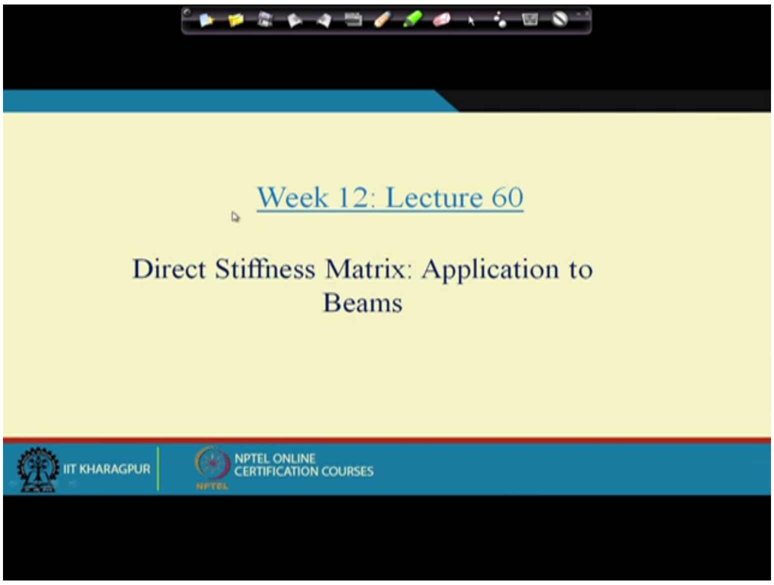
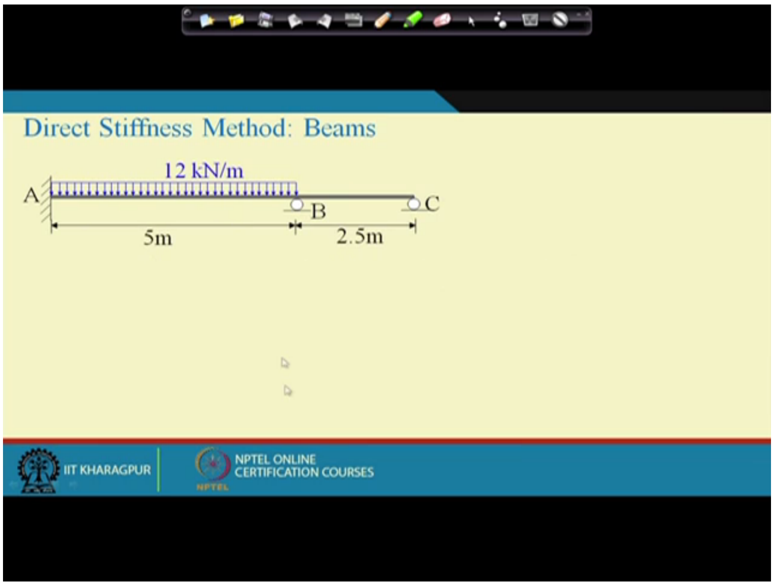
Structural Analysis I.
Professor Amit Shaw.
Department of Civil Engineering.
Indian Institute of Technology, Kharagpur.
Lecture-60.
Direct Stiffness Method (Continued).

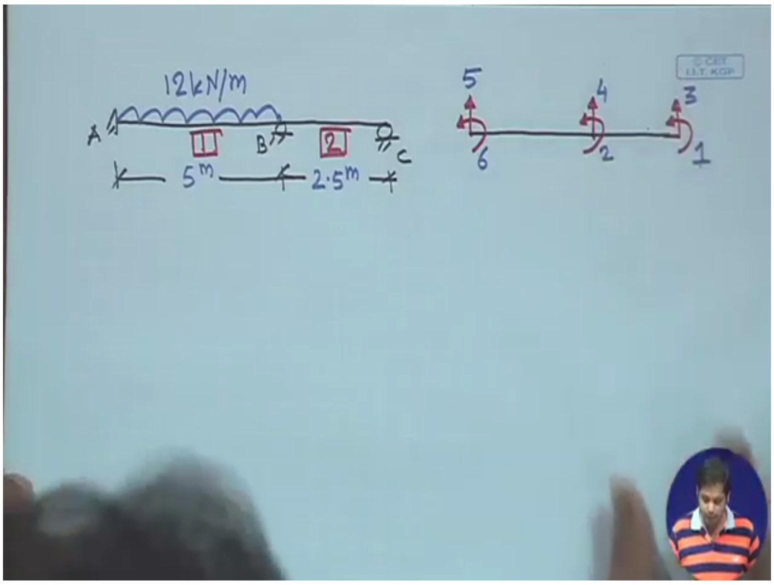
(Refer Slide Time: 0:48)





Hello everyone, welcome, what we will do is we will, in the last class we had seen the application of direct stiffness method for analysis of truss. Today we will see how direct stiffness method can be applied to beams the background that is equate, that already we have discussed in 1st 2 classes in this week. Today we will just demonstrate how it can be applied and demonstrate the ent different steps in the application. Okay. So let us see this example. It is a continuous beam, 2 span continuous beam, one end is fixed at A and B and C are roller supported and then we have external applied load, all the dimensions are given here.

(Refer Slide Time: 1:35)



Now what we have to find out, we have to find out the , we have to analysis means we have to find out the displacements and then internal forces, the displacement reactions and the internal forces in the beam, okay. Let us let us do this. So beam is something like this. The end is fixed, this end, this is a roller support and this is a roller support. And then we have an externally applied load here 12 kilo Newton per metre, okay. And then this length is 5 metre, this length is 5 metre and this is 2.5 metre, 2.5 metre, okay. Now the next, 1st step is the numbering of beam, node and beam I mean joints, numbering of joints and numbering of members.

