Course on Structural Analysis 1
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Lecture 42
Module 9
Analysis of Statically Indeterminate Structures by Force Method

Hello everyone welcome to week 9, this is the first lecture of this week what we will do this week is we will apply we will see how the force method can be applied to beams to for indeterminate beams and frames. And the we have already introduced force method, method of consistence deformation and then we already have applied method of consistence deformation for trusses those therefore the concept of method of consistence deformation is already known to us.

What we will do is we will just apply that concept in beams and frames this week, okay. But before that there is very interesting theorem which is called the theorem of three moments or also called three moment equation that is again derived based on the concept of method of consistence deformation and this three moment equation is very useful for continuous beam and for the beams where the support may undergo settlement. What we will do today is we will derive this equation and then see how it can be used to determine the internal force for continuous beam which are statically indeterminate, okay.

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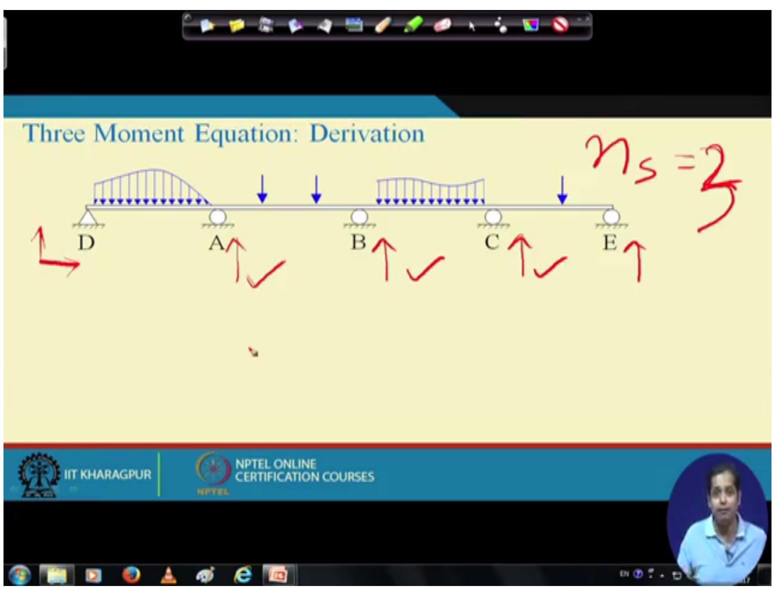

Now suppose consider a continuous beam, okay it is a four span for demonstration I have taken a four span continuous beam but it is true for any continuous beam which is subjected to some arbitrary loading, okay some external loading like this. So this is an indeterminate structure for this structure what is the degree of static indeterminacy? The static indeterminacy for this problem is 3 we have a 2 reactions at here, 1 is this and then one reaction upward By, Cy and Ey so total 5 reactions and number of equations available 3 so ns is equal to 2 so it is ns is equal to 3 ns is equal to 3 total 6 reactions and 3 equations available, okay.

Now, then suppose since it is it has 3 static indeterminacy is 3 then we need to in order to solve this problem we need to analyze this problem we have we need to identify 3 redundant forces, okay or moments we need to identify 3 redundants, okay. Identifying redundants means then we identify force or moments and then in indeterminate structure we release the associated constraint and once we release the associated constraint then this structure becomes determinate, okay.

So after releasing the associate constraint the structure what we get that is called primary structure, okay. Then we need to as method of consistence deformation we know demonstrated in the previous week then we need to analyze the primary structure subjected to the external load and then primary structure subjected to the redundant force get the corresponding displacement apply the compatibility conditions written in a flexibility form but and then solve it get the redundant forces, okay.

Now one possibility in this case is choose this A Ay one possibility is choose Ay, By and Cy as redundant, okay. Now if you do that then the structure becomes statically determinate structure but we do not do that as the equation of three moment suggest that this equation actually gives how the moments are related to each other at different points. So what we do here is we take the moment at A, moment at B and moment at C as redundant. Now if we take moment at A, B, C are redundant then what we need to do is we need to release the corresponding constraint and what is the corresponding constraint for moment is the rotation. So in this so we need to provide internal hinges at point A, at point B and at point C.

