, try to formulate this strain try to define strain and try to calculate strength with geometry of that small element ok.

Now, but will come to this point, will define this map there is a term called Deformation gradient. Deformation gradient, will come to this point will come will define this deformation gradient which is essentially gives this projection from one undeformed space to deform space. But will come to this point probably in the lecture 11 when we talk about non-linear Elasticity ok.

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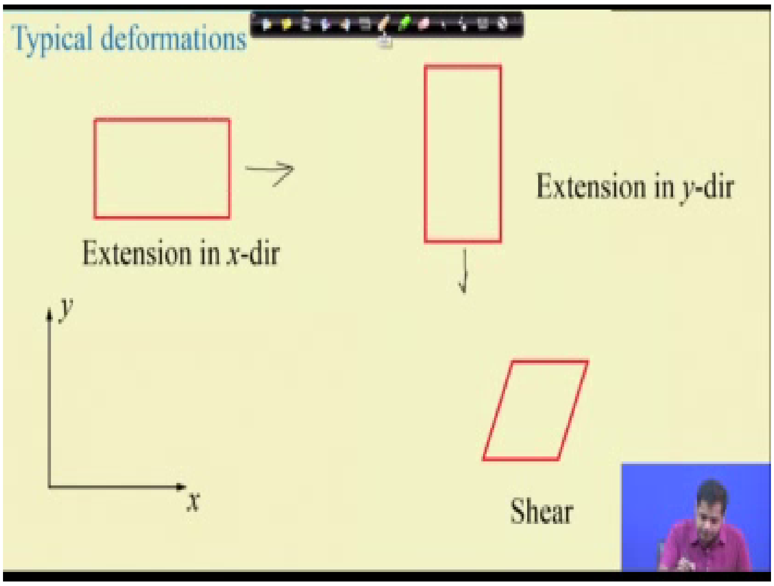

Now, take this is a possible deformation mode a typical deformation mode that can that as a two-dimensional object can have; all the deformation are in plane deformation.

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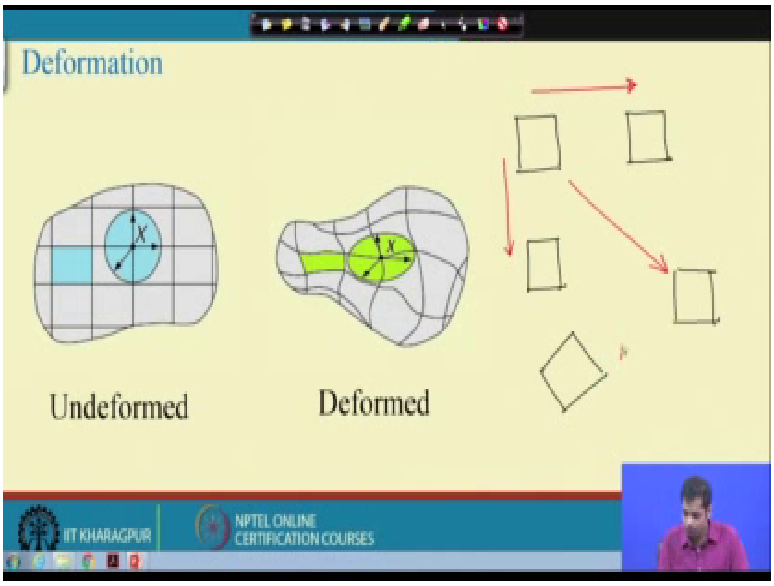

But remember, here the rigid body deformation, rigid body deformation for instance is if rigid body deformation is if an object is like this; if an object is like this the object rotates as it is and the object translates as it is. This is so, from this point object comes here from this point, object comes here. This is called rigid translation these translations do not change do not create any strain in the body ok. It does not cause any change in shape or change in value, no deformation in the body only the translates. Similarly, this object can I can come to this point can come, can translate in this direction and have and take a take a position like this, take a position like this or this object can translate in any arbitrary direction and can take a position like these are called rigid body translation ok.

