Structural Analysis 1
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Lecture 14
Analysis of Statically Determinate Structures: Method of Virtual Work (Contd.)

Hello everyone. What we are going to do today is today we want to see what is principle of virtual force and which will give us a method which is called unit load method, okay.

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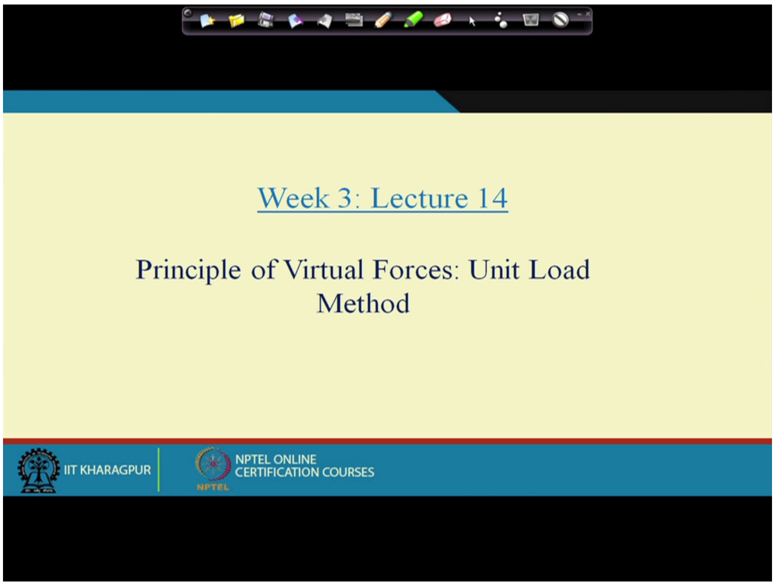

Now this is the summary of that we have discussed so far in last two lectures. We have discussed the principle of virtual work for rigid bodies and essentially principle of virtual displacements. Then we also discussed we need to go beyond this principle to analyse the kind of structures and the deformable body. And then we need to understand what is virtual force and then what is external and internal virtual work and that is the thing that we will do today, okay.

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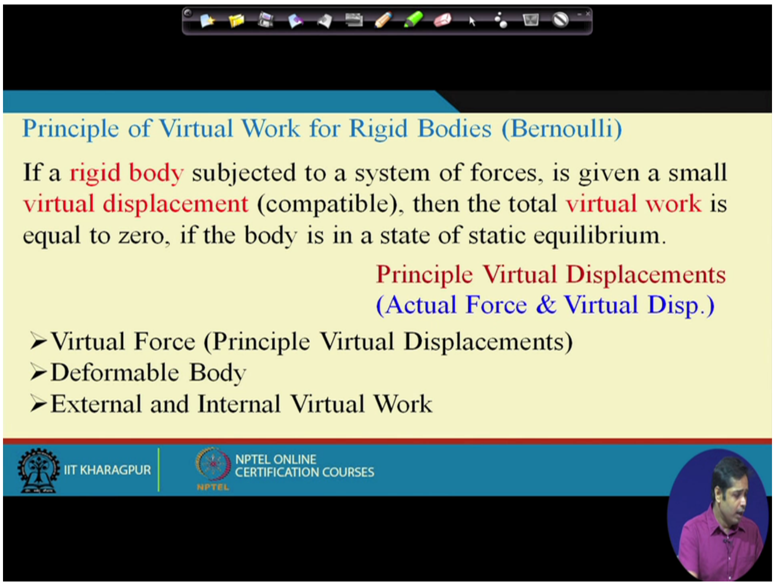

Now let us first understand what is virtual force and the associated virtual work, okay. Now suppose again consider the same object, then suppose for some load the object undergoes deformation like this and this deformation is the actual deformation.

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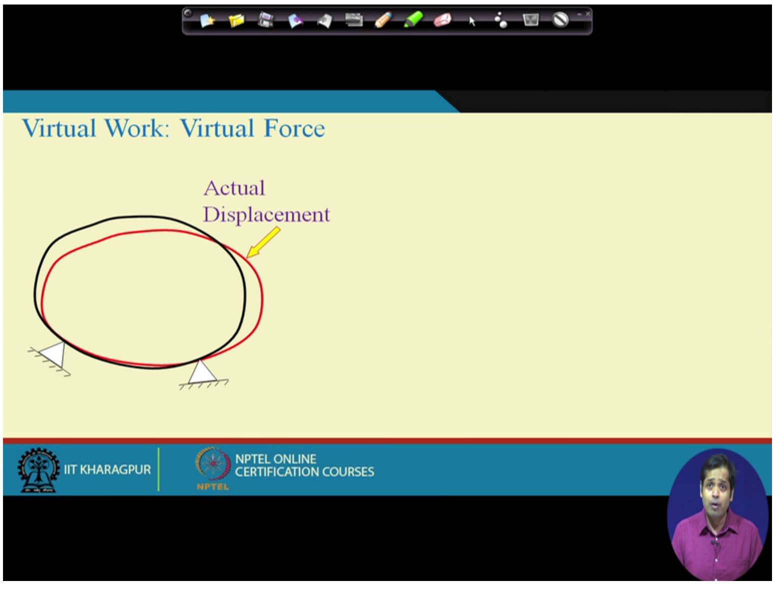

If you remember when we demonstrated the virtual work through virtual displacement and the load acting on the structure was actual load. Now suppose for some load the displacement of the structure of this object is this.

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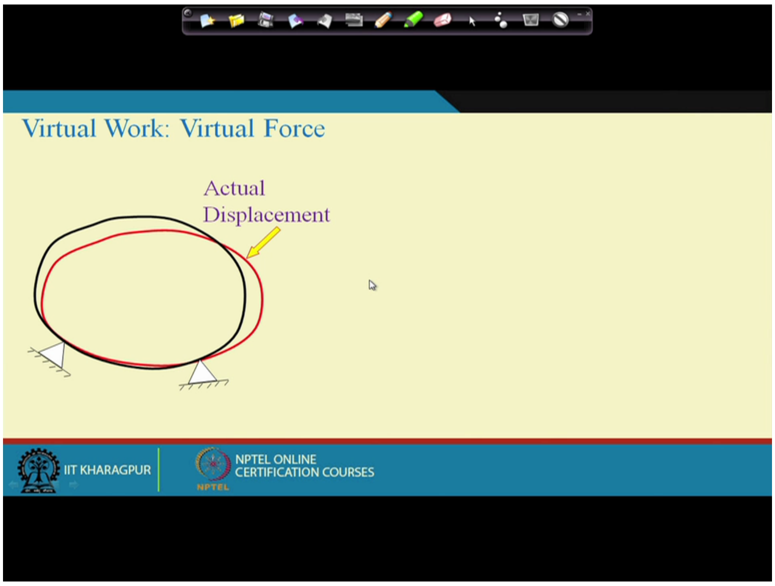

Now let us apply some virtual forces on this object. These are some virtual forces. Now remember one thing again, similar to the previous case where the actual force and the virtual displacement they do not have any cause of any relationship. The virtual displacements were not due to the actual forces, okay. Similarly here this actual displacement and this virtual load they do also not have any cause effect in relation.

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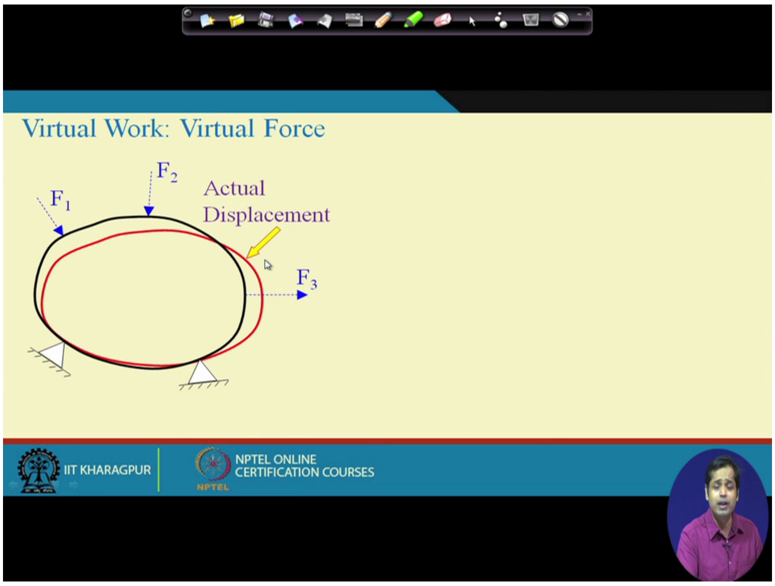

They have a similarity in a way they have the same sense of direction but the actual displacement is not due to this virtual loads, okay. Now again considerate D1, D2, D3 are the associated displacement at the point of action of F1, F2, F3, the virtual forces. Then work done is equal to, this will be D1, okay, please correct it. This will be D1, this will be D2 and this will be D3.

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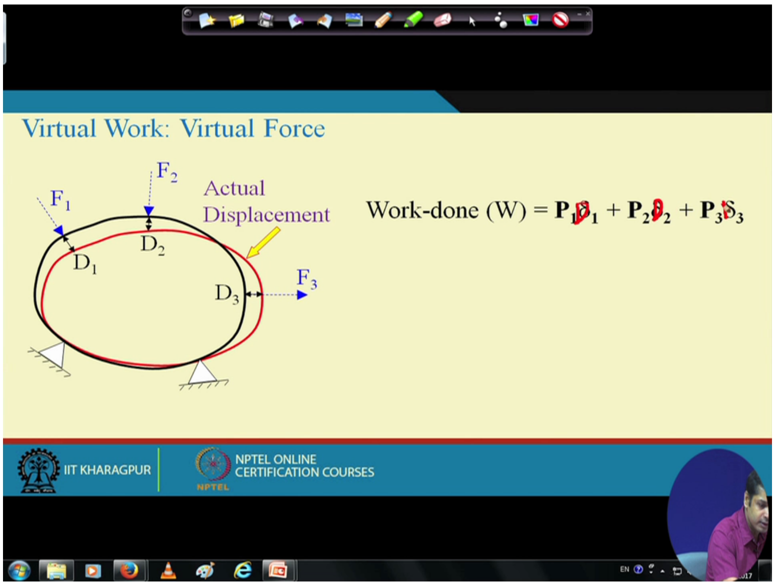

The work done will be F1, F2, F3. Again this is F1, this is F2 and this is F3, okay.

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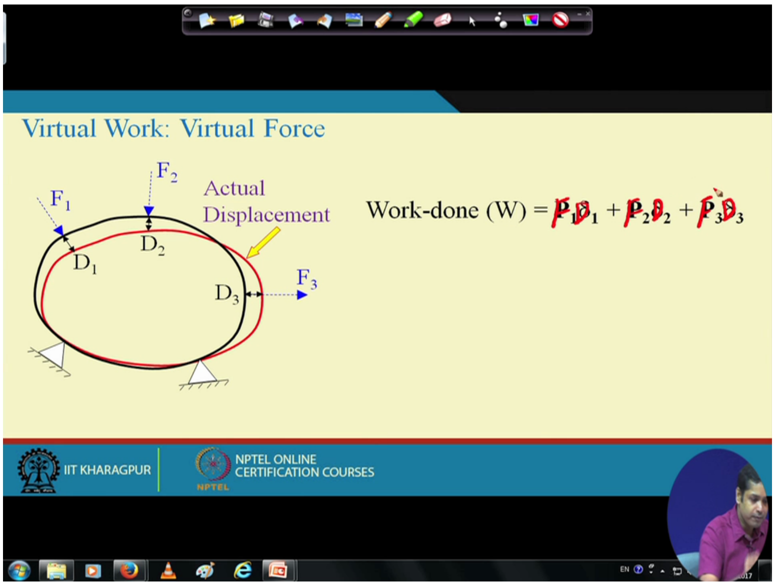

Now this is the work done. Now then virtual work here is the work done by the virtual forces. The forces are virtual acting on the body which moving through actual displacement, okay.

It is just the opposite. The difference between the virtual work due to the virtual displacement were the forces were actual and the body undergoes that displacement was virtual but here the forces are virtual but the body is moving through the actual displacement, okay. Now this is virtual work.

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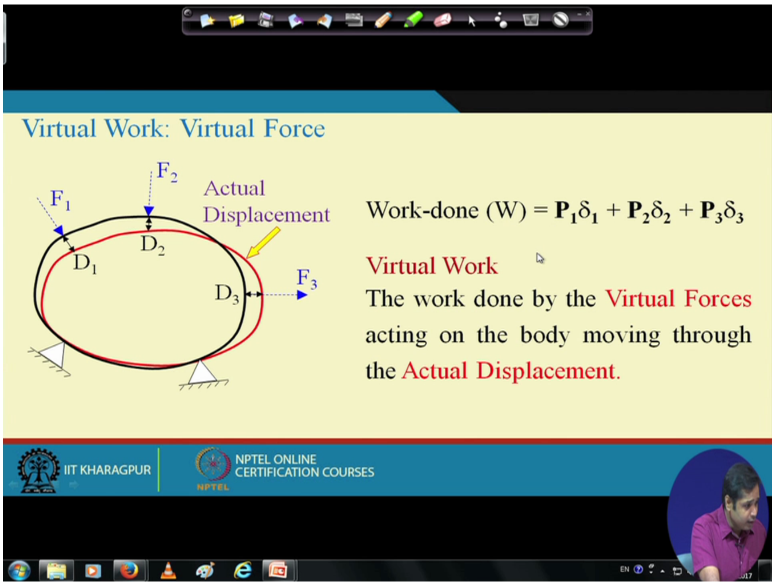

Now again in the previous case the force displacement field was conjugate to the force field but in this case the force field is conjugate to the displacement field, okay. But again this displacement field is not the effect of this force or this force field is not the cause of the displacement. They are totally independent, okay.

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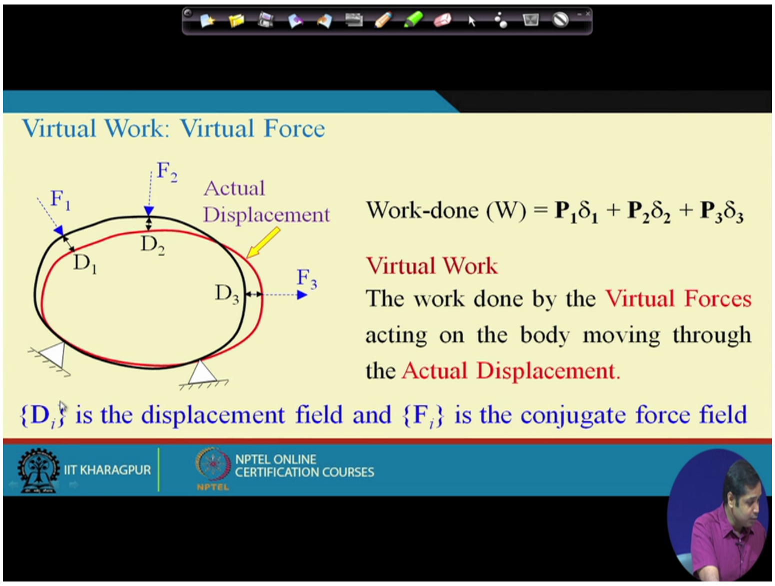

And this (displa) only thing is the force field should satisfy the static admissibility. Static admissibility means they should satisfy the (compatibi) equilibrium condition. If you remember we discussed what is static admissibility? And the displacement field they should be compatible. They should (consi) satisfy the internal compatibility of the object, okay.