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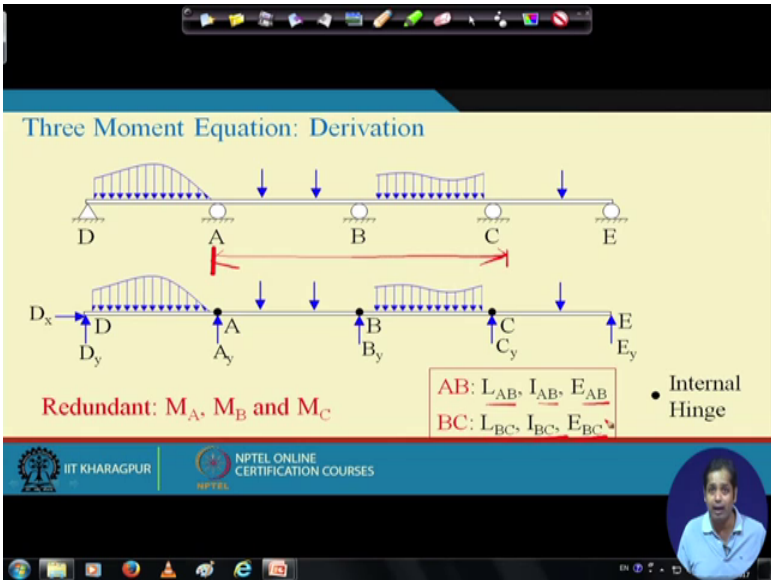

So if you do that then what happens so this is the primary structure once again shown all the reactions are shown here and then if we insert 3 hinges, 1 is at A this one, and this one and this one. If we insert 3 hinges then what happen they essentially then our redundants are these are internal hinges then our redundants are Ma, Mb, Mc.

Now so if you take Ma, Mb, Mc as redundant and provide and release the corresponding constraint now once we have introduced hinge at A means what the moment at A will be in the primary structure the moment at A will be 0 and then moment at B will be 0, moment at C will be 0 that what is the primary structure, right? Now what happens now this structure is statically determinate structure, okay. Why statically determinate structure? Just by looking at this you can say we need 6 equations because there are 6 unknowns, then 1, 3 is equilibrium conditions and 3 additional constraints are moment at A is equal to 0, moment at B and moment at C is equal to 0 and these 6 equation we can solve for the unknown so these are statically determinate structure, okay.

Now next now suppose what happens suppose let us make let us derive the expression very general suppose consider this span this consider this span entire span A, B and C this part, okay. And suppose length of AB is L length AB and then second moment of area is Iab and young's modulus of AB is Eab. Similarly the Lbc, Ibc and Ebc are the length second moment of area and young's modulus of BC and, okay. Now so there is no restriction that the beam has to be of same beam has to have the same second moment of area same young's modulus, so there is no such restriction when we have this general expression, okay. Now what happens consider only this part this span A, B, C, okay.

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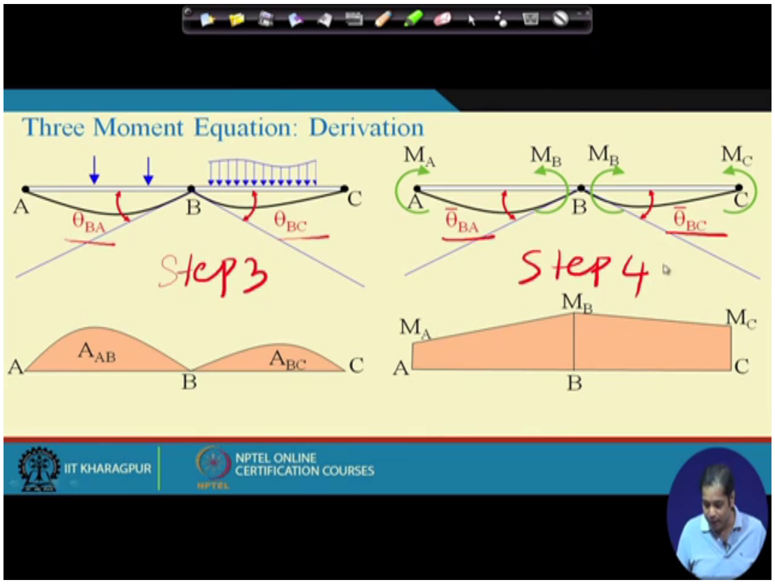

Now if we take only this span A, B, C this is the primary structure for primary structure means if we only consider this part of the part A, B, C of the entire structure. So this is the primary structure and this primary structure is subjected to external load, okay. Now due to this external load suppose the beam deflect like this, okay this is a deform shape of the beam.

And then corresponding slope at B is 3 B theta BA and corresponding slope at B due to in segment BC is theta BC. So theta BA means that slope at B in member AB in segment AB or in span AB and when we say that theta BC it means that slope at B in span BC, okay. Now again so this is the primary structure if you remember the steps we discussed in the method of consistent deformation the first step is the determination of static indeterminacy we have already done in this case we are seen it is ns is equal to 3.

And the second step is identification of the choice of the redundant forces we have chosen Ma, Mb and Mc as redundant forces and the step 3 is solve the primary structure subjected to external load and the step 4 is the solve the analyze the primary structure which are determinate structures anyway analyze the primary structure subjected to the redundant force, okay. Now this is the primary structure subjected to external load step 3 and then this is the primary structure subjected to the redundant forces. Redundant forces are moments at A and moments at B and moments at C, okay. Now suppose under the action of this redundant force redundant forces the beam deflect like this and the slope is denoted by theta bar. So theta bar BA means again slope at B in segment AB and that theta and this slope is due to the in the primary structure due to the redundant forces.