This is rigid body translation. Similarly, this can object can rotate as well. This can object, this object can rotate are they without causing any deformation rotate about this and this is called rigid body rotation. So, rigid body rotation. So, when in the in this in this figure, when I showed these three-deformation mode, they are not rigid body rigid body translation and rotation in all the deformation all the deformation causes strain in the material ok.

Now, if we have two 2 direction x and y direction; one possible is extension in x direction or the or the contraction in x direction either increase the length or decrease the length. Similarly, in y direction, you can have change in length; in y direction either increase or decrease and this is called, these are called normal this is the this the movement is in normal direction in the x direction.

This movement is in x direction; this movement is in a y direction. Now, the another moment is this Shear movement. We will come to this point, what is Shear? So, these are the 3 possible deformation note that a two-dimensional object can have and all the deformation mode called strain in this object. Now, let us consider. So, whenever we if we go back to the previous, whenever we have this object this point, this goes here.

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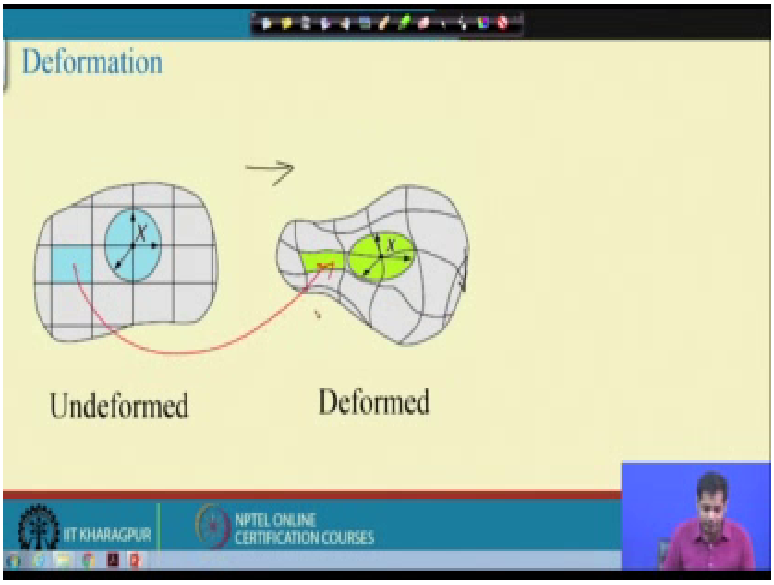

So, it is just not only not only extension in x direction, y direction or only shear; it has all the all the all different combinations of this typical deformation mode that act together and that all act together and cause deformation in a in an object ok. Now, take a very general case.

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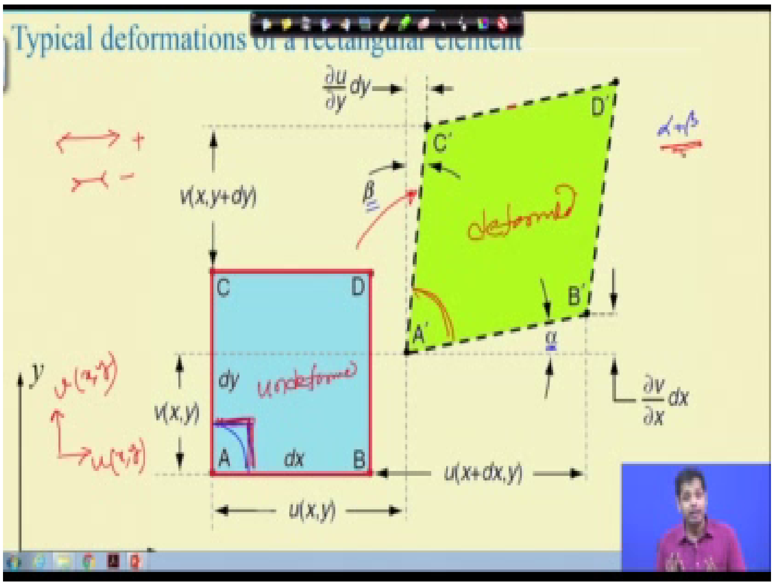

Consider this one is an undeformed configuration, this the red rectangle is an undeformed configuration and these rectangle moves here, moves here this is deformed configuration. This is undeformed and this is deformed configuration ok.