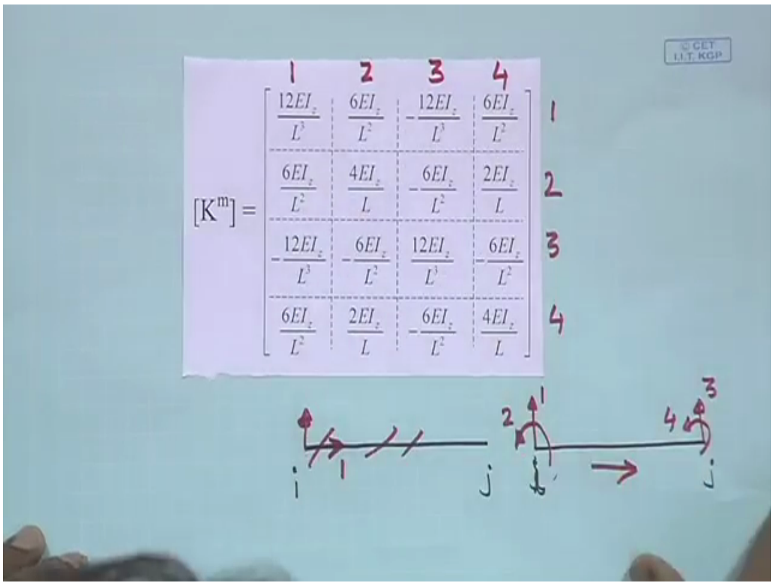
Suppose this is number number 1 and this is member number 2. Okay. And then this is beam number 1, this is beam number beam number 2. Then what we need to find, what we need to do, we need to name the degrees of freedom. You see it has 2 joints, 3 joints, joint A, joint B and joint C. And at every joint we have 2 degrees of freedom, these degrees of freedom are vertical transverse displacement and rotation. Okay. Let us say in this case, so this is the problem restriction. Now suppose this is the beam and this is point A, this is point B and this is point C. And then we have, we have vertical displacement C, degrees of freedom, vertical displacement and vertical displacement here.

And then we have moment and moment and then a moment. Right. So at every node we have 2 degrees of freedom, total 6 degrees of freedom. Now the numberings are, say this is 1, this is the, this rotation at joint C is 1, then this rotation is 2 and then this is 3, this is 4, this is 5 and this is 6. Okay. Now you could have, you could have numbered like this, this is 1, 2, 3, 4, 5, 6 something like this. It will be clear by I chose to, chose to number this rotation at this joint protection at this joint 1 and rotation at this joint 2. The reason is, you see if you look at this culture, this is fixed end, there is no, all the come all the degrees of freedoms are constrained here, so there is no translation, no rotation at this point.

In this case, so there is no in terms of displacement degrees of freedom, in terms of displacement there is no unknown here. Now at joint B, only, there is no vertical movement, only rotation at joint B can take place, similarly there is no vertical movement that joins C, only rotation and joint C can take place. So the unknown degrees of freedom in this case are rotation at point C and rotation at joint B. Just to make the unknown equal, if we number it 1, so when we say degrees of 1 is the 1st unknown, degree of, degrees of freedom 2 2nd unknown, it will help us to, help us while partitioning the matrix.

Okay, other than that there is no specific reason for choosing this way, numbering like this, you are free to choose barring the way you want. Okay, as of now. Okay. So this is the degrees of, this is the degrees of your, numbering of the rotation and translation. Then what we need to do is, we need to find out, we need to find out, now element number we need to find out the element stiffness matrix, okay. Now let us, let us tweak for, we have 2 members, we need to find out element stiffness matrix for 1 and element stiffness matrix for 2. Now if you remember, this is the expression for element stiffness matrix, okay. That we have already seen.

(Refer Slide Time: 5:41)

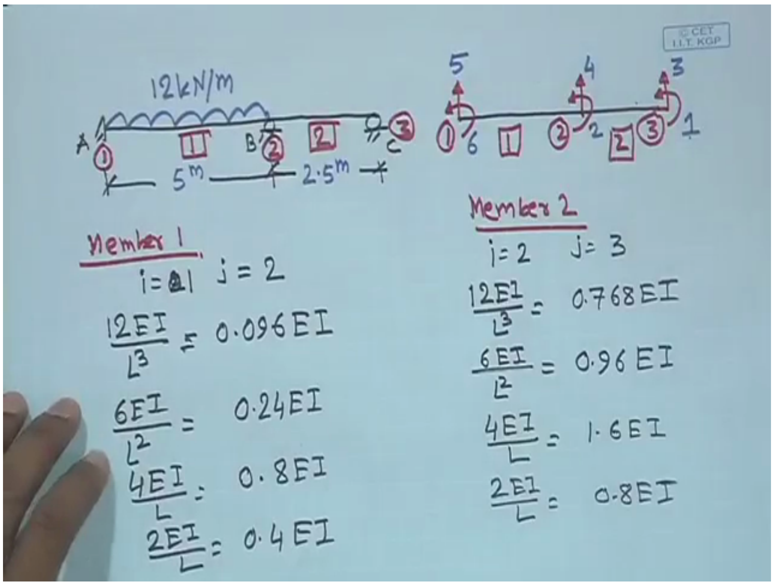


This is the member since this matrix. Now how this member, what is this convention, this member stiffness matrix is written here. If we have a beam element, beam element, okay, suppose this is beam element, this is joint i and this is joint j. And the degrees of freedom like this. You have degrees of freedom, this is 1, and this is 1 and this is and degrees of freedom, sorry, not like this. Once again draw it here, this is node, this is node i, this is node j, node i, node j, at node i the degrees of freedom is vertical direction and then this rotation. And node j, this is the degrees of freedom in the vertical direction and then rotation.