Similarly theta bar BC is again slope at B in segment BC when the primary structure is subjected to redundant force, okay. Now then suppose the moment bending moment diagram for this is this you have taken a very arbitrary shape this is the bending moment diagram for span AB and span BC. And similarly this is the bending moment diagram for the primary structure subjected to redundant force. This is very obvious because you see other than moment there is no other loads here, so at C the moment will be MC and at B in this case the moment will be MB and between B and C since there is no other intermediate load the variation will be linear.

So MB and MC the variation is linear, similarly for AB at B the moment is MB and at A moment is MA and the variation is linear. So this is the bending moment diagram for this part, okay. Now so what we will do is we will calculate theta bar theta BA, theta BC this are the corresponding displacement, right? This is step 3 if you remember step 3 and this is step 4 in method of consistent deformation, okay.

Now step 5 is application of the step 5 is the compatibility condition. Now compatibility condition means its is it how the associated deformations are compatible to it related each other should be related each other in the actual structure. Now since our constraints are moment here redundants are moments here the associated displacement are rotation. So what we will do next is we will determine what is theta BA and theta BC and in the primary structure subjected to external load then theta bar BA and theta bar BC the primary structure subjected to the redundant force and then apply the compatibility condition which says that how theta BA, theta BC, theta bar BA and theta bar BC should be related each other in the actual structure, okay.

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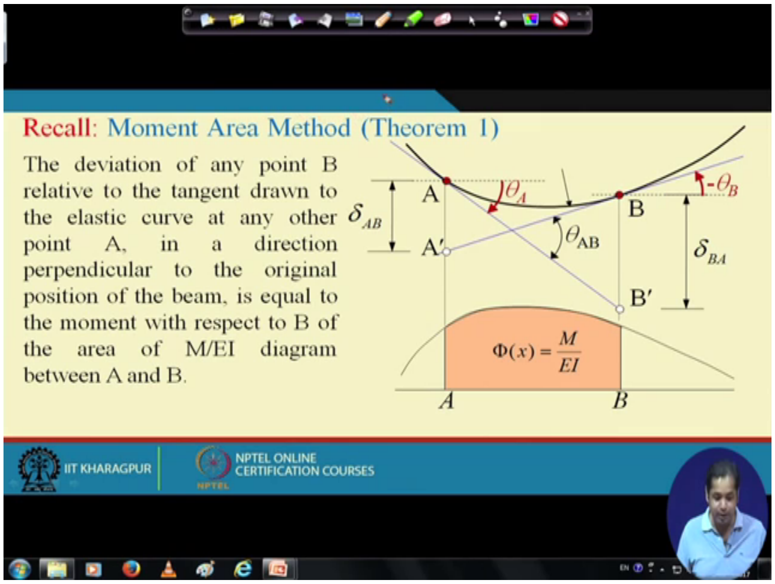

Now, so let us see how to do that now before that let us recall the moment area method we will be using moment area method what it says, it says that moment area method has 2 theorem, okay and it says that suppose if this is elastic curve or elastic line this black blind that you can see this is a deform shape on a beam. Now you take any point A and B now draw a tangent at A and draw tangent at B and angle between these two tangent is theta AB and what this is the bending moment diagram between A and B.

Now what the what moment area method says? Moment area method has 2 theorem the theorem 1 says that this theta AB is essentially the area of this bending moment diagram, bending moment diagram means the area of M M by EI diagram, okay. And then what theorem 2 says is if you take B is the actual point in the beam and B dash is the deviation of or the projection of B on slope drawn at A then delta AB delta BA which is the deviation of B on slope drawn at A is essentially the moment of these diagram about B, okay.

This is actually moment this is theorem 2 this is not theorem 1, okay that is moment area method. So we have already discussed moment area method in details what we will do is we need to find out as we have seen in the previous as I said in the previous slide that we need to find out this theta BA, theta BC, theta bar BA and theta bar BC these angles or the slopes will determine using moment area method, okay. Now this is the moment diagram for this case and this is the moment diagram for this case.

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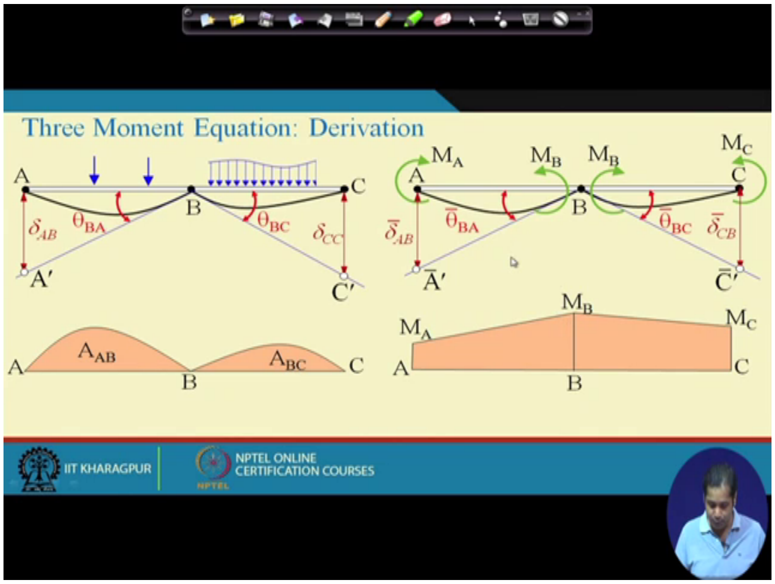