Now then the principle of virtual work for deformable bodies it says that external virtual work is equal to the internal virtual work. What is external virtual work? Of course the virtual work is due to the external forces and the internal forces due to the external forces. Internal forces develop in the structure in the deformable body and associated virtual work is the internal virtual work, okay. So now work done by external forces and then work done by internal forces.

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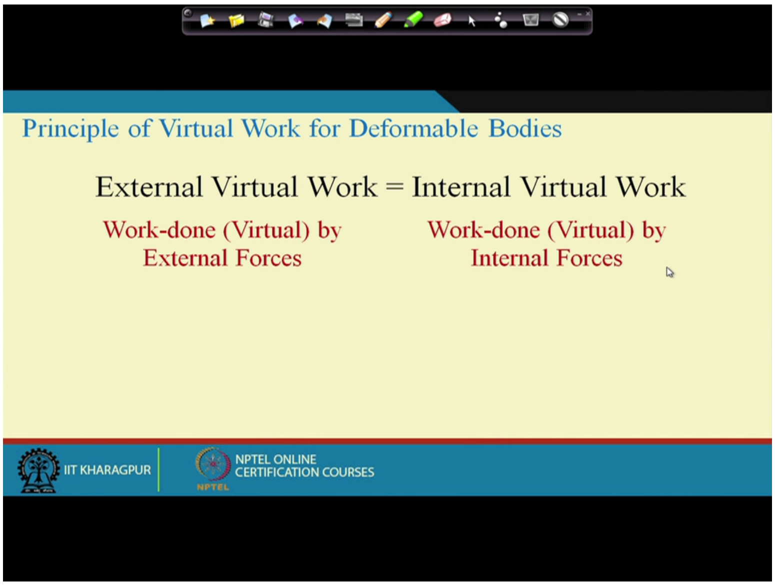

Then whenever we say the work done two important things are involved. One is the force and the displacement and then the associated force field is called as external force and suppose this is the associated displacement field which is D external and the force is F external. And similarly this is the internal force and this is the associated displacement.

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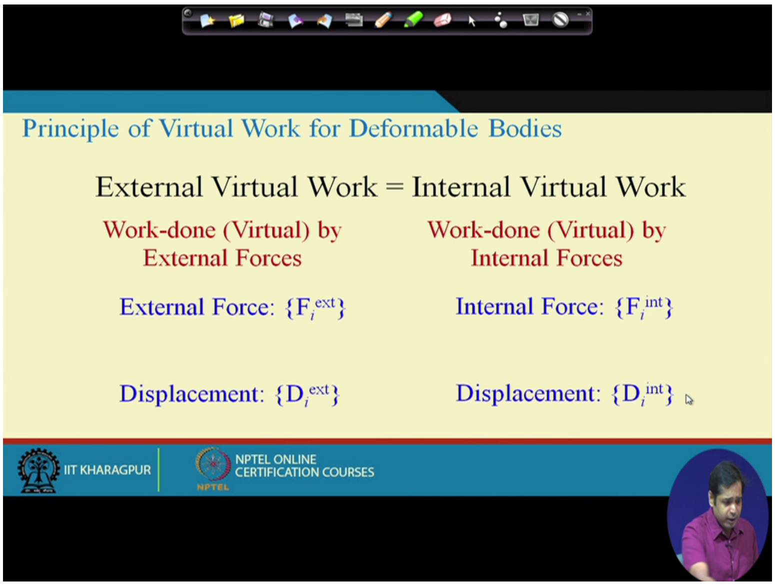

Now we have not made any statement whether the forces are real and displacements are virtual or the displacements are real or the forces are virtual. If you remember we said that we want to derive a general principle of virtual work and then see the principle of virtual displacement that we discussed in the last class. That is a special case of this general principle, okay. So in this case what is happening?

The force could be real and displacement is virtual or the force could be virtual and the displacement could be real, okay. Now but only thing is this displacement and this force field they are conjugate to each other. So they are conjugate to each other, okay.

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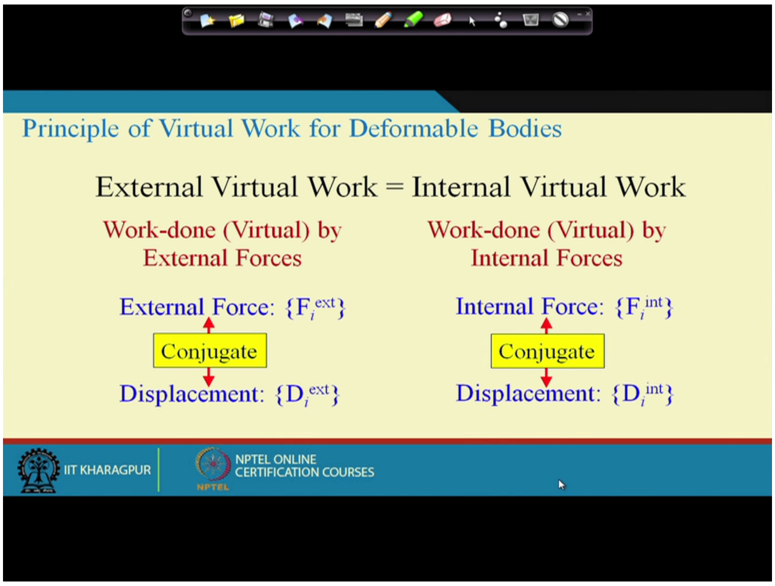

Now before we actually go further in deriving the expression of the principle of virtual work let us understand what we (un) mean by external. There are four parameters involved here, okay. External forces, internal forces and the external displacement and the internal displacement. Let us now understand and these four ingredients they constitute the principle of virtual work for deformable bodies.

So before we derive that, before we get that equation let us understand what are the internal and external forces and what are the external and internal displacements. Now let us consider a truss problem. And if you remember this truss again, last week we solved this truss using equilibrium condition. We obtained the internal forces using equilibrium condition. Now suppose these members are 1, 2, 3 and numbered like this.

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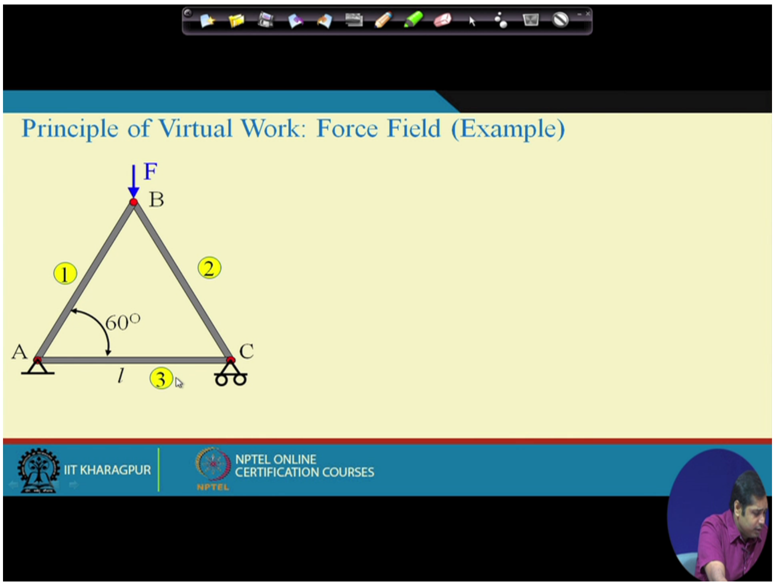

Now if you draw the free body diagram of this object then these are the reactions and then we have external force at B.

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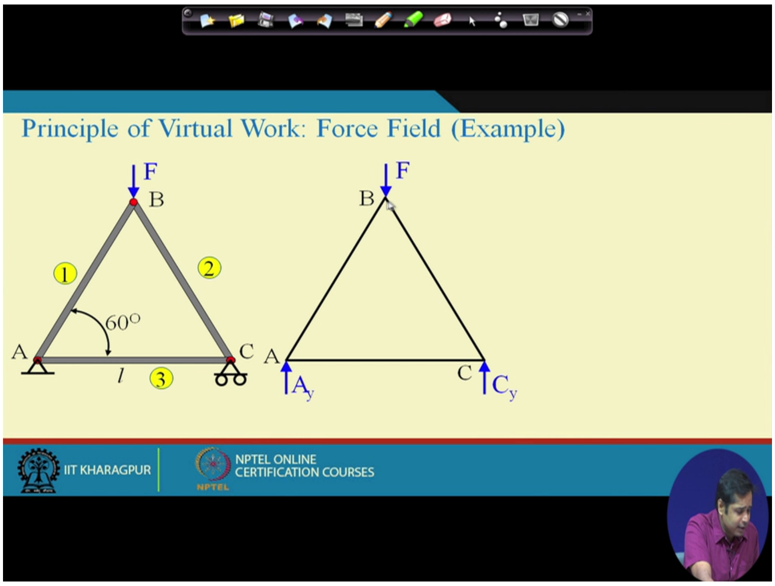

And then because of these external force if we get each member force in all the members then these are the member forces in (mem) different members, okay.

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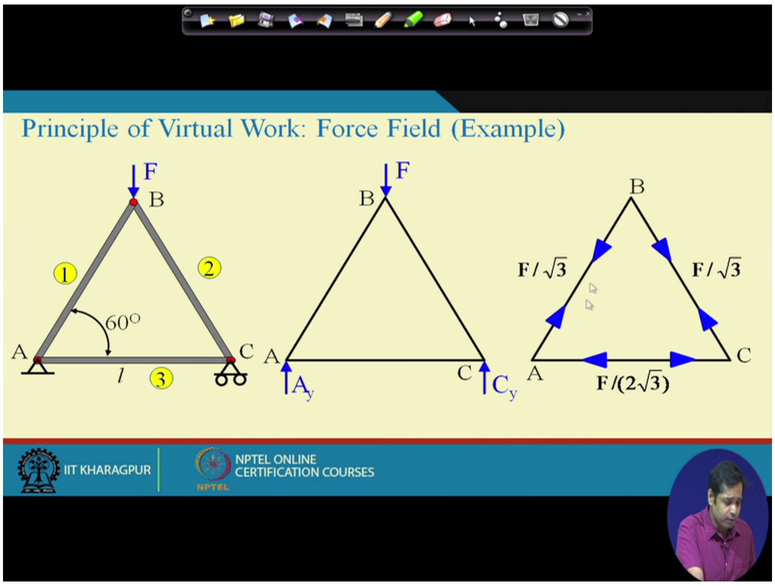

Now if you remember this is how we are showing the member force. This means that if it is shown in the member this means the member is in compression and if it is shown in member this means the member is in compression that is the sign convention we are following for member. Now so these are the forces in different members. Now this is the external force fluid and this is the internal force fluid and this is F external and this is F internal, okay.

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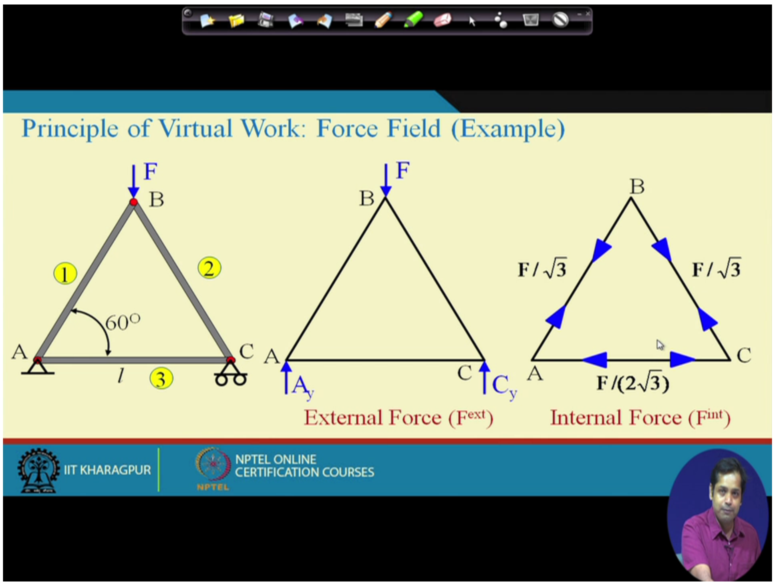

Now suppose this system we call system 1. The system 1 is essentially a force field which has the external force and the internal force, okay. Now these internal forces are due to this external forces, okay.

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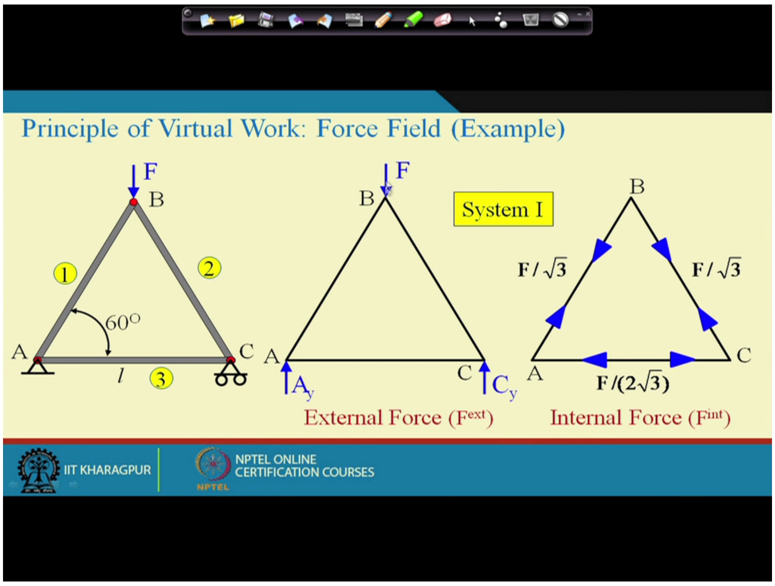

Now so (dis) this was the force field. Now let us understand what is displacement field? Now again this is the same truss. Suppose the point B undergoes some displacement in this direction, okay. And this dotted line shown here is a displacement in the truss, okay. And how do we get the displacement? We apply some displacement at B and the B goes to B dash and this corresponding displacement is a, okay.