So this is 1 and this is 2, this is 3 and this rotation is 3, this is the sign convention we chose, please remember the way this stiffness matrix is derived here, it is written here. The sign of the stiffness matrix is written here, this is the convention used, okay. And we have already discussed in 1st 2 classes while deriving this matrix. Now if this is the case, then this row is for this, this row, this is for 1, 2, 3, 4 and this is 1, 2, 3 and 4. So any take, if we take any beam element, so 1 and 2 is the degrees of freedom for 1st node if we go in this way and the 3 and 4 are the degrees of freedom at 2nd node.

And the 1st degrees of freedom is the translation and the 2nd degrees of freedom is the, is the rotation, okay. Now if we, now let us find out what is the, what is the what are the things we need here, for any element we need 12 EI by L cube, then 4 EI by L, 6 EI by L square and then, that is what. So let us compute these elements separately and then substitute in this matrix to get the element stiffness matrix.

(Refer Slide Time: 7:55)

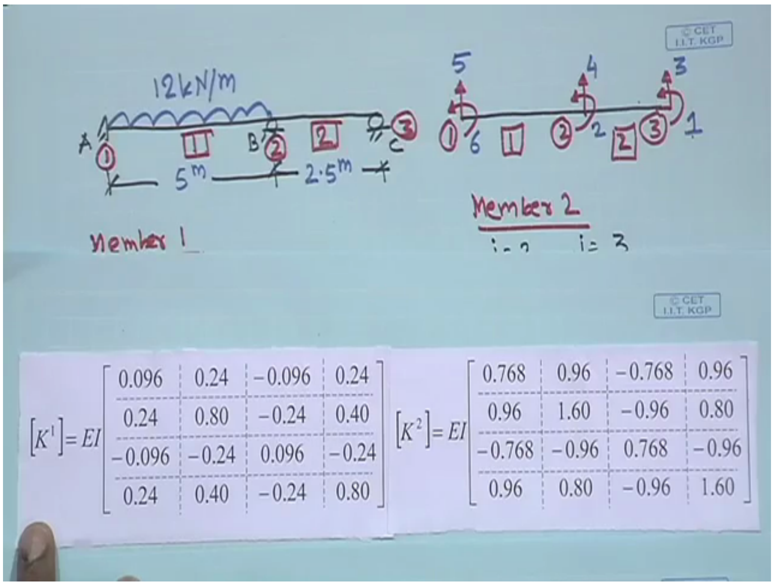


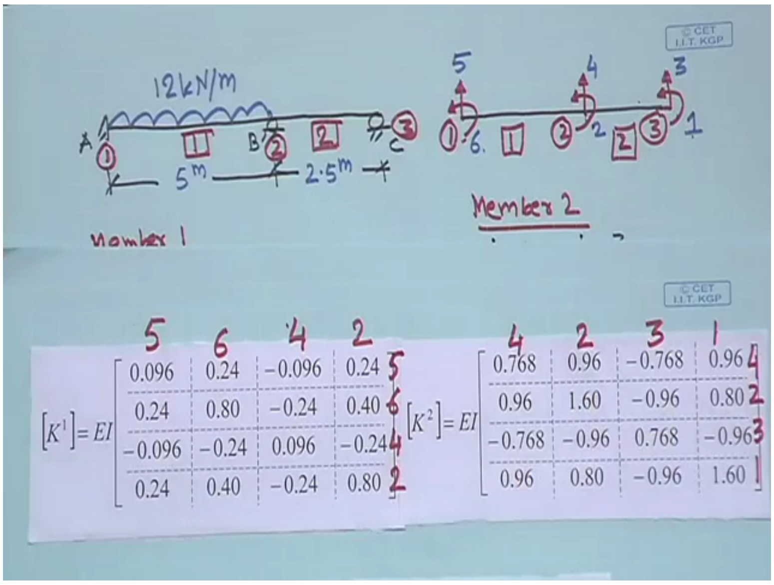
So now for member 1, so, 1st is member 1, member one is i is, i is this node number, suppose this is node number 1, this is node number 2 and this is node number 3. So this is , this is node number 1, this is node number 2 and this is node number 3. This is member number 1 and this is member number 2, okay. Now for node number 1, i is equal to, i is equal to 1st node, i is equal to 2 and j, i is equal to 1 and j is equal to 2nd node, 2nd node, okay. Now let us calculate what is 12 EI, then 12 EI by L cube, L cube is, L is 5 metres, so this will become 0.096 EI, let us write it in terms of EI.

Then once we know the value of EI we can substitute them. And then similarly 6 EI by L square is is equal to 0.24 EI. And then finally 4 EI by, 4 EI by L becomes 0.8 EI and 2 EI by L becomes 0.4 EI. Okay. These are the things we need to calculate this stiffness matrix, okay. Now similarly let us find out for joint member number 2, member 2, for member 2, member 2 span between joint number 2 and joint number 3. So for member 2, i is equal to joint number 2 and j is equal to joint number 3, okay.

Now similarly we can, if we calculate them, then these values will be 12 EI by L cube, this becomes 0.768 EI, it is, in this problem it is assumed that EI is constant for both the, both the span but if it is not constant, you have to take that variation, okay. So this EI is for member number 2 and this EI is for member number 1. But here it is same for both the members. The 4 EI by L, this becomes 1.6 EI and then finally 2 EI by L, 2 EI by L 0.8 EI. Okay. So we, we have already calculated the elements of the member stiffness matrix for of the members, member 1 and member 2.

(Refer Slide Time: 11:23)





Then what next we need to do is, we need to substitute these values in this, in this matrix. And to get the element stiffness matrix for both the members. And if you do that, then what we get is, we get the expression like this, okay. We get an expression like this, now you, you recall for member number 1, for member number 1, this was the member number 1 and this is the member number 1 and member number 2, member number one member number 2. This is the member number stiffness matrix for member number 1 and this is the stiffness matrix for member number 2. And that we obtain just by substituting those values that they calculated just now into, to the member stiffness matrix expressions. Okay.