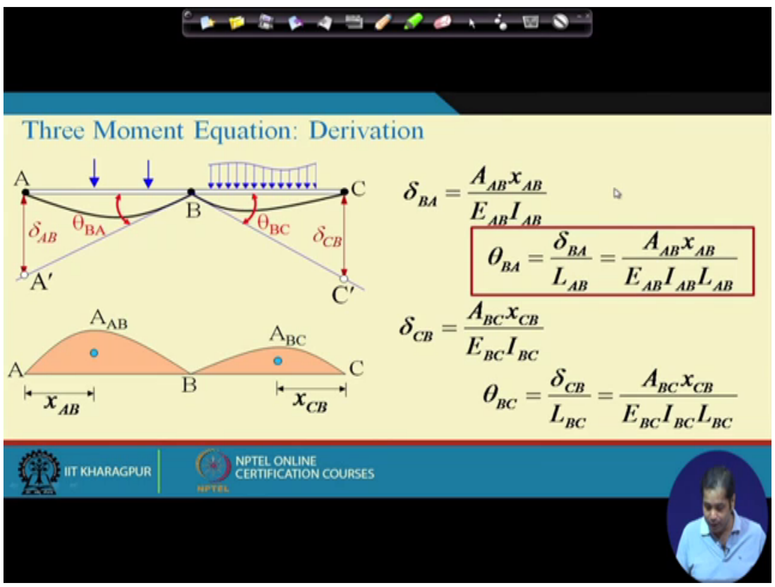

So let us do that, now suppose the deviation of point A is A bar A dash and this distance is delta AB and similarly deviation of point C is C dash and this distance is delta CC delta CB it should be B this should be delta CB, okay. Now means it is deviation of C on slope drawn at B, now similarly we have delta AB delta bar AB here and delta bar CB here. So what so delta AB will be the area of this bending moment diagram from A moment of this bending moment diagram from A and delta CB will be moment of this diagram from C.

Similarly, delta bar AB will be moment of this diagram from A and delta C bar will be moment of this diagram from A. And suppose let us first consider the first primary structure subjected to external load then this is say Xab which is the centroid of this and this is Xcb distance from C of the centroid of this part and suppose Aab is the area of this diagram and Abc is the area of B. So area of this diagram, okay.

Now then moment area method says that delta BA, delta BA will be moment of this area about point A, so moment of this area point A will be and divided by EI. So moment of this area would be Aab into the centroid distance of centroid divided by E and I of segment AB. Now once we know this distance and if we assume that this displacement is small that is what our assumption right from the beginning, then this slope can be written from this displacement as this, okay.

So this angle is we can write this divided by this length, okay. So delta BA divided by Lab, now delta BA we have just know found this using moment area method, so theta BA becomes this. Now similarly delta CB will be moment of this about point C divided by EI moment of this will be area of this multiplied by centroidal distance area of this multiplied by the centroidal distance divided by EI, EI in this case it is EI of BC and then again assume that this angle is very this displacement is very small we can write theta is equal to delta this distance divided by L and finally we have theta BC is equal to this, okay.

So we have already determined what is theta BA and theta BC. Now next we need to determine what is theta bar BA and what is theta bar BC which are this corresponding slopes in the primary structure due to the redundant forces, okay.

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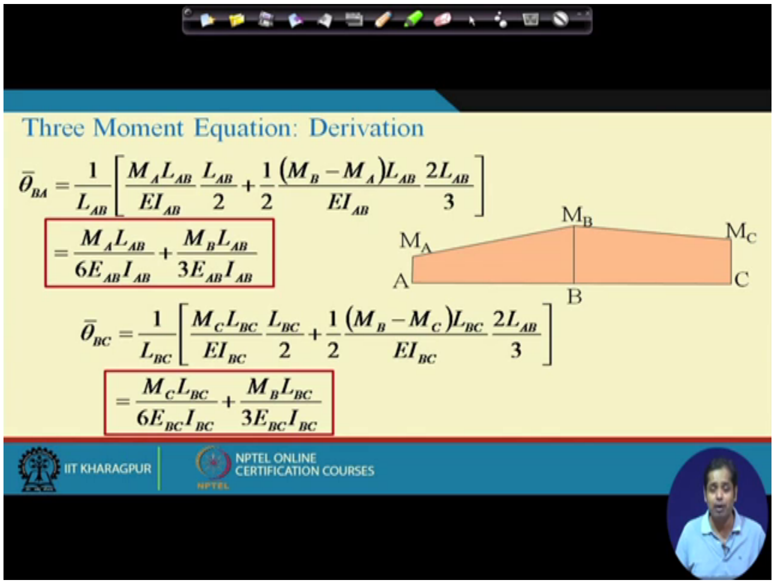

Let us do that, so we have determined this, okay so this is the bending moment diagram in the primary structure which is subjected to redundant force we have just now seen it.