Now, say A, B, C, D at the points and corresponding points are A dash, B dash, C dash, D dash; these of this infinitesimal small area. So, the area the length is d x and d y ok. Now; so let us first try to understand all these different the this u v written here.

Now, this point suppose x direction this in x direction, we call u is u is displacement in x direction and v is displacement in y direction and both are function of x and y ok. So, suppose this point the point A moves here and suppose this distance is u of x, u x u x y and similarly the point B moves here and what is the cordon what was the initial coordinate of point B, initial location of point B? This is d x, the point B was d x right. So, this is called this is u of x plus d x and y ok.

So, similarly, so this point is point in y direction develop the displacement is v. So, point A moves to A dash. So, corresponding y displacement is v and then, corresponding y displacement of C, C moves to C dash. These become this distance was d x. So, this become so this is this is v of x y and this distance was d x. So, and this therefore, this becomes v of x. This distance was d y, then this becomes v of x and y plus d y ok. Now, so, will come to this point now then, the change in angle this angle initially was this angle initially was 90 degree ok, 90 degree and this is the change in angle, the change in angle is alpha and then, here it is beta. So, total change in angle is alpha plus beta.

Now, let us 6 our sign convention. The sign convention is tension is if the change in length, elongation is positive means if it is in this direction, these if these this elongation is positive and then, this is negative. Change in length, decrease in length is negative and then, for angle what happens if it is initially these 2 points are orthogonal to each other, this is 90 degree.

Now, now this angle is less than 90 degree. If the angle decreases between 2 line AB and AC here; 2 initially orthogonal line angle decreases, this is considered as positive. If the angle increases, this is considered as negative. So, change in angle is alpha, beta ok. Now, we have 3 as just in the previous slide, we discuss there are 3 deformation; the typical deformation mode translation, elongation in widely in x direction, elongation in y direction and then, Shear.

Now, elongation in x direction means that initially the length is d x. So, there will be change in length in x direction. Initially in the length was d y in y direction. The elongation in y direction means the change in length in y direction will be there will be change in length in y direction which causes strain in y direction and then, shear is this there will be change in length change in angle, this angle which is this angle. So, the alpha and beta, they are the measure of shear strength ok. So, let us first understand what is the let us first; let us first find out what is the strain in x direction means what is the change in length in x direction and then, corresponding strain in x direction ok.

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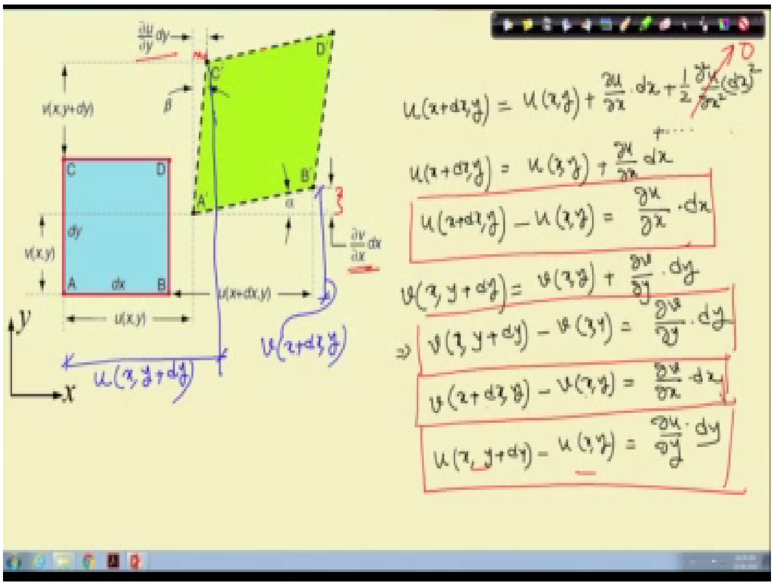

So, this is the we can that is better ok. Now, you see suppose this u u that u at x plus d x and y ok. If we expand it with in Taylor series, then we have x y plus del u del x into d x and then, plus half of del 2 u del x 2 d x square and then plus so on ok. Then we have all other higher order term.