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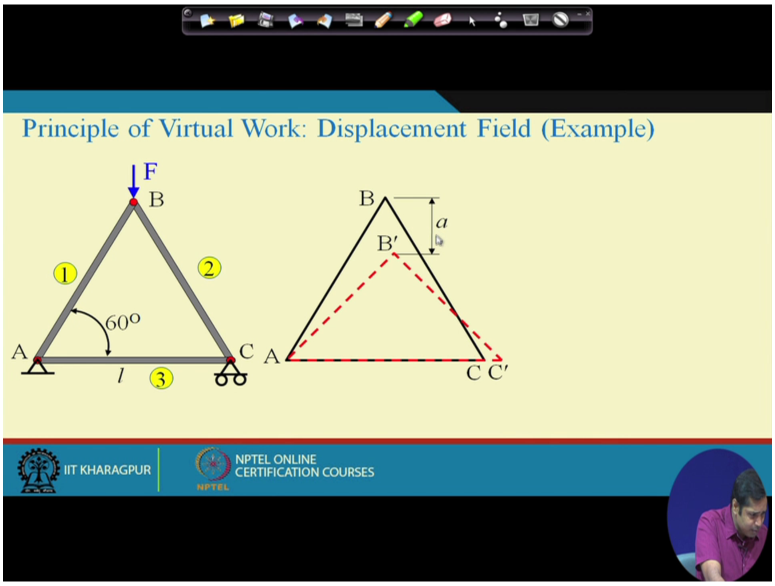

Now what happens because point B goes to B dash then the point C will go to C dash and we assume that there is no change in length in member AB and BC. Therefore there will be a change in length in member AC. Member AC will be in tension and the elongation of the members will be C-C dash and that is equal to 2 by root 3a, okay. And you can obtain this value through some simple geometry, okay.

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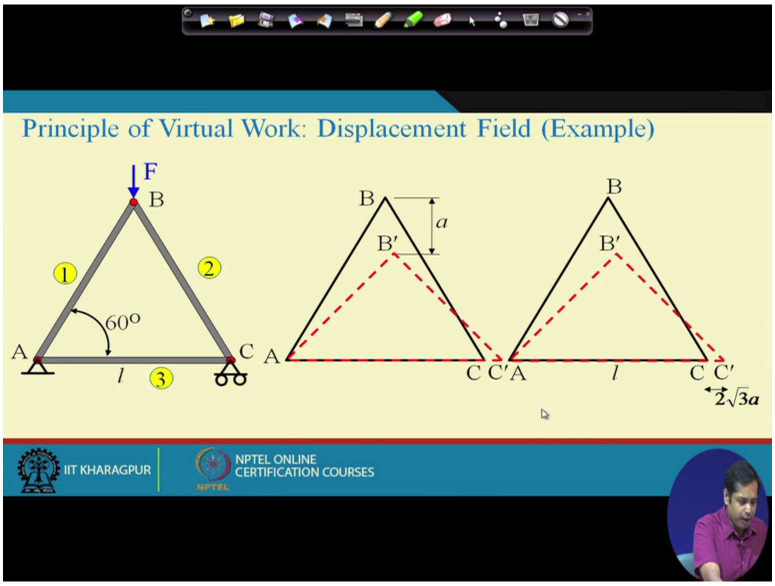

Now this is called external displacement because you are applying a displacement at joint B, okay, externally. And this is internal displacement. Internal displacement it means that this C-C dash this displacement this due to the change in length in member AC.

So when we apply a load every member has internal forces and if the members are deformable then (mem) because of the internal forces the member undergoes deformation and that deformation is the internal displacement.

In this case the member (bec) AC will be in tension and because of this tension the length of member AC will increase and the initial length of the member was AC and (af) after the elongation the member become A-C dash and change in length is C-C dash and that is due to the internal forces developed in the member and this displacement is called internal displacement, okay.

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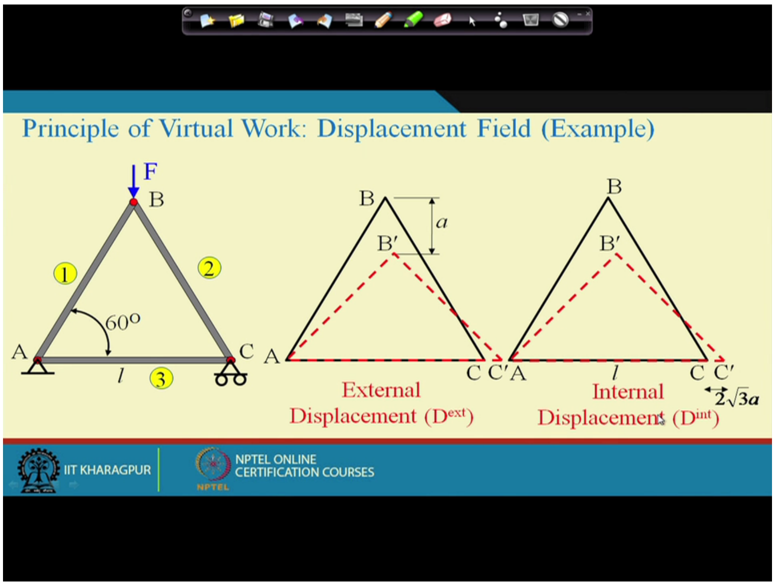

Now then this is called system 2. So system 1 is the force field and the system 2 is the displacement field. And we have two forces in system 1, external force and the internal force. And in system 2 we have two displacements, external and internal displacement, okay. And how the external forces and internal forces are related?

We apply a force on the object, we get the (inter) corresponding forces in different members and it gives us some displacement at the joints and because of this displacement within the member itself the member undergoes some deformation and that deformation is the internal displacement. And this displacement field is say it is system 2, okay. Now then go back to our previous slide.

So this was the external virtual work and the principle of virtual work says that external virtual work is equal to virtual work. And now the work done by the external forces means the external force into external displacement. And then the internal virtual work, internal forces into internal displacement. So they are equal, okay. So this is the principle of virtual work for deformable bodies, okay.

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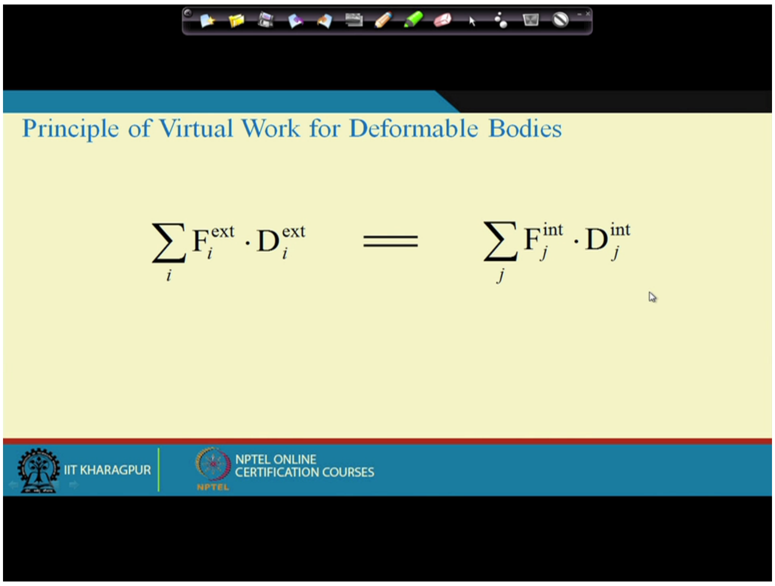

Now I said this is system 1 and this is system 2. But please note system 1 and system 2 they are not related to each other. It is not they do not have any cause effect relationship. It is not that because of system 1 we have system 2 or because of system 2 we have system 1, okay. System 1 is independently to system 2, okay. But only thing is system 1 gives you the force field which is the external force and corresponding internal forces and the system 2 gives you the displacement at the joints and corresponding displacement in different members, okay.

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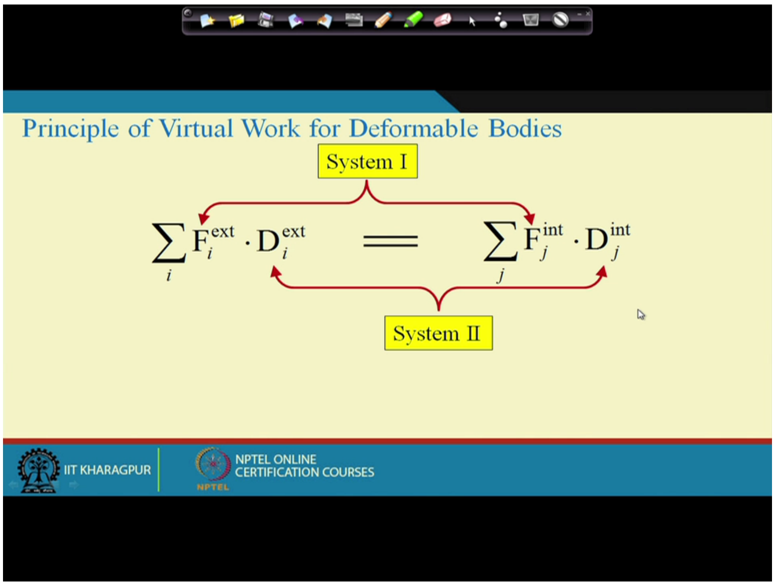

Now so what constitutes system 1? The equilibrium constitutes system 1. Because you see when we apply some external force in a structure some internal forces develop in different members and the summation of the internal forces should be equal to the summation of the external force, right? And that is equilibrium condition. So what constitutes system 1 is the equilibrium, right?

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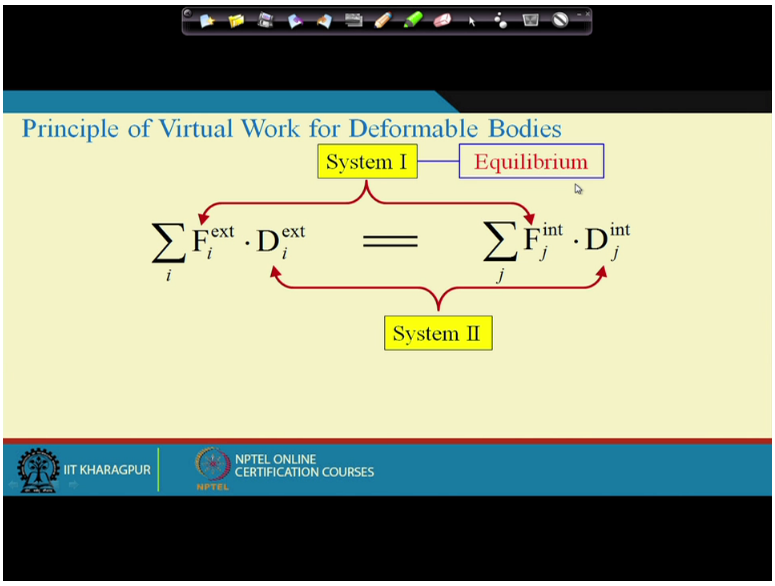

Now similarly what constitutes system 2? It is compatibility. Now if we give some displacement at any joint the member undergoes deformation and when the member undergoes deformation or the structure undergoes deformation all the members undergoes deformation. But this deformation in the member should satisfy the internal compatibility, okay. So the compatibility which is the basis of system 2, okay.

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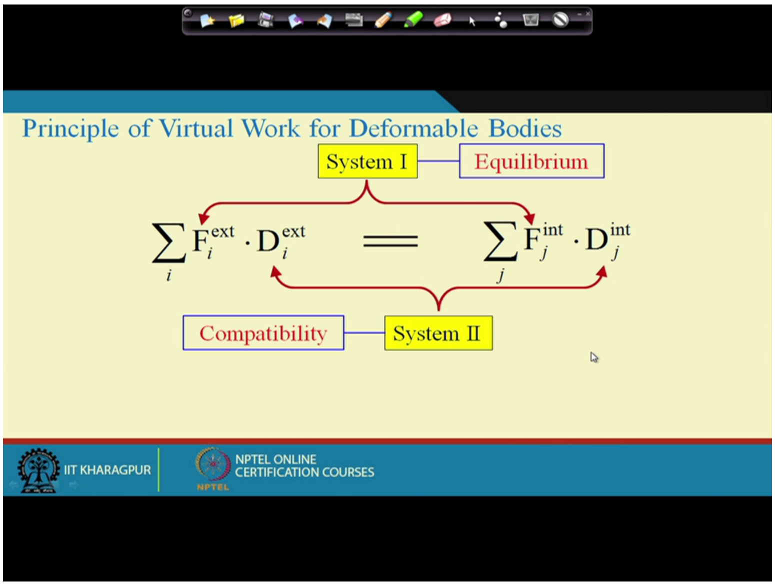

Now this is called principle of virtual work, okay. And the general principle of virtual work. Now as I said we will see that principle of virtual displacement that we discussed in the last two classes is a special case of this generalized principal. Now we have two situations, one is if system 1 is real means the force field is real but system 2 is the displacement field is virtual then we have principle of virtual displacement which is exactly same we discussed in the previous classes.

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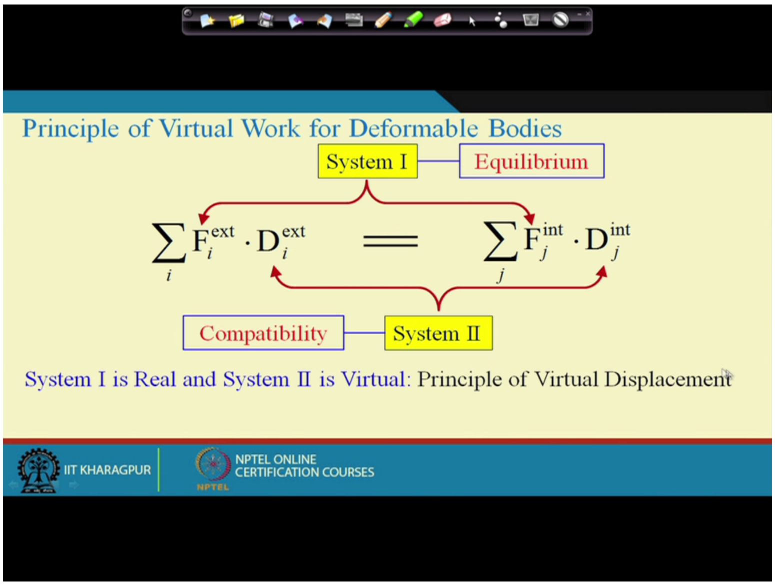

Now if system 1 is virtual means the force field is virtual and the displacement field is actual then this is called principle of virtual force, okay.

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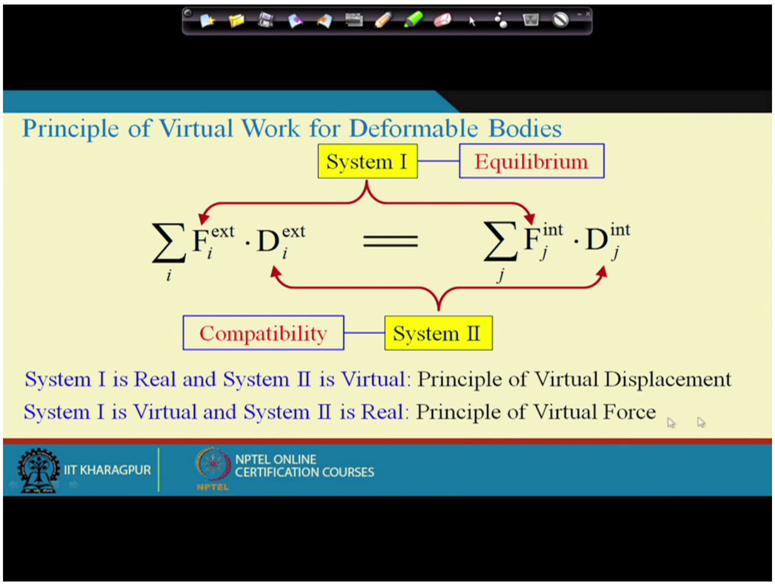

Now let us see how the principle of virtual force from the application point of view how they are different? If you remember in the last class we demonstrated that principle of virtual displacement is essentially a problem in statics or solved using through geometric. And also we said that principle of virtual forces their problems in geometry are solved through statics.