Now in this is bit simpler because in both the any segment you take this bending moment diagram is a trapezoidal, okay. So we can determine this the moment of this bending moment diagram very easily. So theta bar BA will be again moment of this diagram about point A and if you do that for any trapezium and this will be the expression finally expression and divided by EI. Similarly theta BC will be moment of this diagram about point C divided by EI and then again it is trapezium so finally what we get theta bar BC is equal to this. Now we have obtained what is theta bar BA and what is theta bar BC, great. So step 3 and 4 are done, next step is to apply the compatibility condition, now what is compatibility condition in this case?

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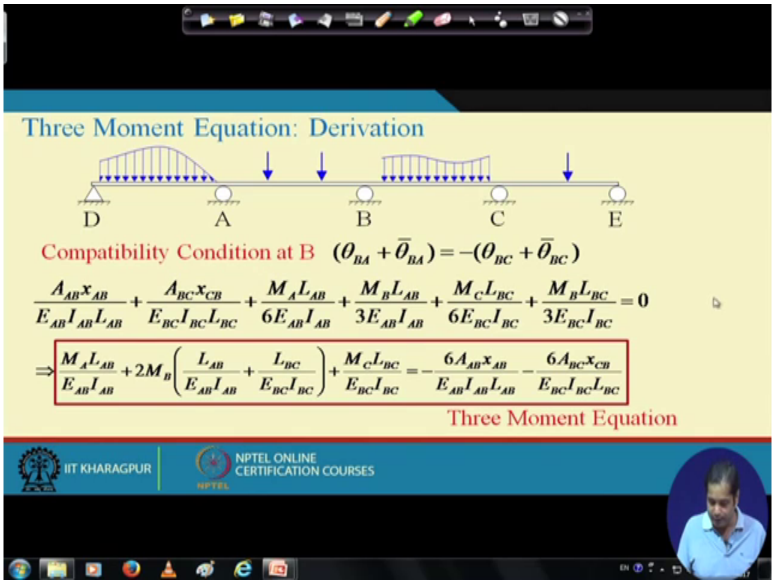

You see now compatibility condition is compatibility condition will be you see suppose take any continuous beam say in this case it is A, B, C for instance suppose this deflect like this so under the any action of load, okay and we draw a slope like this, so this is A, B, C, right? This is A, B, C so this is the deflected shape and this is the slope drawn.

So what is this angle? This angle will be theta this angle is theta BA, okay. And then what is this angle? This angle will be again theta BC, okay. Or now or you can take a deflected shape like this, okay let it be like this, so theta BA and theta BC. Now what compatibility say is since the slope needs to be continuous at that point, otherwise the governing equation itself is not varies so this should be equal to minus of this, the minus because how the slope is being calculated, okay.

So this becomes so theta BA is minus of theta BC or we can say that theta that is equal to minus of this that is the compatibility condition or we can say that this plus this is equal to 0. Now so in this case also the compatibility condition will be same if we apply the compatibility condition at B, then it says that theta BA is the, so what is the total rotation at point B? At point B total rotation is rotation due to the in the primary structure due to the external applied load and rotation due to the redundant forces this is the total rotation.

And similarly what is the rotation at B in segment BC? This is due to the externally applied load and plus the redundant forces, okay. And the compatibility condition just now we should that the total rotation at B in segment AB should be equal to total rotation at B in segment AC negative of that, okay. Now we have already obtained the expression of each all these expression just now we have obtained just substitute those expression in these equation and if you do that, then our final expression will be this.

And this is now we just slightly if we manipulate it, then this is the expression and this expression is called, now what this expression tells us? Now in this expression all this values are known what is the length of AB and then length of AB this value, then value is known because this is the length and second moment of area length and second moment of area this are also known, this are also known, this are also known, this are also known, this are also known, only unknown in this equations are Ma, Mb and Mc this Ma, Mb and Mc, okay.

So and right hand side is known. So what this equation tells you this equation tells you how this 3 moments at moment at A, moment at B, moment at C if you take two conjugative span and how the moments at two three conjugative supports they are related to each other and this equation gives you that relation and therefore this equation is called three moment equation means how these three moments, moment at B, moment at A, moments at B, moment at C are related to each other, Now if you do not take now we have obtained this expression because we took span AB and BC. Now instead of that if you take span DA and AB, then what this equation will give us? This equation will give us what is so moment at D is Md, moment at M is Ma and moment at B is Mb.