Now, just now a in it was beginning of this lecture I, we discussed that we are these this is a this is a small area infinitesimal area d a d x the infinitesimal area. So, all these higher order terms will go away ok. So, essentially we are left with u of x plus d x, y; this essentially become u of x y plus del u del x into d x ok. Now this will vanish, this is so small that we can neglect, this we can neglect, we can make it 0 ok. If this is this, then u of u of x plus d x y minus u of x y that is equal to del u del x into d x. Now look at this and what is this? What is this expression? This is x, this expression is essentially your change in length in x direction right; change in length in x direction.

Now, similarly if we do the similar exercise for say y. So, if it is we can write say v of, v of x y plus d y following the same exercise, it can be written as v of x y plus del v del y del v del y into d y and this gives us v of x y plus d y minus v of x y that is equal to del v del y into d y right ok. Now, so, let us find out what is? So, let us find out, what is this distance and then, what is this distance ok? We are not we have not a defined what are the strain in different direction; what we are trying to understand it if an object a rectangular object moves in this way deforms in this way, then just from the geometry we are trying to understand what are the different dimensions ok, what are the different.

Now, let us find out this length. How do we get this length? Now, this was this B was, this B moves from this B moves from B B was there and this B moves from this. So, this distance this total distance is if I write it, this is if I this distance total is that is v of x plus d x y right. So, this was v of x y and this B was at a distance of delta x d x and therefore, this becomes v of x plus d x into x plus d x of y.

And similarly your, the point C if the point C is this, so this distance this distance this distance become u of x y plus d y because the point C was at a distance d y. So, when it moves, so point C dash, the corresponding coordinate of point C dash will be x y plus d y. Now similarly if we if we express v x plus d x in Taylor series like this and u x plus x of y plus d y in Taylor series like this, what we get is we get this. v of the same thing we get, v of x plus x plus d x y minus v of x y is equal to del v del x into d x right and then, similarly we get u of x y plus d y minus u of x y that is equal to del u del y into d y.

Now, we are almost. So, these 2 are important. So, what it tells you that the movement of point B dash will be this which is equal to this minus this which is del v del x into d x which is del v del x d x and similarly, this tells you the movement of point C in x direction which is this minus this del u del y into d y which is del u del y into d y ok. So, these 4 are important.

Now, let us once we have once we have done this now let us let us define what is what is this strain in direction. Now strain in x direction will be what? Strain in x direction, if you recall in your we discussed strain is essentially what? Strain is essentially the change in length divided by the original length. Now, if we are measuring changing length in x direction with respect to the original in then x direction that gives us strain in x direction which is written as epsilon say x x, very similar to sigma x x.