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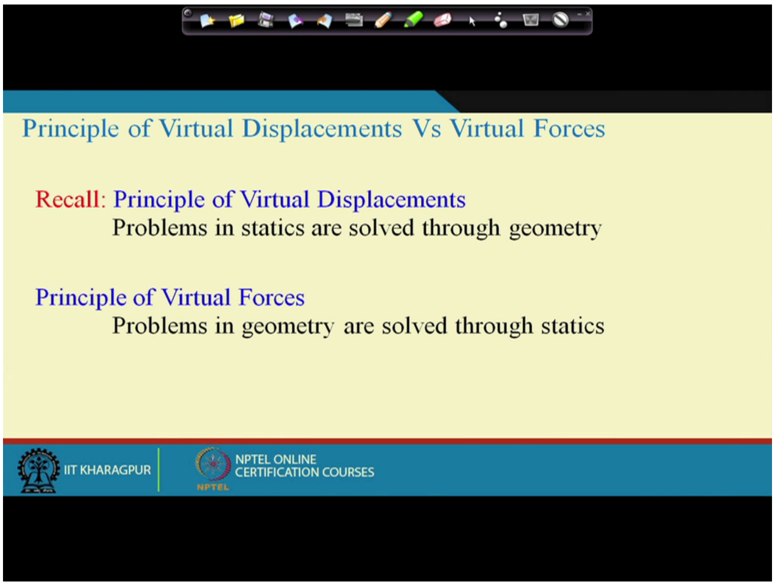

Let us demonstrate this quickly. Let us consider this example once again. Now point B goes to B dash and this displacement is a and the member AC undergoes deformation in C dash and this deformation recalls C-C dash is equal to 2 root 3 and this can be obtained easily by geometry.

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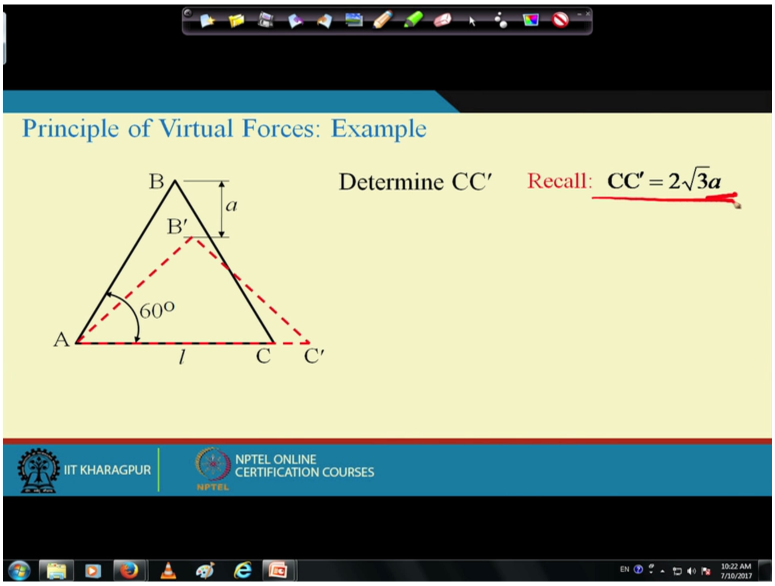

So it is essentially a problem in geometry. There is no statics required here, okay. You apply a simple geometry, we get C-C dash is equal to 2 root 3. But we will not apply geometry to find what is C-C dash? What we will apply? We will apply the concept of statics. Let us do that quickly, okay. Now this is the problem and then we have a force like this which is F and if you remember this is your Ay and then this is Cy, okay. And then this is the external force, right? This is F ext, external, okay.

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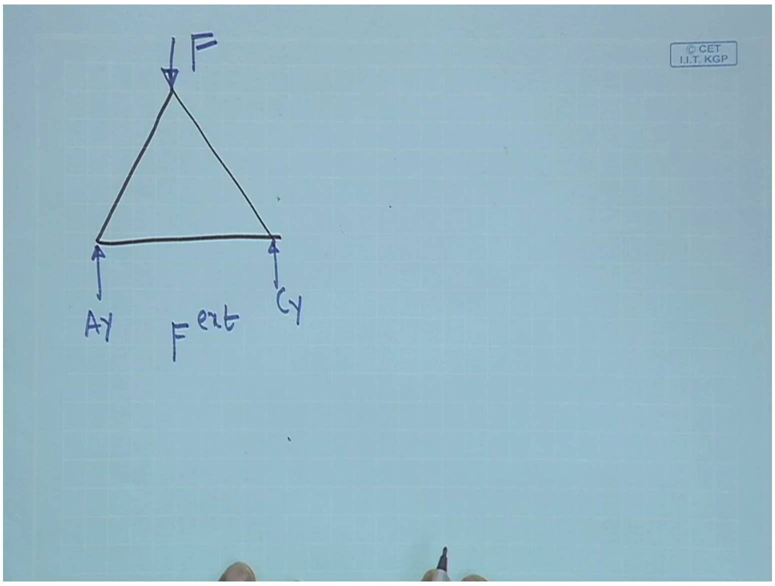

Now let us draw the internal force system. If you remember the forces in these members are this is minus F by root 3, this is also minus F by root 3. You can just easily find out. This is F by 2 root 3. This is the internal force in the member, okay.

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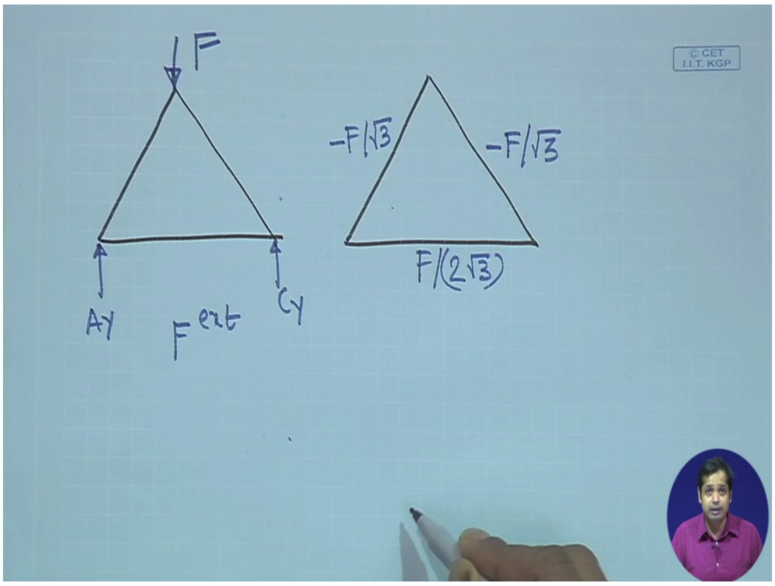

This is because of the external force. So this is F external and this is F internal. So this is our system 1, okay.

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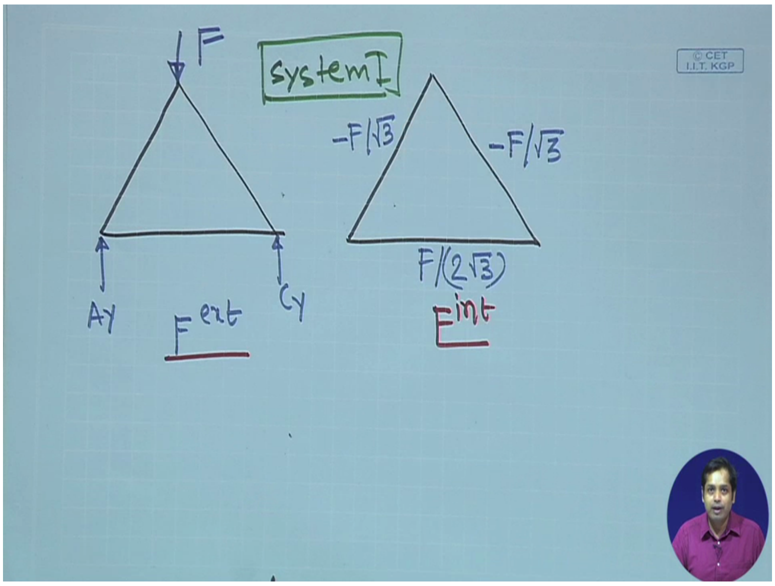

Now let us let us find the displacement field. So this was the actual structure, okay. And then this is the displacement, so this distance is a, okay. This is A, B, C.

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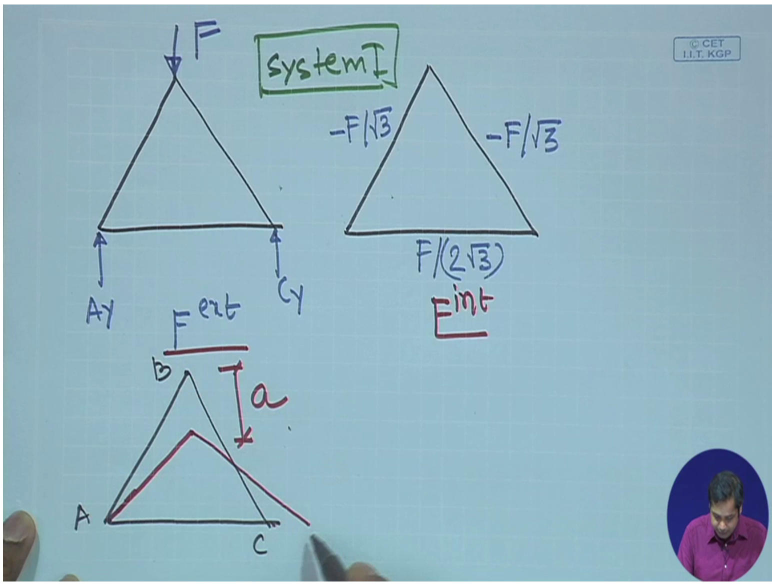

Now the external displacement go mostly to C dash. This is the displacement applied to joint B and this is D external.

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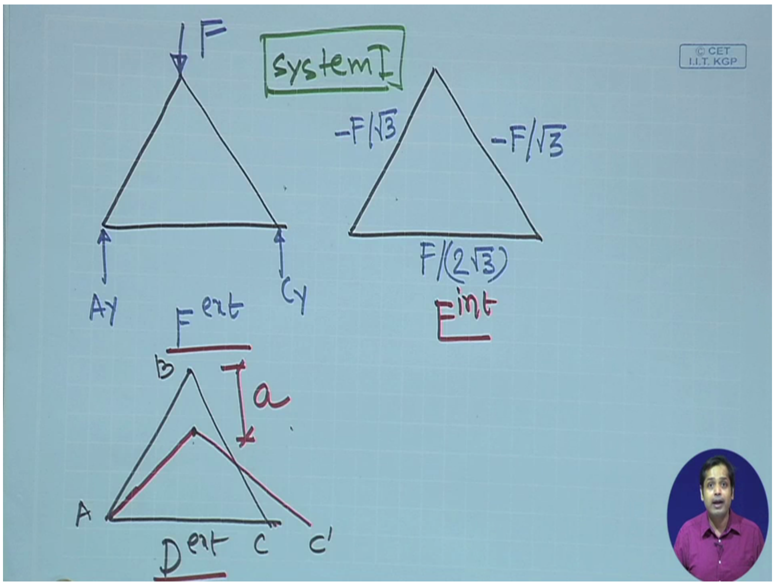

And then what is the corresponding internal? Internal will be, so this is A, B, C and this go to C dash and this distance is say delta, okay. And this is your (sys) D internal, okay.

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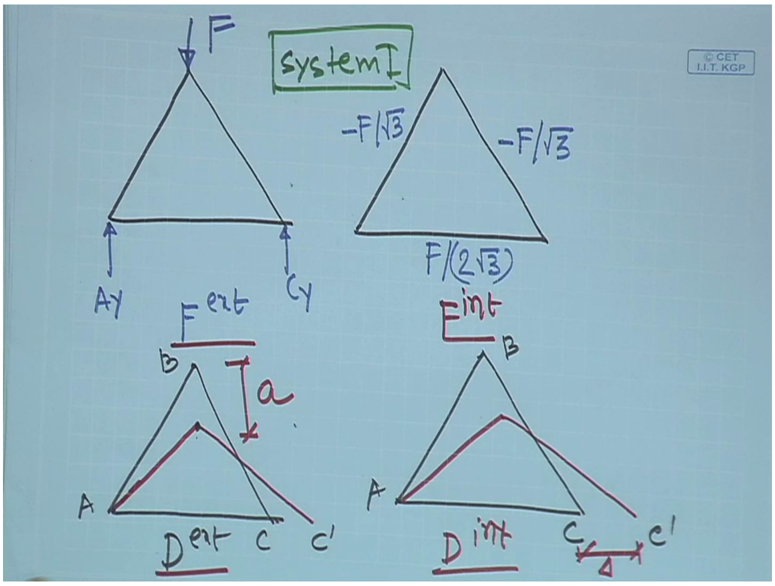

This is D internal because this delta is due to the internal force developed in AC. But because of this a, we have this internal force, right? So this is system 2.

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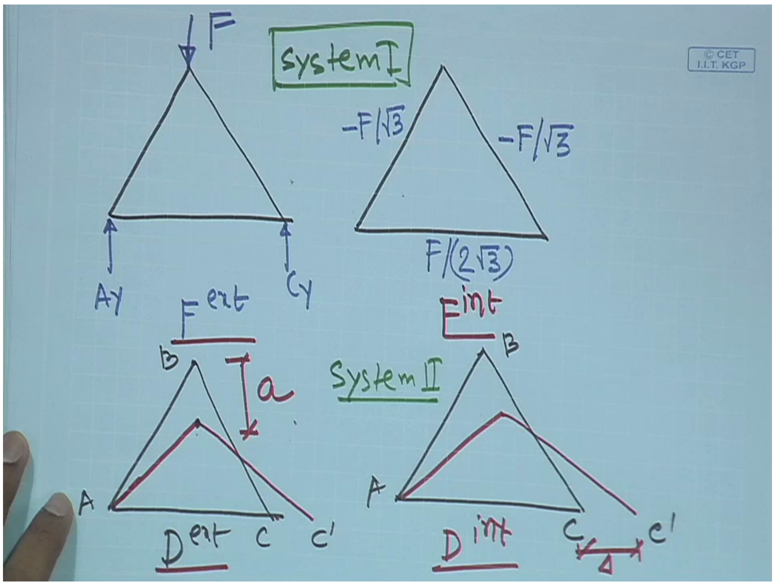

Now what principle of virtual work says? Principle of virtual work says that summation of F external into D external is equal to summation of F internal into D internal, okay.