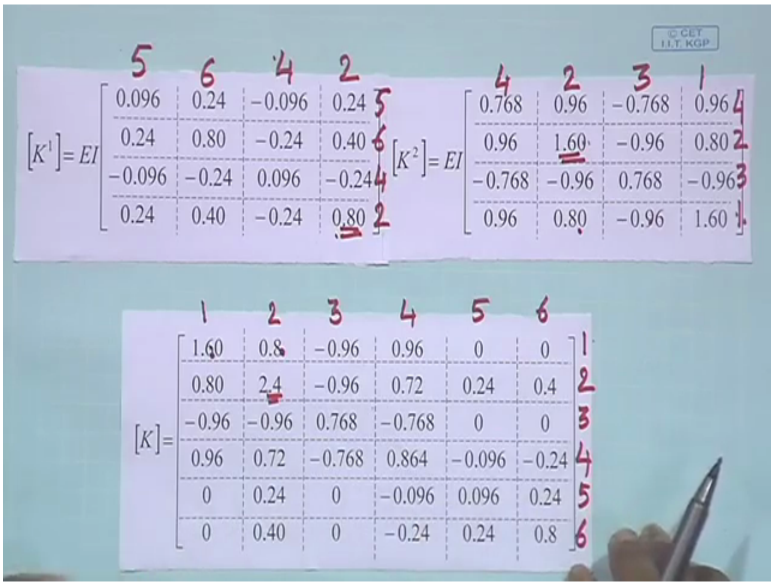
One thing is important here, you see member number 1 span between, we assume that it is ith point and this is jth point, okay. And i point degrees of freedom are 5 and 6 and at jth points

4, 2, right. So this is how, if we remember, if this is the stiffness matrix and the 1st column is the, this degrees of freedom, vertical transverse direction, the displacement in transverse direction at ith point and then 2nd is the rotation at ith point and 3rd is the transfer of direction at jth point and 4th is the rotation at jth point. Okay. And then similarly now for member number 1, member number 1, degrees of freedom are 5, 6, so this is for, this is for 5 and this is for, this is for 6.

And then for member number, joint number 2, this is for 4 and this is for 2, okay. Similarly along the row also, this is for 5, this is for 6 and this is for 4 and this is for 2, okay. Now for member number 2, member number 2 spans between joint number 2 and joint number 3, degrees of freedom at joint 2 is 4, 2 and then 3, 1. So this becomes 4, this is 2, this is 3 and this is 1. And similarly here it is for 4, it is 2, it is 3, it is 1, okay. So these 2 are member stiffness matrices. This numbering is very important because the expression is, this expression we have derived based on these, based on these arrangements of degrees of freedom, right.

If you change the degrees of freedom, the numbering of degrees of freedom, then this stiffness matrix will also change and you have to use that accordingly. Okay. Great. So we have these 2 member stiffness matrices, then next step is to assemble this, assemble these stiffness matrices. Now for each member, for each member stiffness matrix, it is, it is a 4 by 4 matrix, all are 4 by 4 symmetric matrix because for every member and we have 4 degrees of freedom, 2 translation and 2 rotations at both the ends. Now when we assemble them, total degrees of freedom we have is 6, 3 joints, per joint 2 degrees of freedom.

(Refer Slide Time: 14:43)

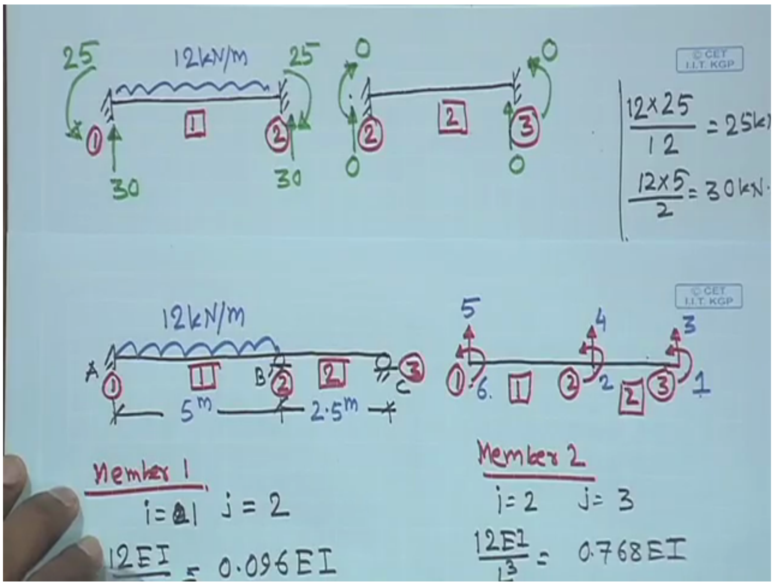


So assembled stiffness matrix, the global stiffness matrix will be 6 by 6 stiffness matrix. And how to do that, the 1st step is , okay, how to do that, let us find the assembled stiffness matrix. Now this is the assembled stiffness matrix, okay. So it is done for you, now let us, let us see how this is assembled. The numbers are, this is for 1, this is for 2, this is 2, this is 3, 4, 5 and then 6 and again this is for 1, this is 1, 2, 3, 4, 5 and then 6. Okay. Great. Now one, one will be, 1 1 point will be, one 1, 11 point, one 1 here and + 1 1 here. So 1, 1 here, so there is no 1, 1, so this should be 1 and then 1. 1, one is 1.6. So it is 1.6.