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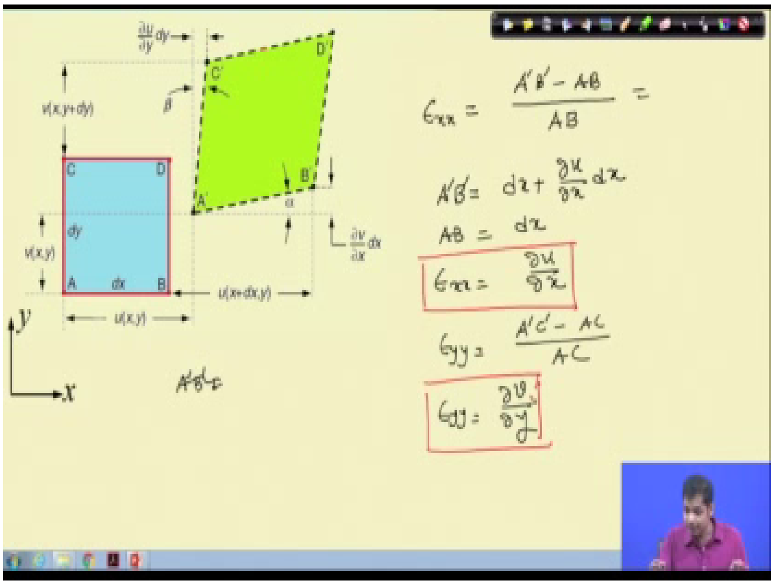

Now, that will be so initially the length was A B and that A B become A B becomes A dash B dash. So, the change in length will be A dash B dash minus A B and divided by A B is the original length ok. Now, you recall just now we wrote that ok, then A B A B was what? A B was A dash let us find out what is A dash B dash.

A dash B dash just now in the previous in the previous slide if you recall the change in length in A dash B dash was del u del x into d x ok. Then, so A dash B dash will be initial length plus change in length; del u del x into that is A dash B dash ok. Now then, then this is ok. Now and then what is initially A B? A B was A B was initially A B was d x and then, what we have finally, epsilon x x is equal to a this from this we have del u del x ok.

Now, in some places you will find that A dash B dash is obtained as A dash B dash is obtained as ok, I can write it that as well in some places you will see that A dash B dash is obtained as this square plus this square means and then, ignore the higher order term, essentially will get the same result ok. So, now once we have epsilon x x x, similarly we can define epsilon y y epsilon y y, if we do the same thing like in this case A dash, A dash C dash minus A C divided by A C A C and substitute what is the A dash change in length in A C; from this we get the strain in y direction and that that becomes epsilon y y become del v del y ok, del v del y.

Now, lastly once we have epsilon x and epsilon y y, now shear strain is shear strain is gamma x y which is written as alpha plus beta because total change in total change in length is denoted by alpha and beta ok.

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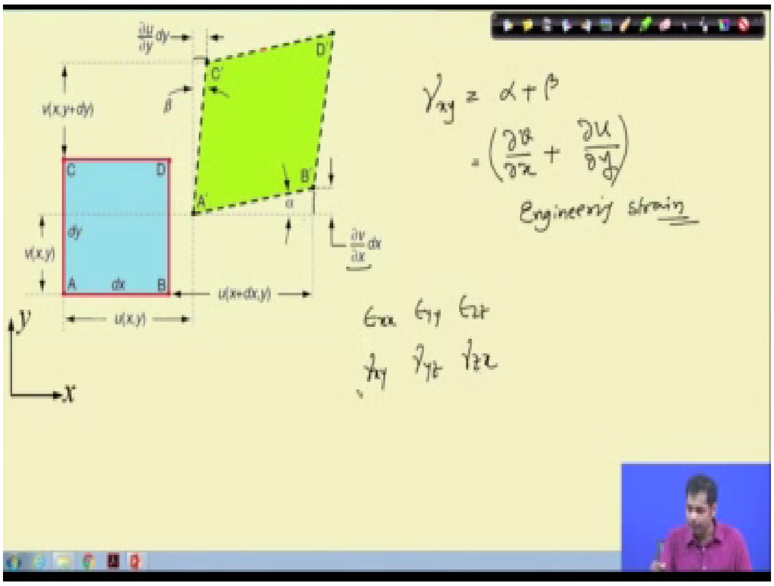

Now, you look at this, this distance is del v del x into d x and this distance is del u del y into d y assuming again a small length where A dash this is the projection is almost similar, we have alpha is equal to del v, del v del x del v del x means tan alpha is equal to alpha that is what we are and beta is equal to del u del y, del u del y.