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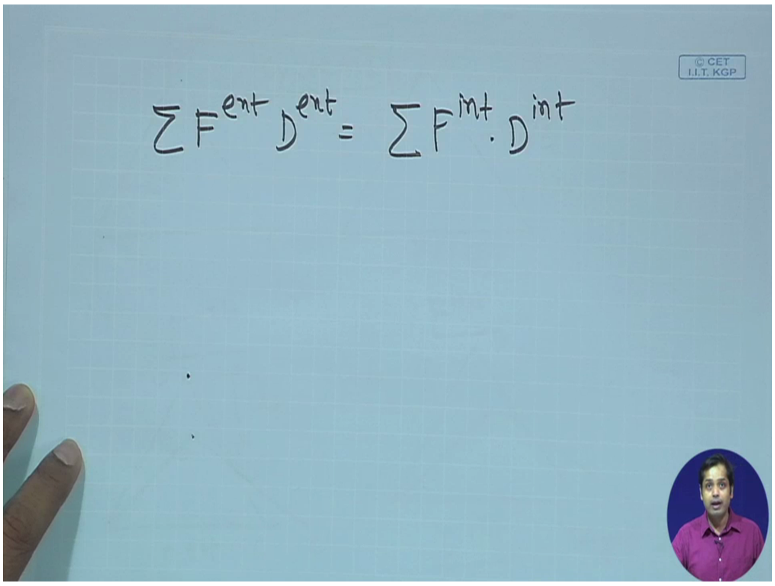

Now what is F external? Now F external you see here, F external we have this into this. So what we have is the external virtual work should be equal to this into this.

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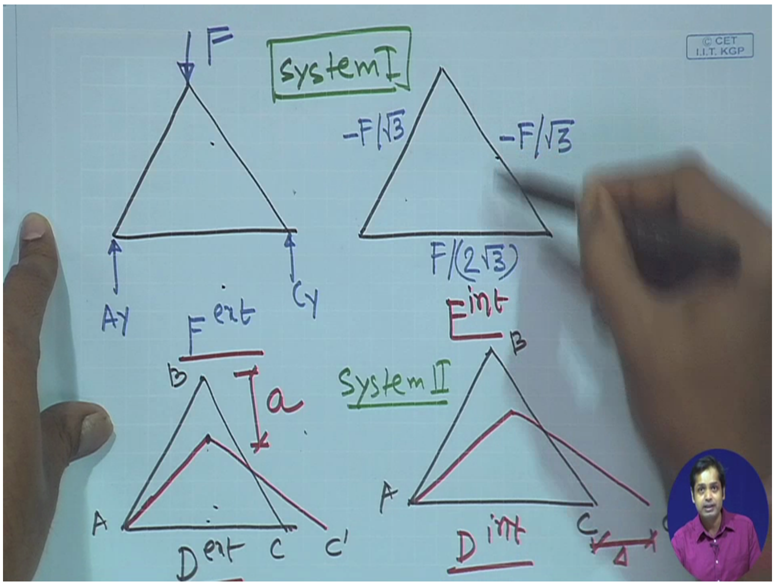

Now this into this if we check then these two forces will not contribute because in this direction there is no moment at point A and point C. The only force that will contribute is F.

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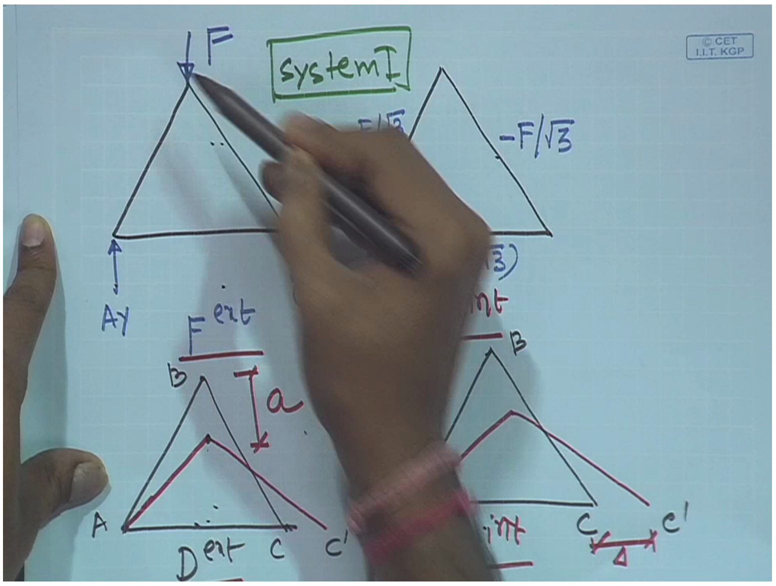

So this will be F multiplied by A, right? So this becomes F into A.

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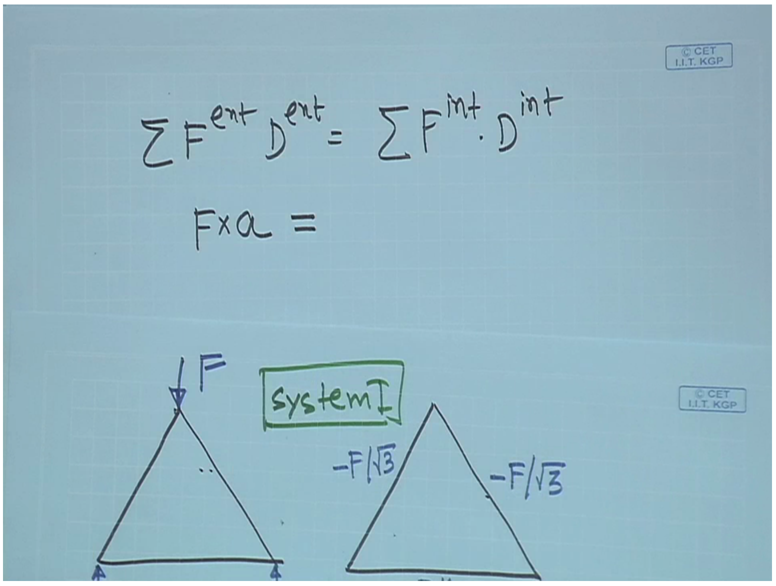

This is the internal forces in this member.

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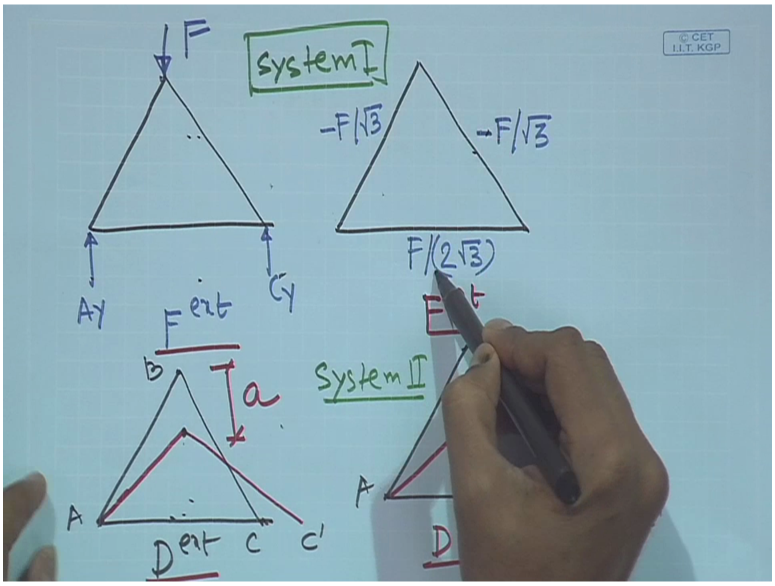

Now here it is given that there is no change in length in member AB and BC. Only change in length express in member AC. So this is F internal AB into D internal AB plus F internal BC into D internal BC plus F internal AC into D internal AC, right?

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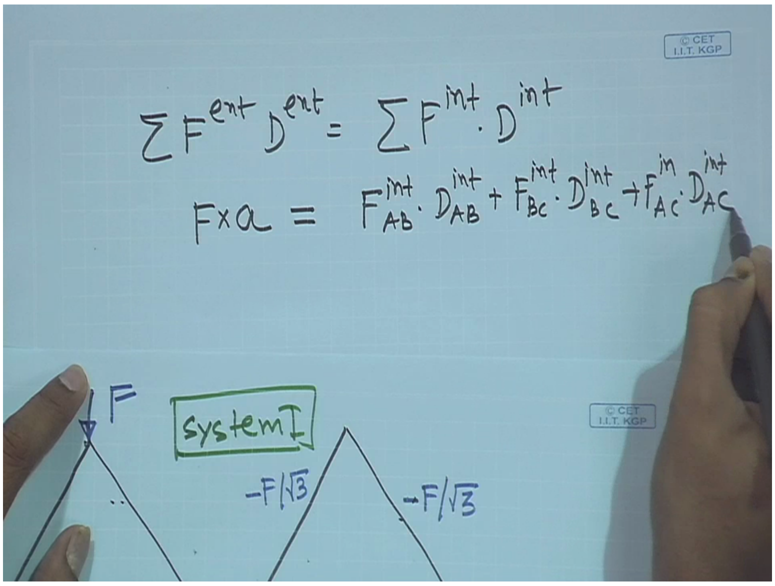

Now F internal is minus F by root 3 but this D internal is zero because there is no change in length AB, so this is zero plus this is again minus F by root 3 and this is zero because again there is no change in length in member BC. Then plus this is plus F by 2 root 3 into this distance is delta, okay.

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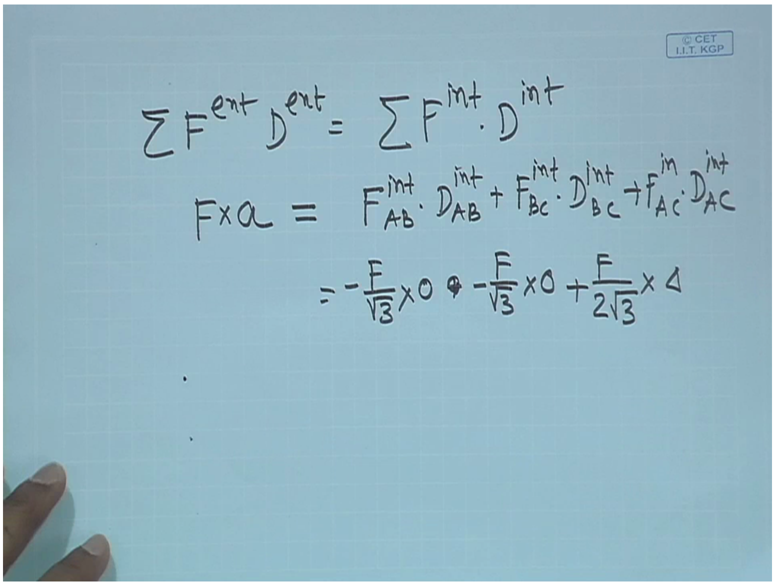

Now if we have this then what we have finally? F into a is equal to F by 2 root 3 into delta. And this gives us, now F, F gets cancelled. This gives us delta is equal to 2 root 3a. So delta is equal to finally we have 2 root 3a, okay.

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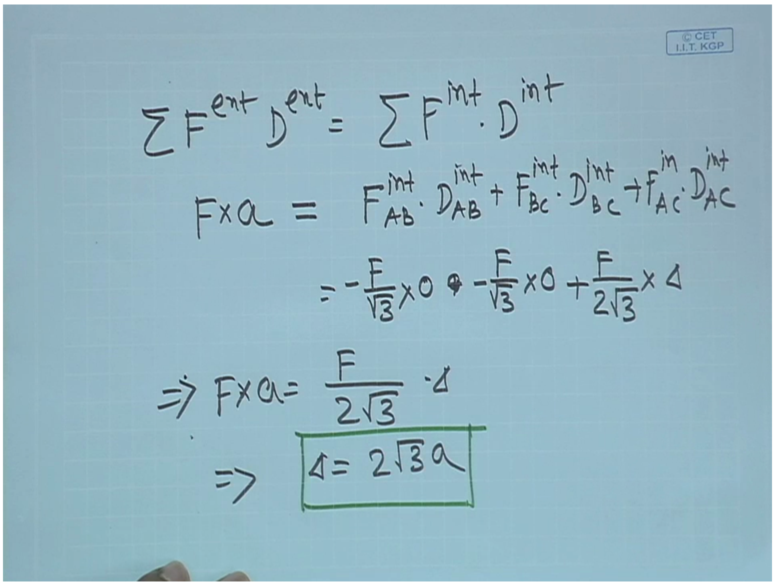

Now if we use the geometry then you can get this value. So essentially what we have done is we have not solved any geometry problem. It is a purely static problem. But we actually solved the geometry problem. This we can simply obtain from geometry but that problem is solved using statics. So that is the difference between principle of virtual displacement and principle of virtual forces, okay.

There in principle of virtual displacement we solved the problem of statics through geometry and here we have solved the problem of geometry through statics, okay. Now here you see the F gets cancelled, right? Here F gets cancelled. Since F gets cancelled any value of F if you give then still we get delta is equal to root 3, delta is equal to this value, okay.

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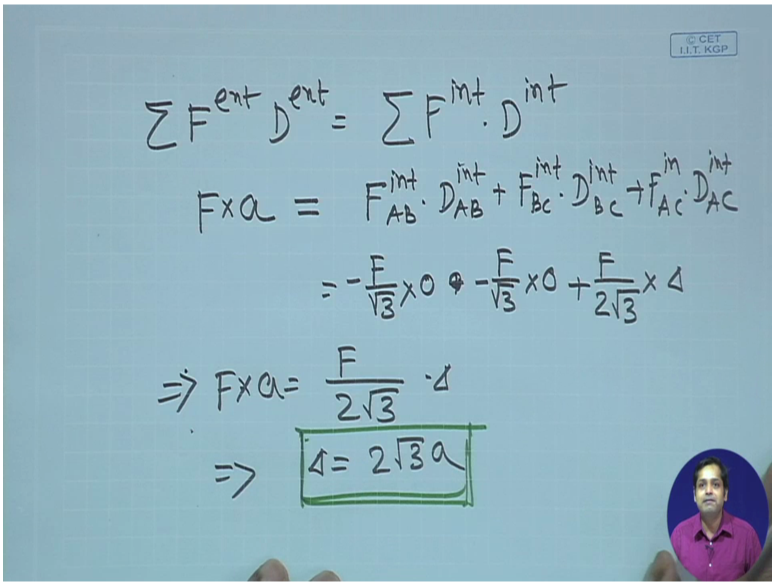

This now gives us a clue of developing a method. The method is now since any value of F this we can obtain the same delta. Give F is equal to 1, okay. So we apply F is equal to 1, get the F external D internal and then another system we have D external D internal which is conjugate to (ex) force field and then apply the principle of virtual work and get the corresponding displacement, okay.