Similarly 1 2 2 will be 1, and then 2, 0.8, 0.8. 1 3 will be, 1 3 will be 1, 3 will be 1, 3 minus 0.96 - 0.96. 1, 4 will be 0.96, 0.96. Similarly 1, 5, there is no 1, 5 here, there is no 1, 5 here, so 1, 5 is 0, 1, 6 is similarly 0. It is the same way we assemble the stiffness matrix for truss, truss problem. The assembling is same but the only difference is in the truss, in the truss, only the, in the truss we had only the translation at 2 joints, here the one translation and one rotation. Okay. Now let us see what happened to member number 2, then 2, 1, 2, 1 becomes vertical, your row and column number 1 row number 2, column number 1 is this, there is no rule, rho 2 is 0.8, so this becomes 0.8.

And what is, 2, 2 column number 2 row number 2, column is 2 and row is, row is 2, 1.6 and then here also we have 2 and then 2, 0.8. So this becomes 1.6, this 1.2 to this 1.6 and then 2, 0.8. 1.6 + 0.8 and then this becomes 2.4, okay. Then similarly all other, all other stiffness, all other elements of the stiffness matrix we need to find out. Okay. So this is the global stiffness matrix for the entire, entire problem, okay. Now once we have the global stiffness matrix, next we need to find out what is the, what is the load vector, okay. Let us let us do that, okay.

(Refer Slide Time: 17:37)



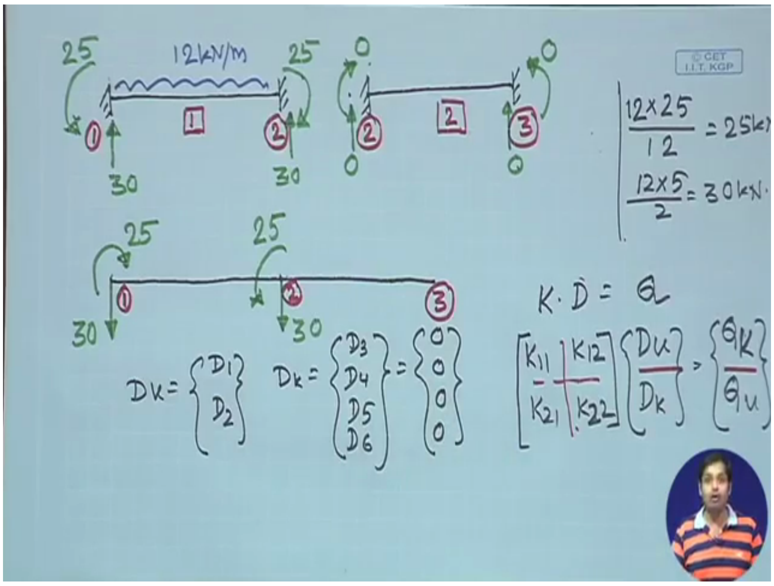
Now the finding load vectors, finding load vector is, 1st what we need to do is we need to 1st step is if this is the problem, 1st up is the need to find out what is the fixed end moment for member number 1 and member number 2. And then calculate the equivalent joint load, okay. Now let us, let us find the fixed end moments for member number 1 and member number 2. Fixed end moments are for member number 1, we assume these, these joints are fixed, okay and then member number 2, again we assume these joints are fixed, okay.

So this is member number 1 and this is member number 2. And this is, this is joint number 1, joint number 2 and this is joint number 2 and this is joint number 3. Right. Now this is subjected to, this is the load, load is, this is subjected to 12 newton uniformly distributed load of 12 kilonewtons per metre, that is there is no load on BC, 2, 3, on 2, 3 is no load. So what will be fixed end moment, fixed end moment we know that for a fixed beam subjected to the uniformly distributed load, then the fixed end moment is W L square by 12.

And the direction of this fixed end moment is, it is W L square by 12, it is, okay, here it is, it is here and then reaction at the fixed end is this and the reaction at fixed end is this. Then this value is WL square by 12, WL square by 12 becomes, this is W is 12 into L is 5, 20 5L square by, by 12, so this becomes 25 kilo newtons metre. And this reaction becomes half, total load Divided by half, so this becomes 12, 12 into total length is 5 divided by 2 is equal to 30 kilonewtons, okay. Now this value is, this value is 30, 30, I am not writing the unit here, this value is 30 and this value is 25 and this value is 25.

So this is the fixed end moment and reaction, this is the reaction at the constraint beam, assume the ends are fixed. Now similarly here also we can find out, this is this and but there is no load acting on it, so this value will be 0, this will be 0 and what happened to the reactions? Reactions this will also be 0 and, the vertical reactions and this will also be 0. So this is the fixed end, reactions for the fixed, fixed beam, okay. Now the next thing is, next thing is we need to find out the equivalent joint load.

(Refer Slide Time: 20:48)



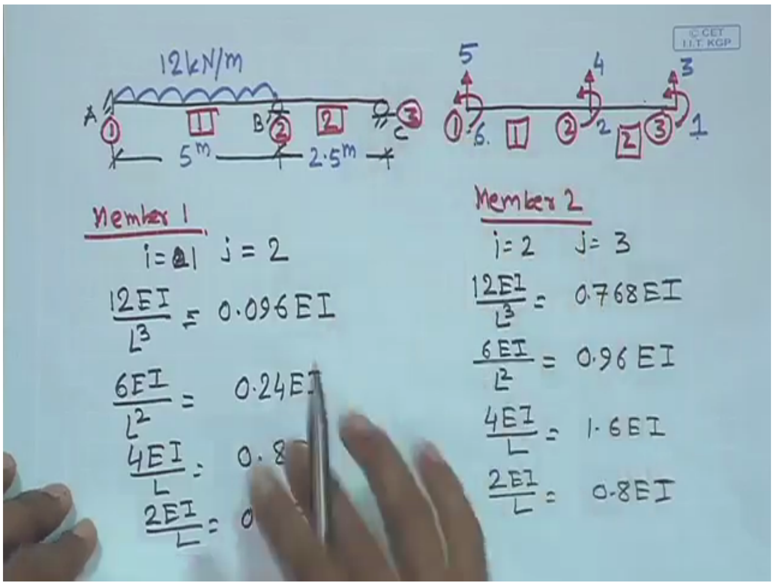
Equivalent joint load will be what, equivalent joint load is just that, we have to sum the moments and the opposite will be the equivalent joint load. Right, so to maintain the equilibrium at the joint. Now equivalent joint load will be, when you take this beam, this is, so at this point, at this point, at this point the fixed end moment is 25, which is anticlockwise and then the equivalent joint load will be, equivalent joint load will be 25 kilo newtons metre but it will be clockwise direction, okay, 25.