So, this is the strain in a gamma x y the total shear strength. So, alpha will be this divided by d x and beta will be this divided by d x and tan alpha is equal to alpha; tan beta is equal to beta, that is how we have gamma x y. Now, these strain is called engineering strain, engineering strain. When we write in a tensor form, we just slightly modify this expression will come to this point shortly.

Now, we can do the same exercise for three-dimension as well. Now is it for two-dimension, we have three strain component; one is epsilon x epsilon y and gamma x y. Similarly for three-dimension, we have 6 strain components; 3 normal component epsilon x epsilon y epsilon x x epsilon y y and epsilon z z normal strain in 3 direction and then, we have gamma x y gamma y z and gamma z x are the shear strain on 3 different plane. So, total 6 strain component we have ok.

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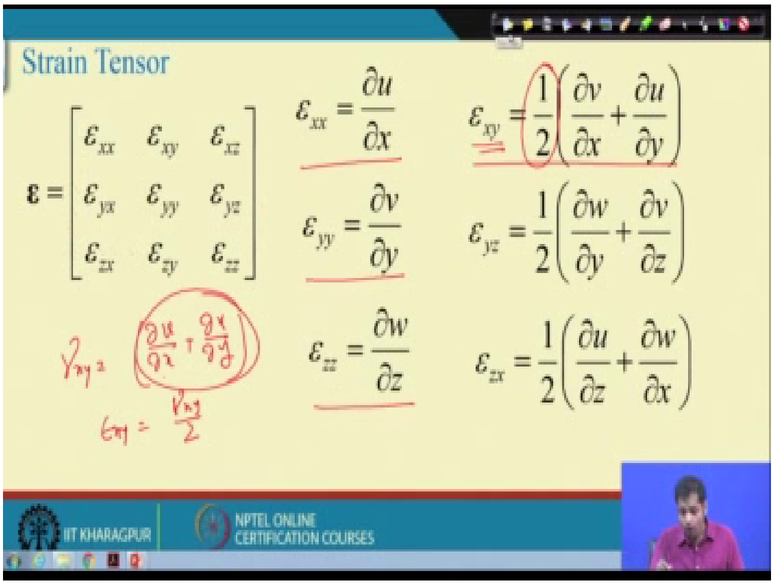

Now, if we write that strain component what we have is now epsilon x s is epsilon x s is this epsilon x s is just now we define epsilon x s similarly epsilon y y del v del y; in 3 dimension following the same exercise we can get epsilon z z.

Now, we remember we wrote gamma xy is equal to gamma xy is equal to del v del x plus del u del y right. Now, that was engineering strain. When we write in a tensor form epsilon xy is taken as gamma xy by 2. So, that is why this half comes. So, you be careful which strain you are dealing with if it is engineering strain, then this is the expression if it is a tensorial strain, then this is the expression. We have one half here.

Now, if we have 6, then strain component and if we substitute that strain in a in this form in a tensor form, then this become strain tensor; very similar to stress tensor, this is strain tensor. Now, not only that all the exercise that we did first strain for stress tensor for instant the transformation of stress tensor, we can do the same exercise for strain tensor as well ok.

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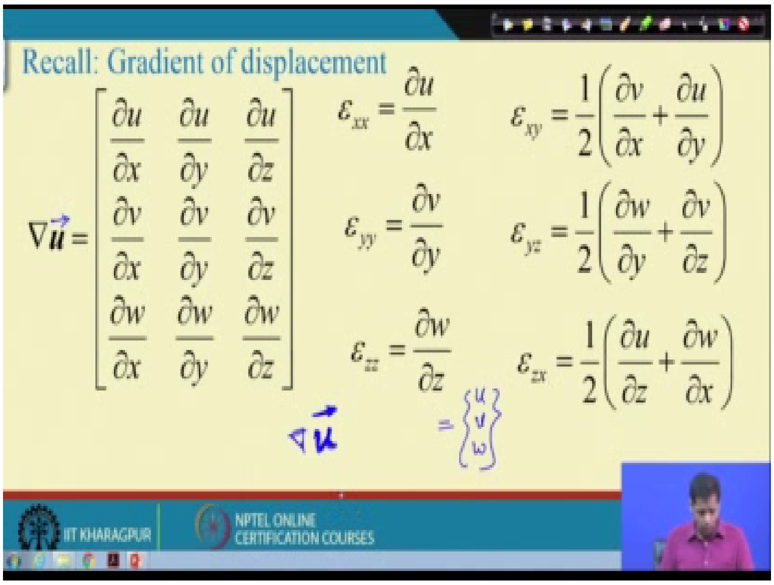