Now we will see how to do this. This method is called unit load method. It is just we have obtained the clue from this example but let us now understand what is unit load method, okay.

Now what we will do is we will demonstrate unit load method through trusses today and then solve some examples of trusses finding displacement unit load method and then after that we will see how this unit load method can be used for beams and frames, okay, plane beams and plane frames, okay. So let us take a truss element, any truss element. Let us take a truss element, okay.

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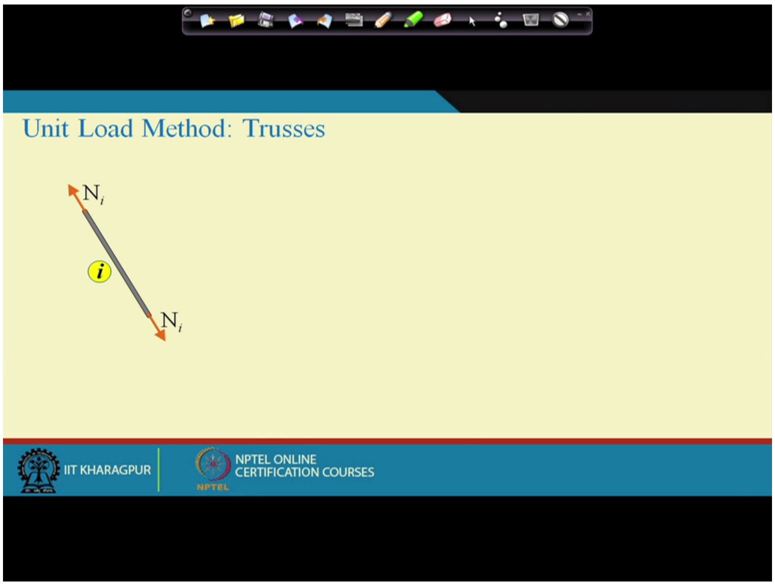

Now we know that any truss element they all are two force member. Now suppose this truss element is identified as i. So it is ith element and the corresponding force in the ith element is denoted as Ni, okay.

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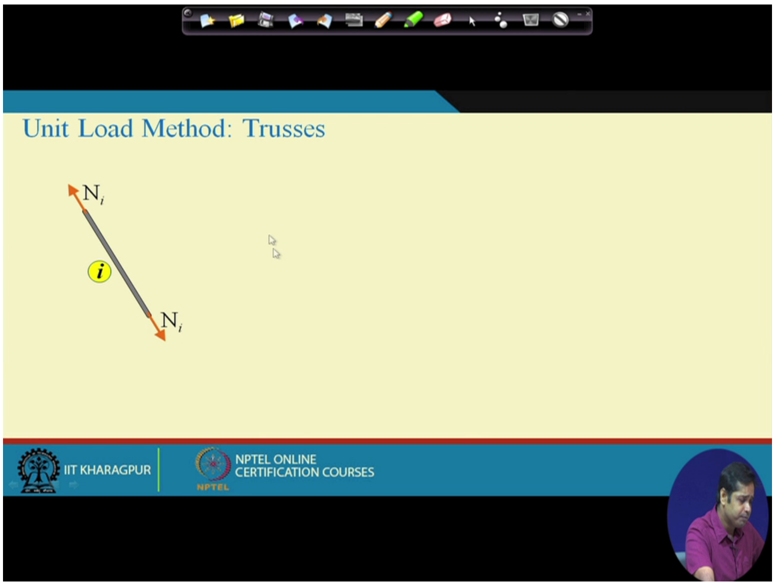

Now suppose length of this member is Li, area is Ai and Youngs modulus is Ei and it satisfy linear elastic area. So this is stress and strain and corresponding slope is the Youngs modulus, okay.

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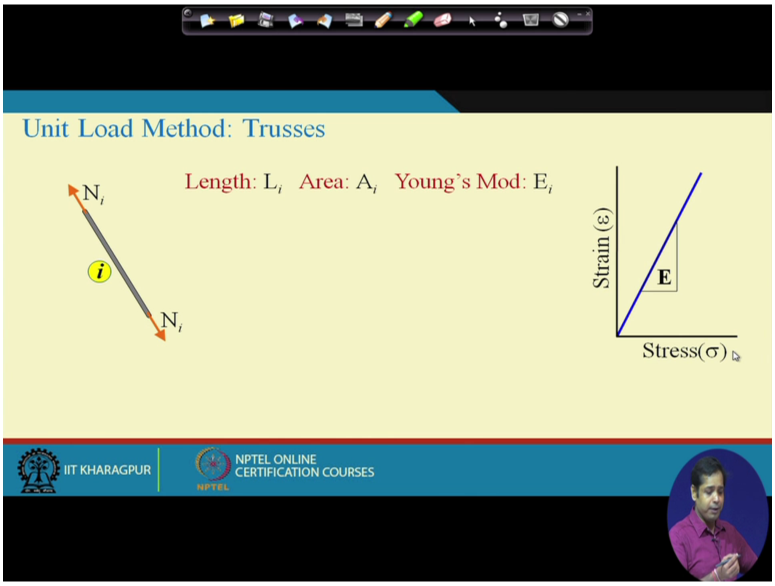

Now if the area of this member is a then what is the stress developed in this? Stress develop sigma is equal to Ni divided by Ai. That is the sigma, okay. Now then if sigma is this then what how this how this stress and strain are related? We know that epsilon is equal to sigma divided by E, right? So stress by Youngs modulus gives us strain.

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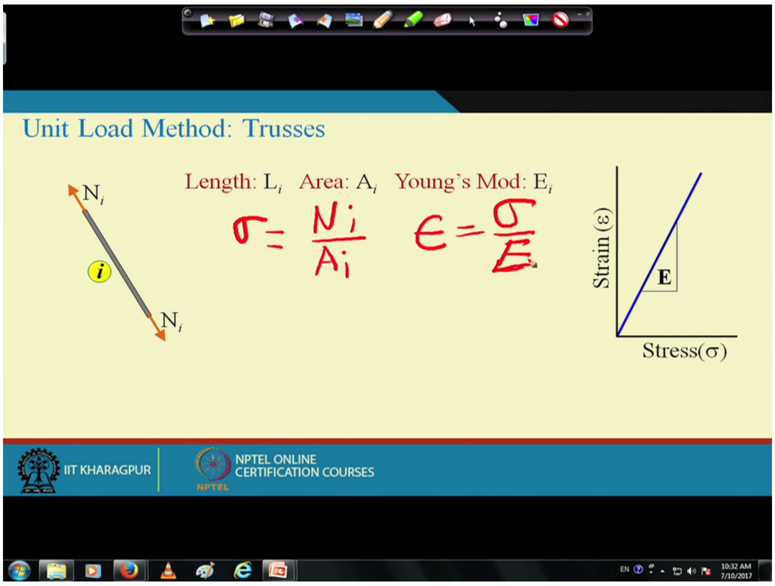

Then if the stress is this then what if the member is subjected to axial load Ni but this could be compression, this could be tension. Then what would be this strain epsilon i in this member. This strain epsilon i in this member will be, then if we substitute sigma here then what we have is Ni by Ai Ei.

If the length of the member is Li then elongation in member Ei is equal to this strain multiplied by the length and this becomes Ni Li divided by Ai Ei. Ai Ni elongation is equal to this, okay.

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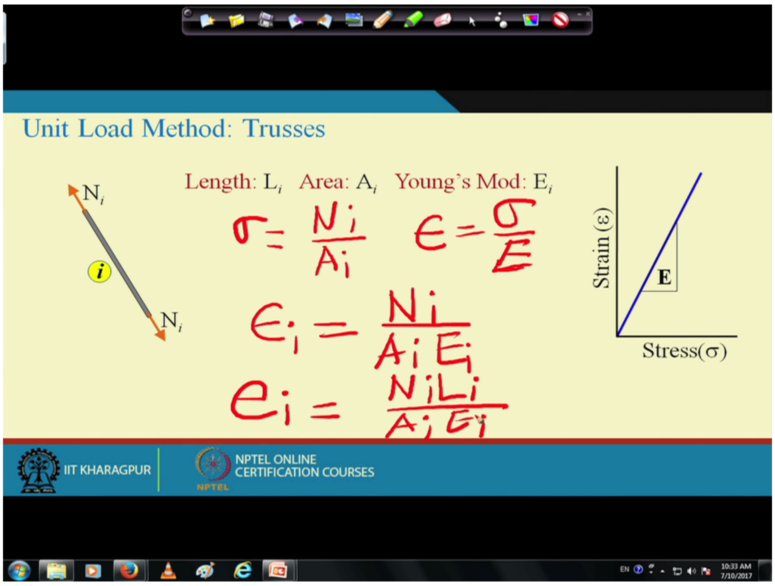

And this is sometimes written as elongation is equal to Ni Li by Ai Ei. This is sometimes written as Ei is equal to Fi into Ni and where Fi is equal to Li by Ai Ei which is called flexibility. We will discuss more on flexibility in subsequent weeks, okay. But only thing we need to remember from this is this expression.

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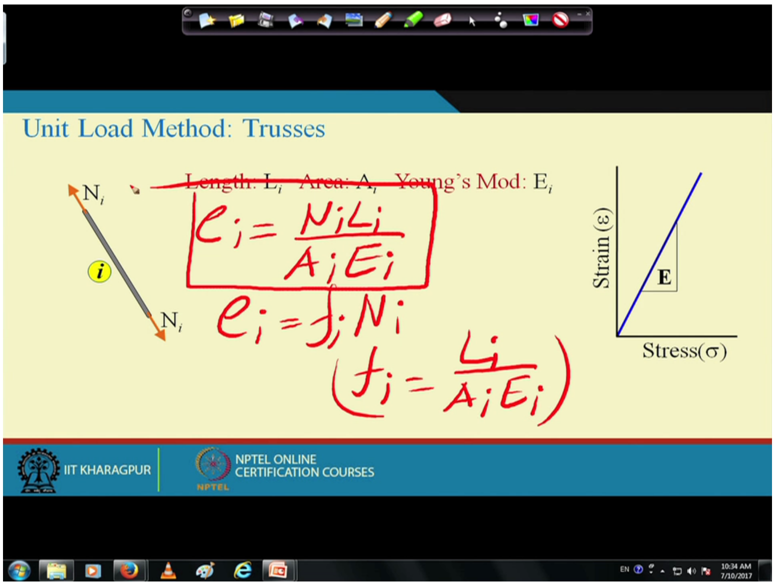

In a truss if the axial force is Ni and then Ai is called axial rigidity, okay. Because you see if Ai is more, then elongation is less and so if Ai is more means the member is axially rigid. So axial deformation in the member will be less. That is why Ai is called axial rigidity, okay.

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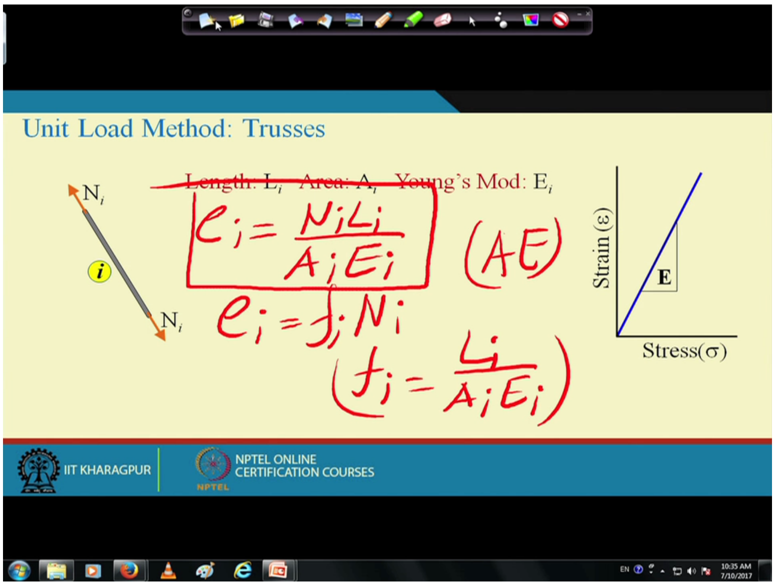

Now great so now next what we do is suppose this is a truss, okay. Now these are the members 1, 2, 3, 4, 5. This is a statically determinate truss and then we first construct the force field, okay.

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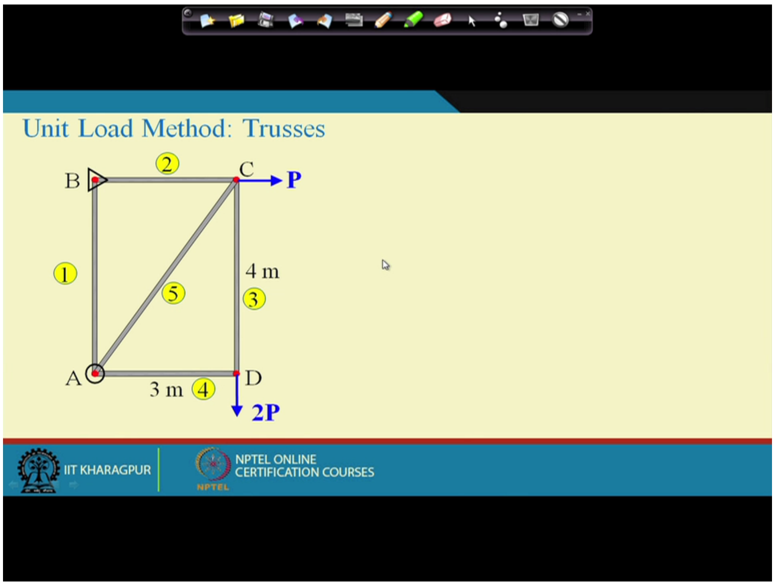

Now suppose this is the reactions and these are the external forces, okay. Now because of these forces suppose these members has internal forces like this, okay. Suppose these internal forces are denoted as Ni, okay.

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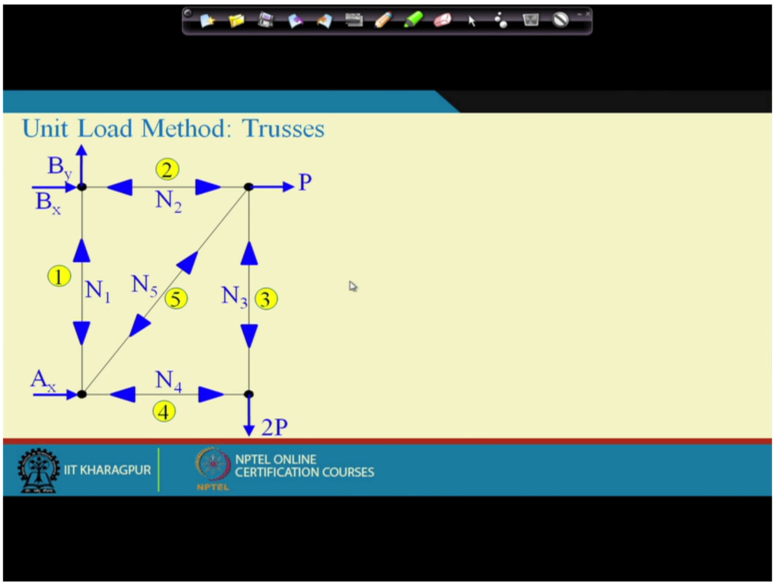

Now this gives us the force field, okay. So this is external forces and the internal forces, okay. Now because the member undergoes a deformation like this, okay.