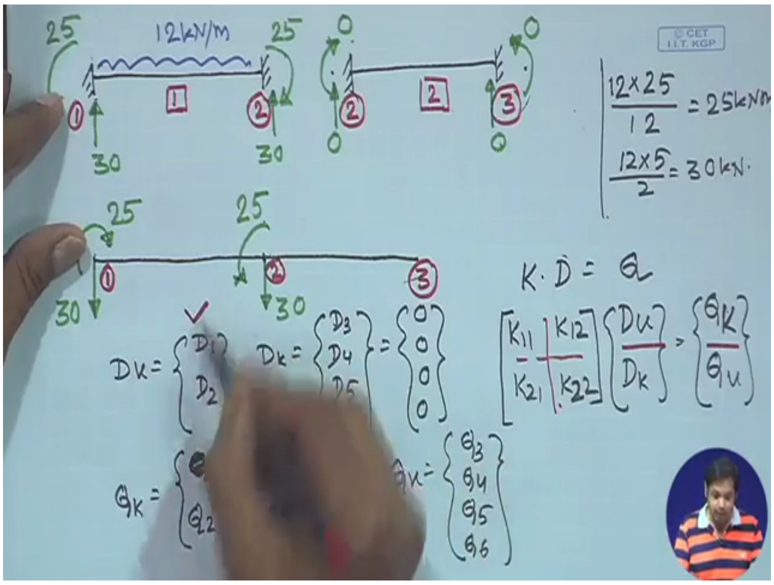
Now this is 30 kilonewtons upwards, so in this case equivalent joint load will be 30 downwards, okay. Now let us see here, here equality joint load will be, this plus minus, this the negative of this plus this and negative of this plus this. So total load here is 25 which is anticlockwise, which is clockwise then again 0, so total load is 25 clockwise. So at this point, at point 2, your equivalent joint load will be 25 but anticlockwise, okay. This is 25 and similarly it is 30 here, so this becomes 30 minus, okay. At point C, there is no equivalent joint load because this is 0, this equivalent joint load is 0, okay.

So this was node number 1, node number 2 and this is node number 3, right. So then what is the, so this is equivalent joint load, okay. Now the next part is, once we, now let us look at the structure if you remember, we know, we have obtained the load vector, we have obtained the stiffness matrix and then the stiffness matrix and load vector is related like this, K into D is equal to Q, right, K into D is Q, Q is the load vector. Now this generally we write like this, D is the displacement, the displacement vector which has displacement all the joints, so it has, K is a 6 by 6 matrix, D has 6 elements is a vector and Q is also a 6 element, is a vector. Now you see these ends are, these ends are, we know all the degrees of freedom, all the displacement at point 1, point 1 is 0.

So means D5 is 0, D6 is 0, right. And then, then we know that D4 is 0 because there is no vertical translation, only rotation can take place, so D2 is unknown. And then D3 is 0 because there is a support here and D 1 is unknown. So only unknown displacements are rotation at 1 and rotation at 2 and all other displacements, they are known and they are 0 here. Okay. So we partition this matrix like this. So this is, this is we write D unknown and D known and this is equal to Q unknown and Q unknown, Q known and Q unknown, right. And this is K 11, K 12, K 21 and K 22, this is how we can partition the matrix, right. This is how we can partition the matrix.

(Refer Slide Time: 25:33)



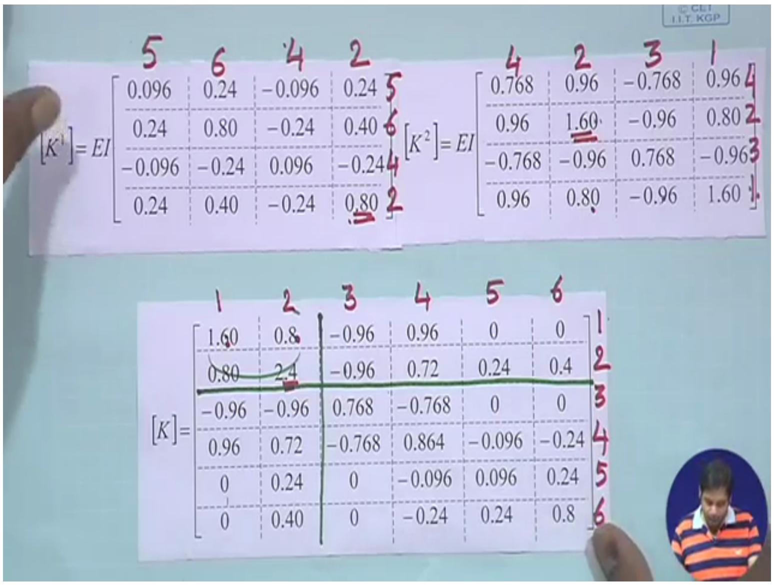


Now what is what is D, DU, unknown, unknown vector DU, unknowns are, DU is equal to, DU is equal to D 2, D1 and D2. DU is equal to, 1 is unknown and 2 is unknown, right. So D unknown is D1 and D2. Okay. And D known is, known is D3, D4, D5 and D6 but these values are all 0, all 0, right. Now let us see what happens, what about Q, K, QK is known, QK known is what, QK known is your, which is which is Q , Q1. We look at, if we look at this, then Q1, it is a roller support here, okay, extreme support, there is no externally applied load.