So if we apply three moment equation at point A the compatibility condition at point A and get the three moment equation for span DA and AB that moment equation will give you how the moment at D, moment at A and moment at B they are related to each other. Similarly if we take span BC and CF, then this gives us how moment at B, moment at C and moment at E are related to each other. Now you see in this this is the essentially the method of consistent deformation and but written in a different form.

Now see in this case how many equations how many unknowns we have? We have total 6 unknowns, 2 reactions same and three all four reactions total 6 unknown and how many equations available equilibrium equations available is 3, okay. Now we need 3 additional equations and what are those 3 additional equations if we forget about three moment equation if we apply the method of consistent deformation this three additional equations will be the compatibility condition at A, compatibility condition at B, compatibility condition at C, right? Now this three moment equations are essentially the compatibility condition at different supports but written in a different form in a very general form, okay.

So you apply get the three moment equation at A, at B and at C you have three additional equations so total 6 equations the you can solve this structure. So it is essentially the method of consistent deformation and this step 5 where you from step 1 to step 4 they are exactly same the step 5 when you apply the compatibility equation this compatibility equation is essentially the three moment equation but it is written in a different form in a very general form, okay.

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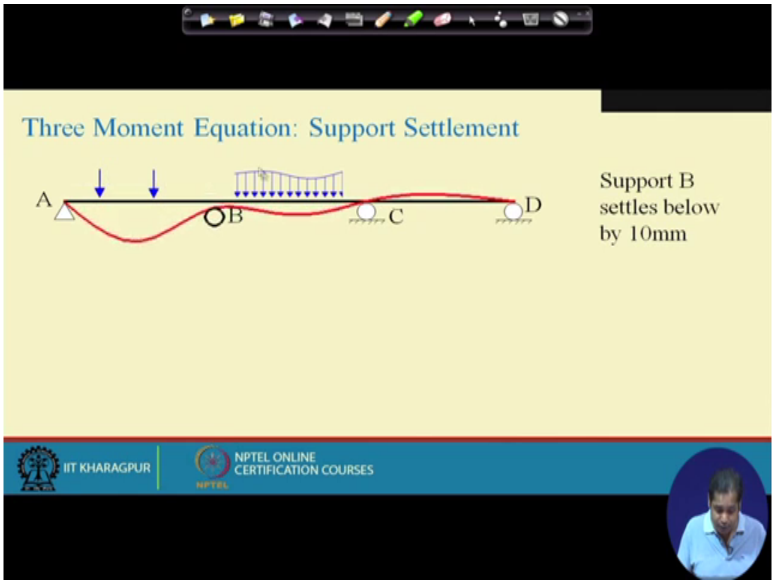

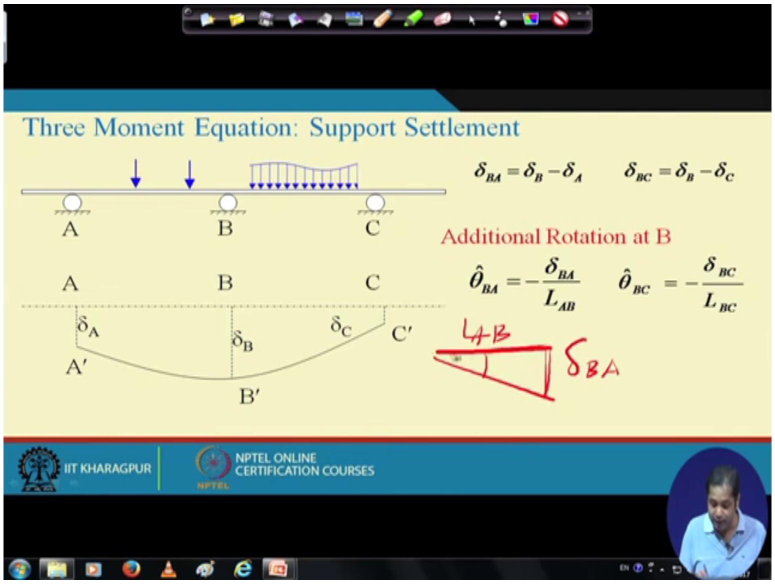

Now so this is three moment equation we are not going to apply this three moment equation today, we will do it in the next class but let us see one more cases as I say it at the beginning that the continuous beam and also the beam where the support may settle in that case also we can three moment equation gives us a very handy expression, okay. Suppose again consider the same beam subjected to same external loading. Now what happens suppose support B at support B settles.

Now if support B settles then what happens because it settles and B point B goes to B dash and this and this settlement is delta B, okay and it may happen I mean if you see any structure sometimes the foundation may settle and the structure get tilted and there are many kinds many practical structures are like this. Now what happens if the support is settle, then because of this settlement you get the internal forces because of the external hesitation external forces, right?