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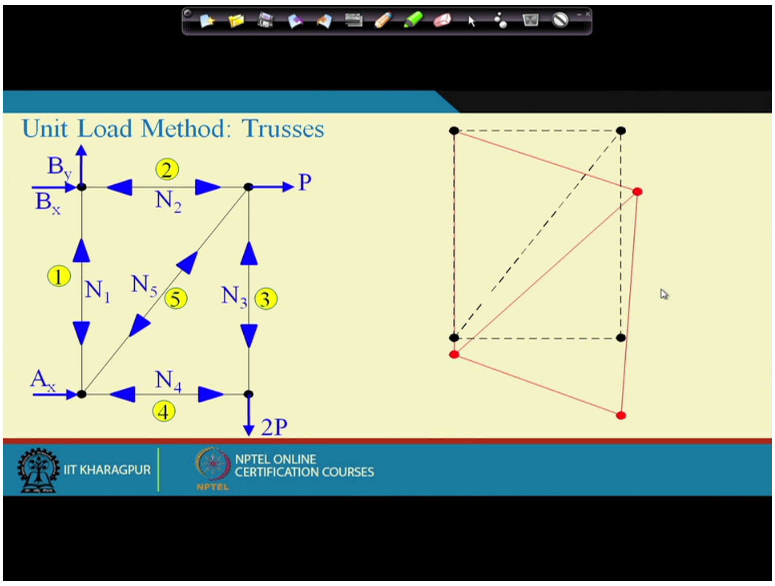

And corresponding (corres) internal length of the all the members now become this.

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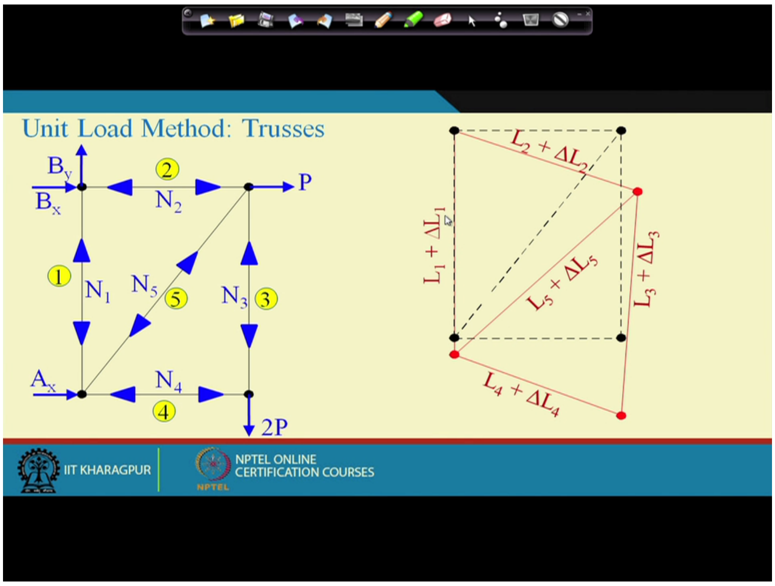

So DL1 is the change in length in member 1, DL2 is the change in length in member 2 and so on, okay. And just now we have seen the change in length can be obtained any jth member, the change in length can be obtained by this. Just now we have seen it, okay.

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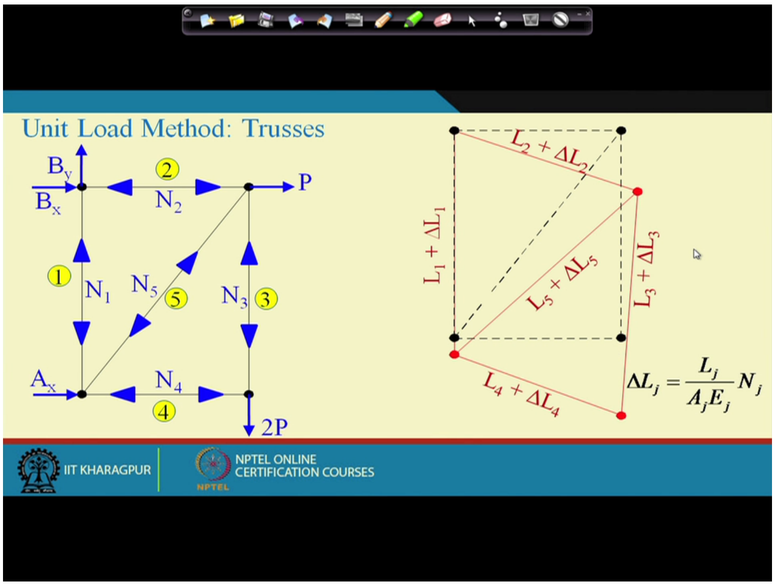

Now consider a point K here and this point K moves to K dash and suppose this distance is (de) DK and suppose our problem is to find out this distance.

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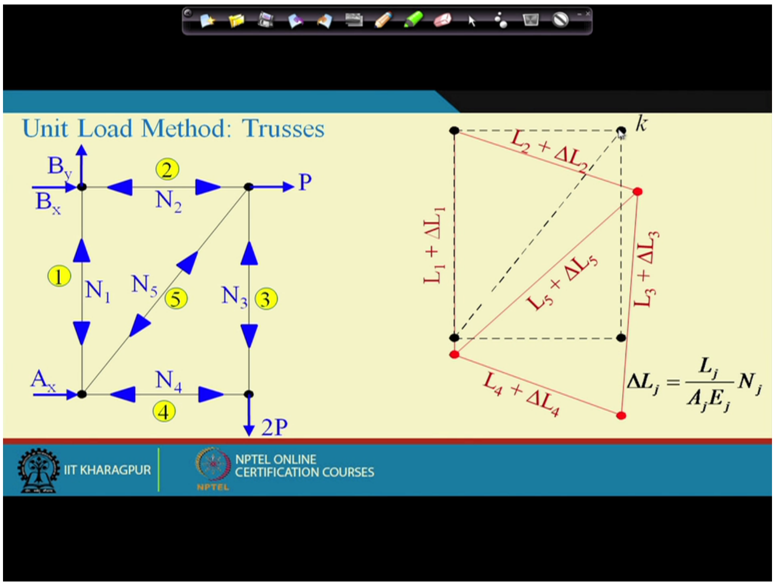

We have already seen method of joints and method of sections to find out internal forces in statically determinate trusses. Now what we want to do is we want to do the deflection. We want to find out the deflection of any particular joints or different joints and that we are trying to do through principle of virtual work. Suppose we want to find out what is the displacement of joint K. So this is DK. We want the displacement in vertical direction suppose, okay.

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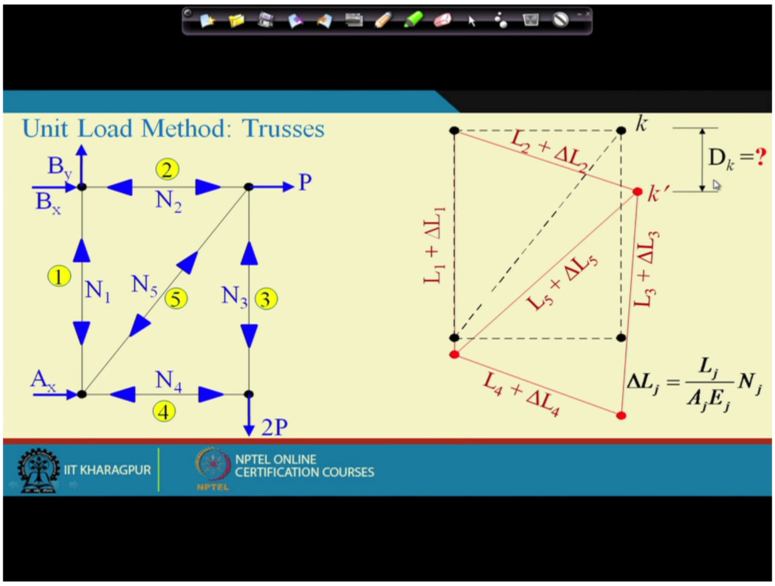

Now so our problem is finding DK. Now let us take the same truss and apply a virtual force at point K in the direction of DK, okay. Now because of this virtual force suppose these are the corresponding reactions.

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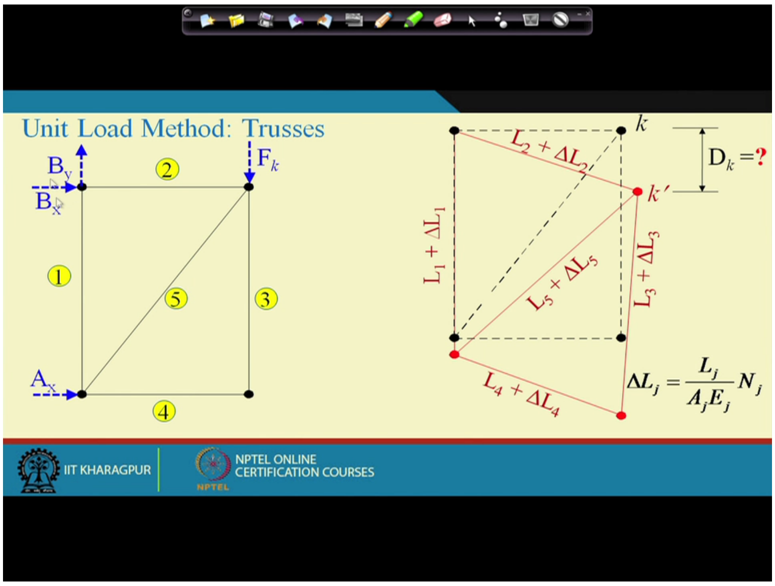

Suppose these are already (al) internal forces in the member and these internal forces are denoted as small n and this small n is the internal forces in the member due to the virtual load, okay, virtual load at K.

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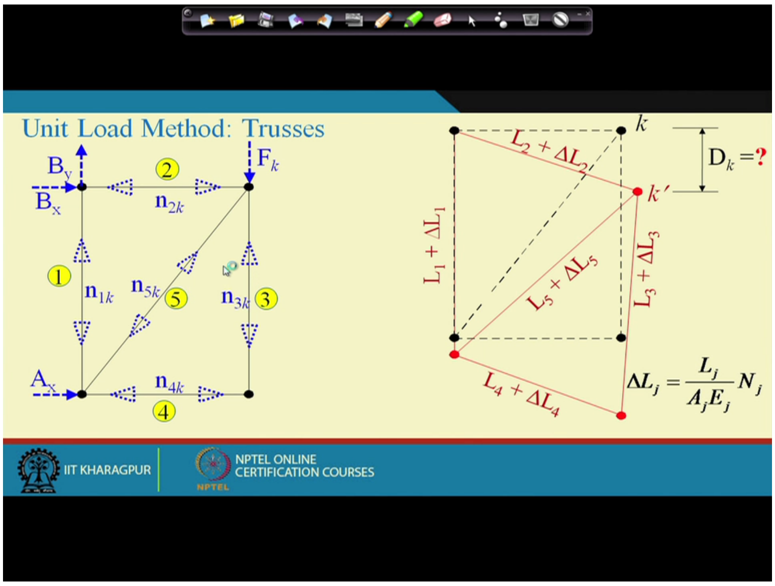

So n1K is the internal force in member 1 due to virtual load FK and similarly n2K is the internal force in member 2 due to virtual load FK and so on, okay. Now if we take this as system 1 and this as system 2, okay.

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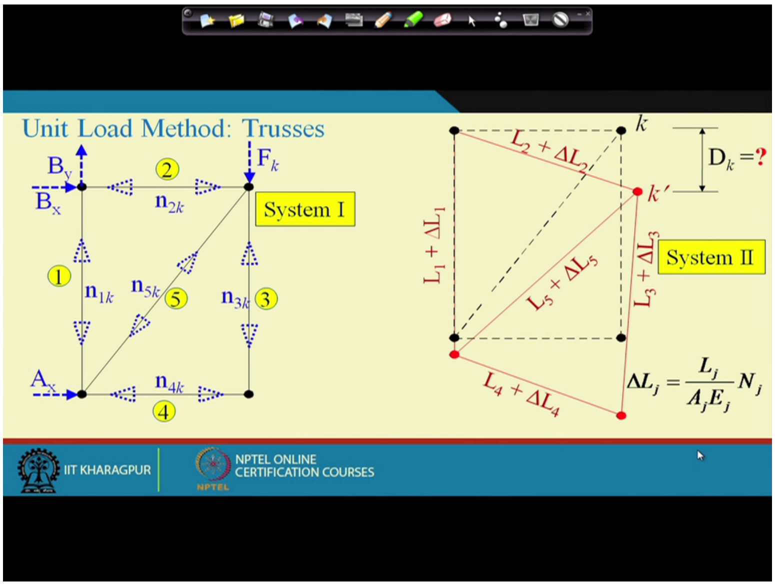

So system 1 is what? System 1 has a (ex) virtual force, external force and corresponding internal forces. So system 1 has the force field in this is virtual force field, okay. The external force is virtual and the corresponding internal forces we have that is due to the virtual external force. So this is system 1.

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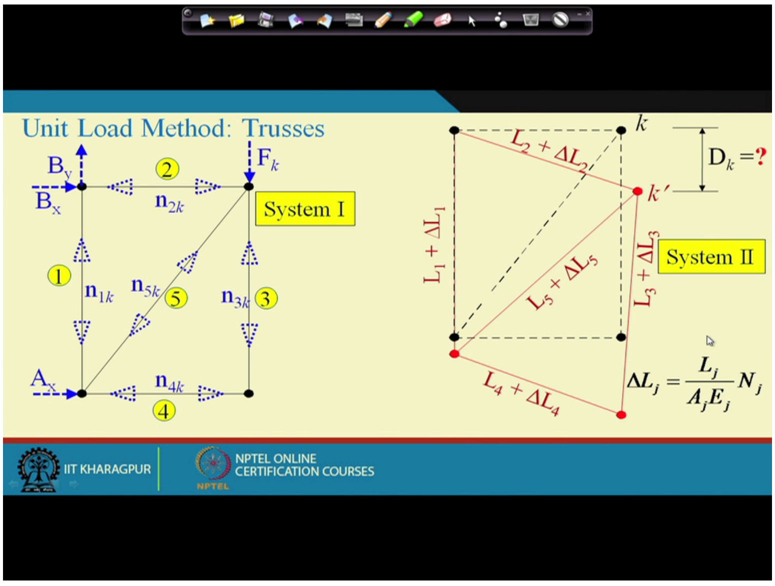

And the system 2 is the real (dis) displacement field, right? Now displacement field is real. So this is essentially principle of virtual forces where the force field is virtual and the displacement field is real, displacement field is real and the force field is virtual. Now if we apply the principle of virtual work considering these two systems and then what it says? This is virtual force field and this is real displacement field, okay.