So we know that Q, it will be it will be 0 here which is evident from this load vector. If you see the load vector, there is this Q1, the corresponding to degrees of freedom 1, this is 0. So Q1 is 0, this is Q1 and then this will be Q2 and these are, Q1 is 0, Q1 is 0 and what about Q2, Q2 is this value, this is 25, this is 25, okay. This is load vector 25. So QK is known, and then what is Q unknown, Q unknown will be, Q unknown is equal to, it is same as we have done for, it makes up to these, up to half to getting the stiffness matrix, it is, there is a difference between different members.

But once you have this stiffness matrix, the solution of the stiffness matrix and this possibility is almost same for, irrespective of the, irrespective of the structure you have. So this is Q3, this is, these are Q4, Q5 and Q6, these are essentially reactions at various joints. Okay. But only thing reactions we know is, at this point the moment will be 0 because it is extreme end and at this point is equilibrium, so your equivalent load vector will be, will be the, will be Q2. Okay. So this is 25, now what we, what we have to do is once, we have to now find out what is the unknown displacement, this needs to be determined, okay.

(Refer Slide Time: 27:16)

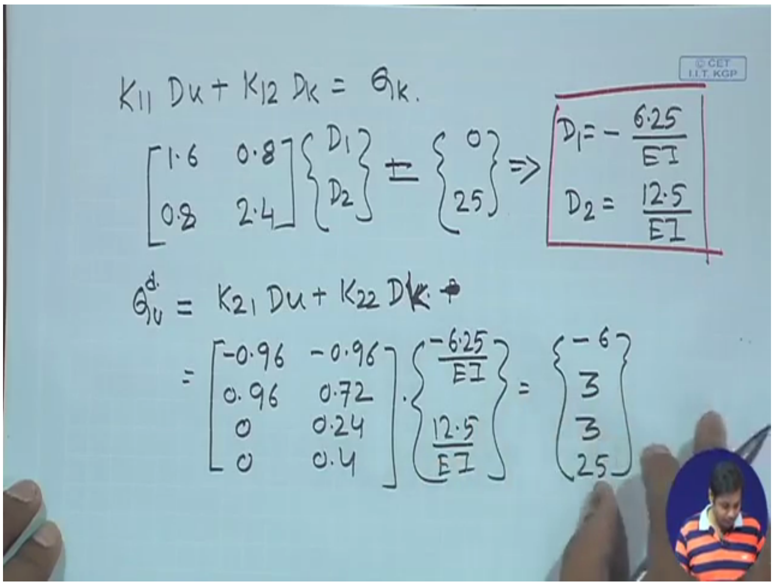


If we write the 1st part, 1st part of this equation, then what we have, we have K11, K11 into, into D, into D unknown plus plus Theta K12 into D known is equal to Q known. Okay. Now D known is 0, now what is, now let us find out what is Q11 and Q 12. That we need to find out from here, now the partition will be here is like this. Partition is this, this is the partition. This is the partition of the matrix, so this part is K 11, this part is K 12, K 21 and K 22. K 11 is a 2 by 2 matrix, K12 is 2 by 4 matrix, it is for by2 and it is 4 by 4. Okay.

So K11, K 11 is this, so let us write K 11, K 11 is, it is 1.6, then 0.8, and then 0.8 and then 2.4. And D unknown is D1 and D2 and plus K 12 is this part but K DK is 0, the known displacement is we have already, we have already seen that DK is, DK these values are 0, so essentially this becomes 0. And this is equal to, and QK equal to, we wrote QK is equal to, QK is equal to 0, 25. And then we have 0, 25, okay, 0 and 25. Now if we solve it, then we get D1 is equal to, this gives us D1 is equal to, D1 is equal to - 6.25 by EI and D2 is equal to 12.5 by EI, okay.

So this is the displacement at, rotation at C and rotation at B. Now rotations are obtained, let us now we have to find out what are the unknown reactions. So unknown reactions will be again, in order to get the unknown reactions what we have to do, we have to get the 2nd part of this equation, okay. So it will be DK21 into D unknown, K 22 into D known. So Q unknown will be, Q, Q unknown will be your, Q unknown will be, this will be , 1st is K 21 into D unknown, D unknown plus K 22 D known, this is due to the, this, now, see total, total force, total reactions will be, see this, total displacement will be displacement in this plus displacement in this, right.

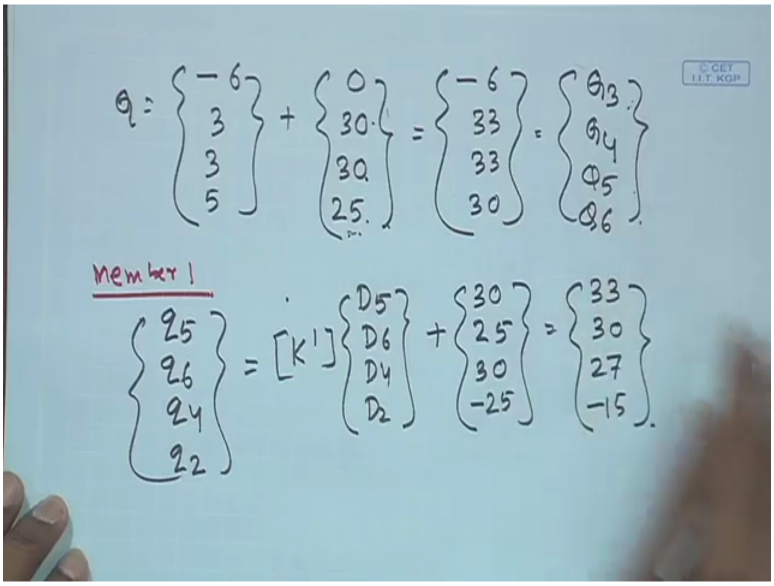
(Refer Slide Time: 29:42)



But since these ends are fixed, and no displacement here. So whatever displacement we get on this, that will be the displacement for the entire structure. Now what would be the moment, what would be the, what would be the forces, reaction forces. This will be the, due to this and due to the displacement of, due to the displacement of, displacement in this and then whatever we have, whatever we have obtained here. So now this will be, then this will be your, this will be, this is, this displacement, this will be for the displacement in the 2nd part.