But in addition to the external addition to the effect of external forces if the support settlement takes place, then this support settlement will also induce some internal forces in there beam. For instance you take a beam or any structure which is subjected to external load, right? Now that external load we will induce some traces in this structure. Now in addition to the those external load written is placed in distributed or concentrated moment and forces.

If the temperature change takes place, okay if the temperature increase or temperature decrease. So effect of temperature if you have to consider, then what happens then the that temperature in addition to the effect of those external forces temperature will induce some additional traces, okay. So total traces will be in any structure is the traces due to the external load plus the traces due to the temperature variation very similar to that.

In this case it is the additional traces are developed in the beam due to the settlement of this support. So what we will do is we will see then how this effect of support settlement can be taken care of in the expression that we derived just now, okay. Now you see suppose this is the beam if support settle be just an animation how this deform shape may look like we will consider this problem in the next class and analyze this.

Now you can see this this total deform shape that you can see that deform shape is the contribution from the external load plus contribution from the settlement of the support. What we are going to do is we are going to find out that contribution and add to the moment equation that we derived just now, okay. Now suppose consider again span A, B, C suppose again it is subjected to the external load suppose again let us derive a very general expression where all the supports where we considered this settlements of these supports A, B, C but in a structure there values may be 0, but the expression we will derive is a very general.

Suppose support A settles by delta A, support B settles by delta B and support C settle by delta C and the corresponding position of point A, B, C are A dash, B dash and C dash, okay Now suppose delta BA is equal to delta B minus delta A so delta BA essentially is the relative settlement of point B. So settlement of point B relative to A. Similarly delta BC is the delta B minus delta C it is settlement of support B relative to support C, okay settlement of support C.

Now because of the supports settlements what happens your when the structure is subjected to external load it deflects some deflected shape at every joints you have some rotation, right? Now what happens because of the settlement of the support the support settlement will also contribute to those rotation, so you get in addition to the rotation due to the external load you get rotation at the joints due to this settlement or support. And what are those additional rotation? Additional rotations are you see if delta B delta BA is the relative displacement or relative settlement of B relative to A, then the rotation takes place at B will be delta BA by Lab, Lab is the length of AB, okay.

And similarly if delta BC is the settlement of B relative to C and it will cause rotation which is very obvious because if it is this additional support is now if we take if it is Lab and this is delta

BA so delta B will cause additional rotation at point B this the additional rotation at that point and this additional rotation will be this divided by this length, okay. Now so this is similarly delta BC, now remember we are use three this is expressed as theta hat, okay.

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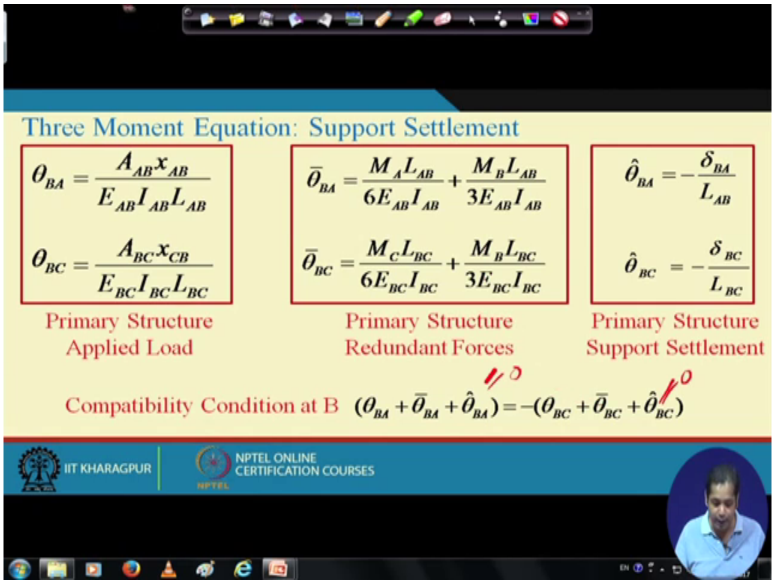

Now what we have right now is this, now this is so now we have we after introducing the redundant forces associated releases in the structure what we have is we have primary structure subjected to external load and because of this external load this primary structure undergoes some deformation or rotation at the joints.

Then we have primary structure subjected to the redundant force and because of the action of the redundant force this primary structure subjected to undergo some deformation and joints are and the rotation of the joints. And then in addition to that we have support settlements in the primary structure and then because of this primary structure the because of this support settlement the primary structure undergoes deformation and rotation at the joints.

So total rotations will be rotations from the effect of the in the primary structure the total rotation will be the contribution from the external load plus from the redundant force plus contribution from the support settlement, okay. Now we have derived it is the contribution from the external load and we have also derived this is the contribution from the redundant forces and then this is the contribution from the support settlement.