Now, this is important you see. Now so, these are the all 6 strain component ok. Now, suppose we have a vector u, we have a vector u which is now be please be careful about the symbol used because this u here what is u v w? U v w are the u v w are the are the displacement in three direction x y and z direction.

Suppose that is represented by a vector u which is written in bold letter ok, this is vector u ok. Now, if this is u; then in the first lecture when you had first week when you had the concept of the reviewer concept of tensor and then, you recall how to how to write the gradient of any vector ok. The gradient of any vector any vector u can be written as this; when u v and w are the components of this vector u ok.

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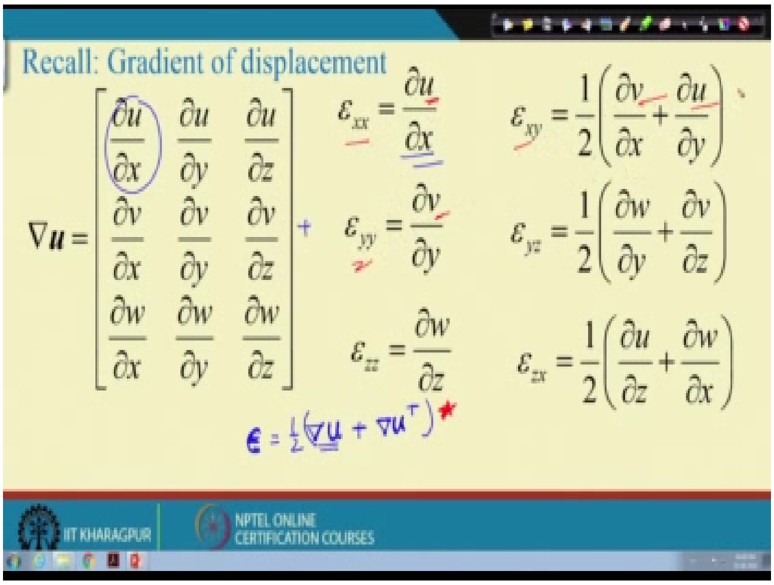

Now, look at this gradient and look at this corresponding expression for strain, what we can write is now from this and from all these definition of strain can we write that strain is equal to as a tensile is equal to gradient of u plus transpose of this half of half of this? You see the first component will be what?

First component in this will be what? This plus transpose of this half of this means this plus again del u del x and half of this become del u del x which is epsilon x s. Similarly, if you substitute del u from this and then, will get all the component of strength like this ok. So, this expression is very important will come to this expression again and again ok.

Now, that was so what we have discussed today is we defined what are the different strain in a two-dimensional object, but remember the same exercise we can extend for higher dimension and get the all the strain component. And while defining the strain, while defining this strain we used we tried to define this strain through geometry with an assumption that the it is a small deformation. So, that higher order terms can be neglected.

These term is very these see you see these all these expression, this gives you what? This gives you a relation between strain and displacement strain and displacement or corresponding displacement. This relation is also called Strain displacement relation. But look at this relation these relations are linear relation.

But as I say it these relations may not be linear always, there are some problem or if there are many problems where we have to take into account the non-linearity in this relation and then, will discuss that in the in a when we talk about non-linear elasticity. Ok, that is what for the day. So, next class what we do is, next class will just have a summary of concept of Strain and Strain. See you the next class.

Thank you.