And then another, then next part will be your, next part will be whatever we have obtained for this. So for displacement, let us find out 1st for displacement Q obtained for displacement. You see K 22, D, D, this is D known. Now D known is 0 here, so it becomes K 21, K 21 is this matrix, K 21 will be this part, this K 21 will be this matrix, this part is K 21. So this is, this is - 0.96, - 0.96, 0.96, 0.72, then 0, then 0.24, and then 0, 0.4. And DU will be D1 and D2, just now we have obtained this and D2 this is - 6.25 by EI and this is 12.5 by EI, right. And this gives us, this becomes minus 6, then 3, then 3 and then 25, okay.

(Refer Slide Time: 33:23)



But this, these reactions what we have, that is due to the displacement, that is due to the displacement. Now what would be total displacement then? Total displacement will be, total reaction will be this plus for this. Now let us find out this, so total Q will be, total Q will be just now be obtained, that is - 6, 3 and then 3, then 5, plus this is, if you look at this, this is 0, the 1st, the 0, then 30, then 30 and then 25. Okay. And this becomes is equal to, this is - 6, then 33, then 33 and then 30. So this is actually your Q3, Q3, Q4, Q4, Q5, Q5 and. Okay. Now you Q3 is 0 here, that is why it is, Q3 is 0 is the vertical reaction, that is why it is 0.

And then Q4, Q4 was the, Q4 was the reaction, reaction 30, that is why it is 30. And then Q5 was, Q5 was again reaction 30, and that is why it is 30 and then Q6 which was the, which was the fixed end moment at joint 1 which is 25 and that is 25, okay. So this displacement, this is the, this is the contribution from the displacement, contributions from the displacement here and this is the contribution from the fixed end. So total reactions will be, total reactions will be this, okay. This is how we can obtain the reactions.

And the last part is if you want to find out the, if you want to find out the forces in any member, says you want to find out the force in member 1, we know the element stiffness matrix and the force in member 1 will be the element stiffness matrix of member 1, element stiffness matrix in member 1 multiplied by the corresponding displacement, that would be the contribution from the displacement. And then plus there will be contribution from the, there will be contribution from this. Similarly if you want to find out the member forces in this number, what we have to do is, the member forces in this member will be, let us find out for number 1.

So member one will be, so member 1, member 1, member 1 this is the stiffness matrix, this is the stiffness matrix and corresponding degrees of freedom D5, D6, D4 and D2. So this will be, so member 4, and so element the element, member forces will be say Q5, Q6 and then you for, Q2, this is the internal force, that is the contribution from this, contribution is K1 into D5, D6, D4, D2. And then plus, plus what we have, we have this 30 at point 5, 30, D5 is 30 and this is 25 and again this is 30 and this is 30 and this is -25. Because the way this displacement, this stiffness matrix are derived is anticlockwise, the direction is taken a positive, this is a negative.

So this is the contribution from the, if we look at the individual member, this is the contribution from the displacement of the individual member, say in this case member 1, plus there will be contribution from this part which is, which we obtained by resuming the end the constraints. So these contributions are 30 and then, then 25, again 30 and then -25. And this becomes, this becomes , this becomes 33 and 30, 27 and -15. Okay. So this is the, this is the forces in member 1, okay. Now quickly, similarly you can obtain the forces in member 2 as well. Okay.

Now let us let us see what is the, whether the forces that we obtained, they satisfy the, at least this is the support, this is the support reactions we obtained, this is the support reactions we obtained, let us see whether this support reaction, they satisfy the, satisfy the equilibrium conditions or not. Our beam problem was like this, beam was like this, now so total, total reactions will be, total reactions will be reaction at 1, reaction at 2, vertical reaction at C. They should be equal to the 2 internal total applied external load which is 12 into 5 metres, 60, 60 kilo newtons, okay.

Of reaction Q1 already we have obtained, Q1 is, so this Q5 plus Q4 plus Q3 should be equal to 60. Okay. Q4 is 33 and then Q5 is 33 and then Q3 is -6, so Q3 plus Q4 Q5, total is 60 and here the externally applied load is 60. So the results that we obtained, they they satisfy the equilibrium, they have to satisfy the equilibrium but the only thing it is a just crosscheck whether while doing the calculation if you make some mistake, then this will not happen, equilibrium may not be satisfied. That, you just crosscheck for that, okay.

So this is the, this has been a very small, very, demonstration by considering a smaller problem, you can solve any number of, you can solve a beam to different kinds of loading and any number of span but the essence remains same, you have to divide it, get the element stiffness matrices, assemble them, get the global stiffness matrix and then calculate the load vector, then partition the load vector and the stiffness matrix, calculate the unknown displacement, with the unknown displacement you calculate the reactions and the member force.

The steps remain same, only difference for different problem will be depending on the number of members you have you have to compute those many stiffness matrices, element, member stiffness matrices and consequently your size of the global stiffness matrix will increase. But as far the steps are concerned, it is exactly same that we have discussed now. So this was for beam, next class we will demonstrate, demonstrate how the direct stiffness matrix, direct stiffness method can be applied to plain frame problem. Okay. We will stop here, see you in the next class, thank you.