Then we apply the compatibility condition, what is the compatibility condition at B? Compatibility condition this is the total rotation at point B in AB this is the total rotation at point B in BC and this rotation will be equal to that total rotation will be equal to 0 where this is equal to minus of this. In the previous case if there is no such support settlement this become 0 this part become 0 if there is no support settlement, okay. And then the compatibility condition will give you the expression that we just know derived, okay.

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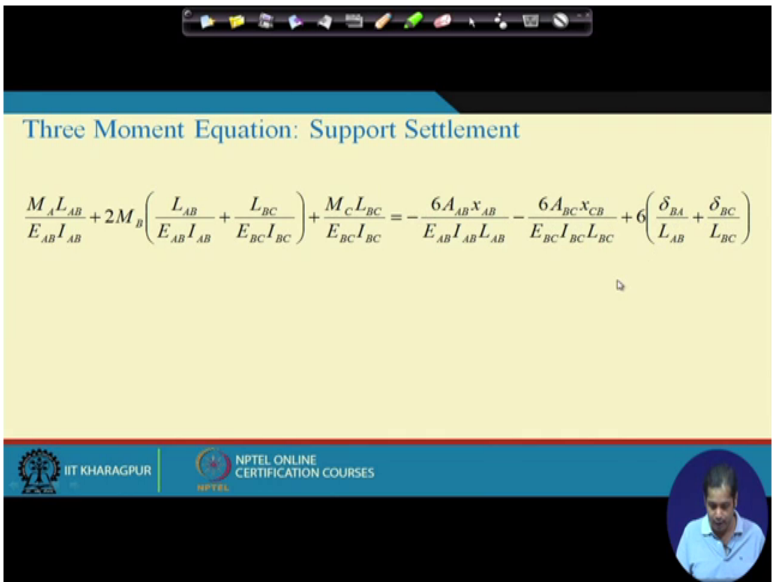

Now if we substitute the values of all these values of theta in this expression what we get is this, okay that you can easily check, okay. So in the previous example this part was not there because delta BA and delta BC where 0 if there is no settlement of support, okay. Now so this is the general expression for three moment equation, okay. So this is the contribution from these are the up to this is the contribution from only the external load and this is the contribution from the support settlement, okay.

Now in the general expression get some special cases first you see in this case we have assume all are different length of different segments are different, young's modulus, then moment of area those are different then settlement at different joints are different so this is the general expression, from this general expression we have some special cases the first special case is if you take EI is constant so if you take EI is constant so if you take EI constant then all these EI we can remove, okay.

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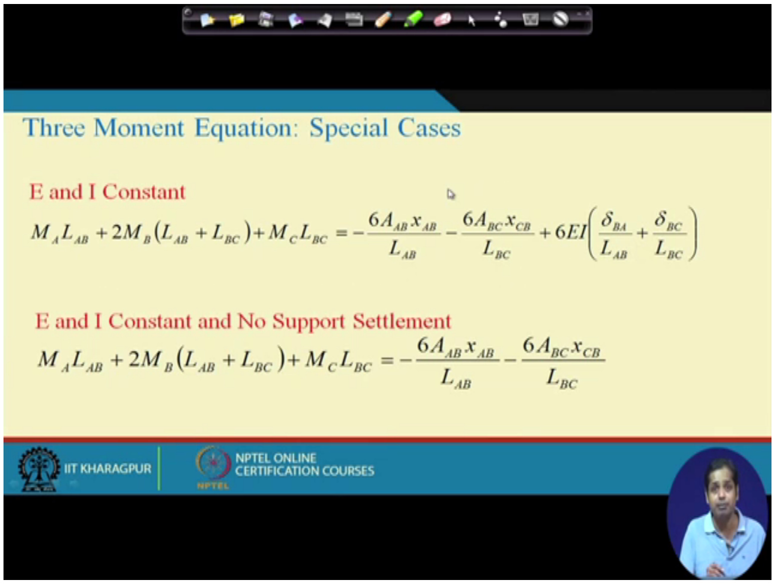

Then if we have EI constant, then your expression becomes this. Now in addition to that EI is constant and then if there is no support settlement then this part become 0, this part become 0 and your expression becomes this, this are very simple expression, okay. So any problem if you find that EI is flexural rigidity is constant and there is no support settlement so you can start with this expression if you find that EI is constant but in addition to that there is support settlement then you can start with this expression and if you find that EI is not constant and then the support is also support settlement also takes place then you start with this general expression, okay.

Now we apply this at different joints get the corresponding equation and solve them to get the unknown redundant forces in this case unknown redundant forces are moments, okay. So this is three moment equation which is very useful for continuous beam and beam which where the support undergoes settlement what we will do is the main purpose of today's lecture has been to derive this equation, now we will see how this equation can be used to find out internal forces for continuous beam that we will be doing in the next lecture so next lecture will be three moment equation some examples, okay see you in the next class, thank you.