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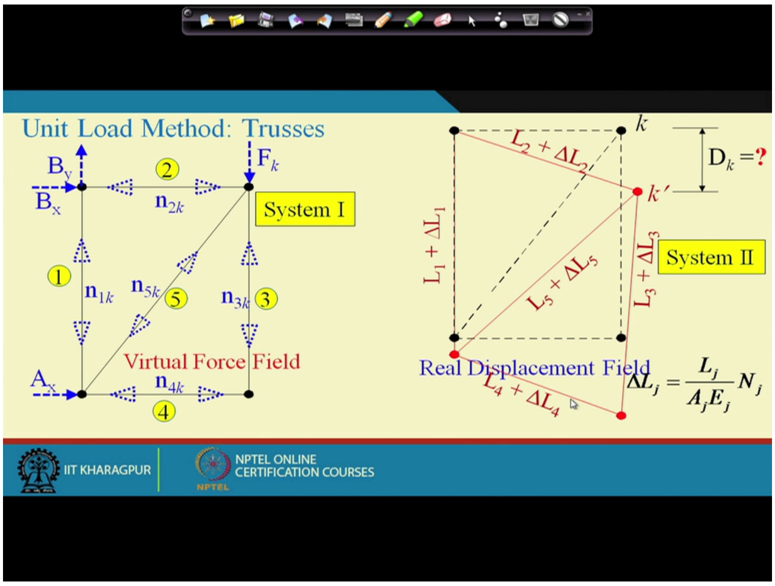

And then apply the principle of virtual work which says that summation of external virtual work is equal to internal virtual work. So this is the external virtual work, internal virtual work, okay.

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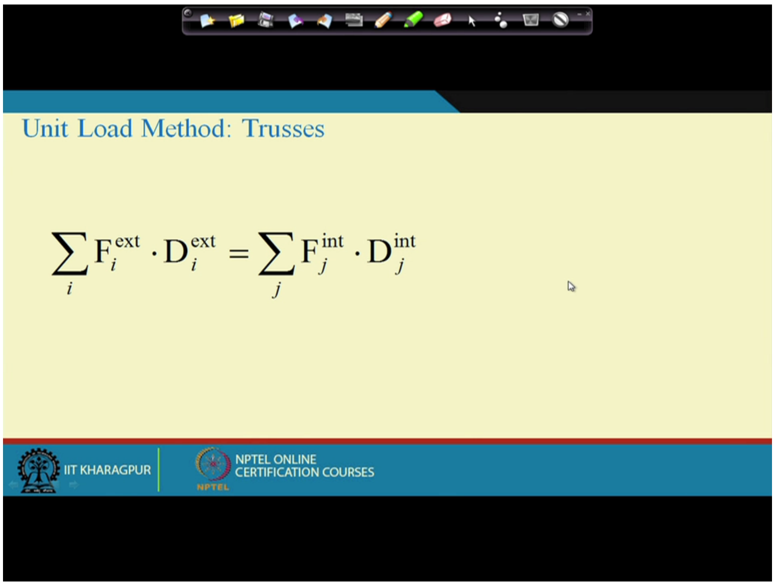

Now what is the external load we have? External load we have in the system 1 is this. That is the external load, okay. The points B does not move so there is no work done by the reaction at point B. Now reaction at point A is in horizontal direction but this point does not move in horizontal direction so there is no contribution of Ax. So in external force field the only force we have is FK, right? Only force is FK.

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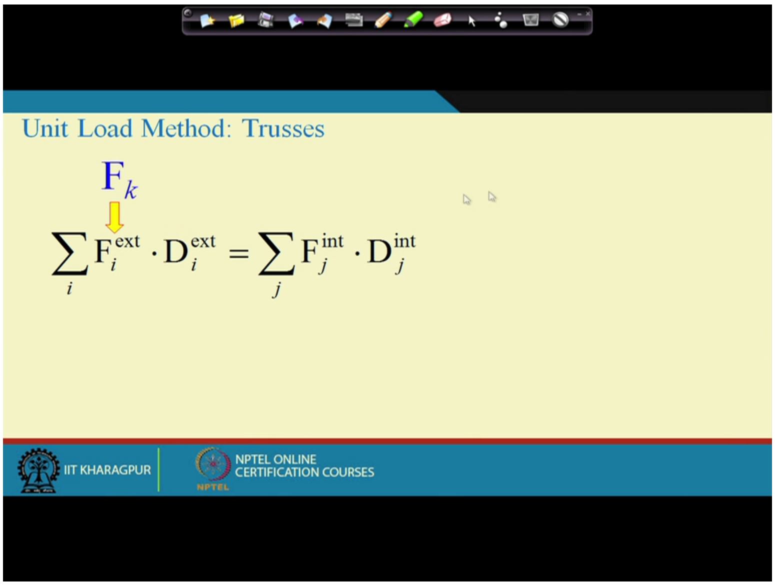

And then what are the internal force fields? The internal force field is corresponding internal forces, okay, the internal force field.

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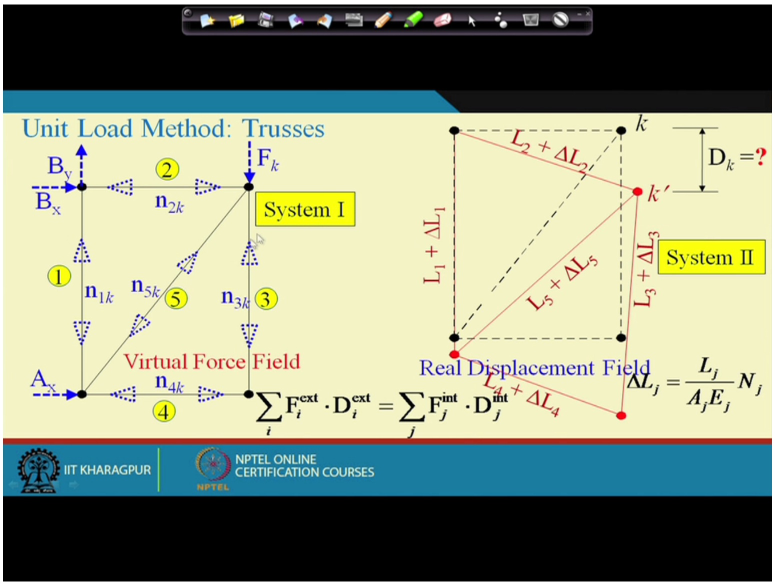

Now this is the internal force field, okay. Now so this is the force in jth member due to force FK, okay. So now what is the external force? External force you see this is the external forces applied in the direction of DK, so this can be treated as external displacement D external and corresponding delta L1, delta L2, delta 3. So this can be viewed in this way. So this is compatible displacement field, okay.

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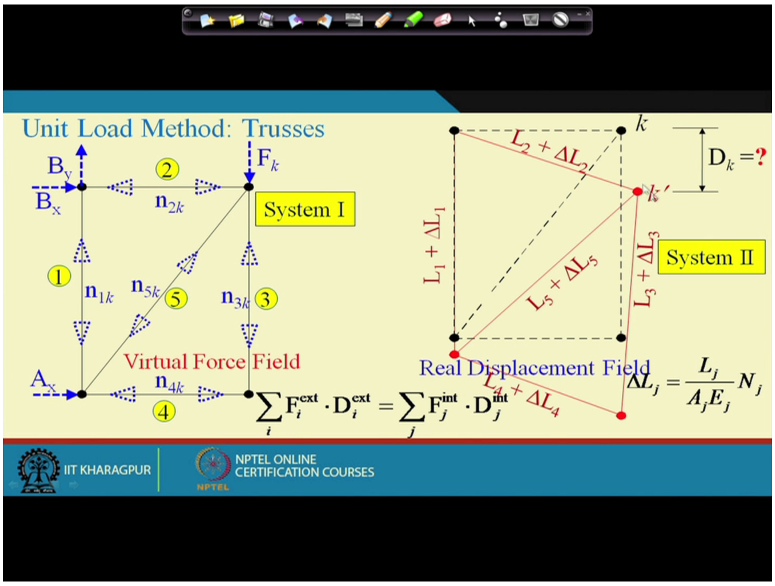

Now if we apply a displacement DK here then all the members will undergo deformation as delta L and the member 1 will undergo deformation delta L1, delta L2 for member 2 and so on if we apply a displacement delta K in member K in vertical direction, okay.

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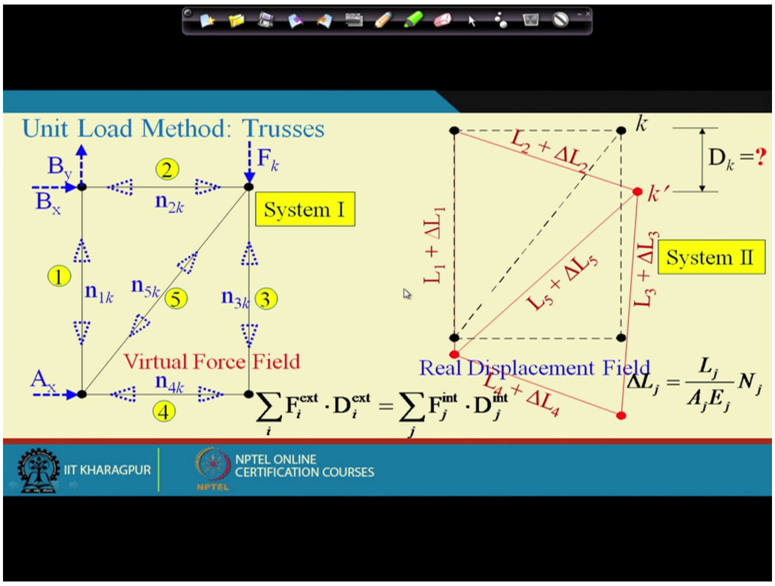

So external displacement will be this and internal displacement will be delta L1, delta L2 and so on. So (in) the external displacement is this and internal displacement will be delta Lj means it is the change in length in member J, okay.

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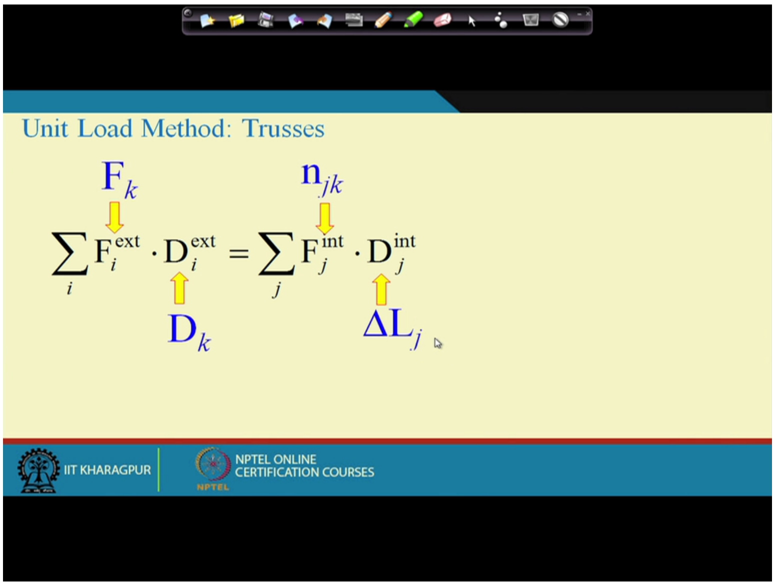

Now so this becomes FK into DK and this becomes summation of nJK into delta Lj. Now we know that delta Lj is equal to this, just now we discussed. Substitute delta Lj in this equation and then FK is a virtual force and this is true for any value of virtual force. Just give FK is equal to 1. If you give FK is equal to 1 and if you substitute this and this both in this equation, okay, both in this equation.

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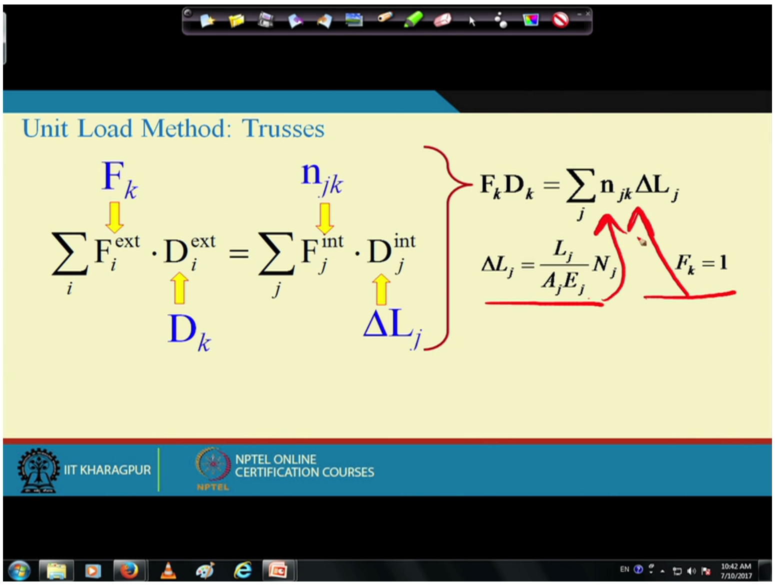

Then what we get is DK is equal to this, okay. So this is unit load method, okay.

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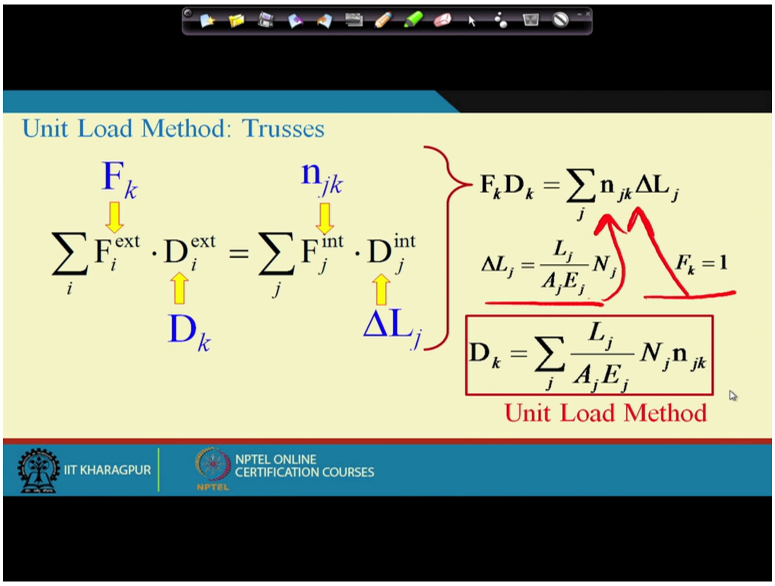

Why it is unit load method because the virtual force that are acting this is unit, okay. Now what unit load method says is? Unit load method says is just quickly let us see unit load method through, now suppose this is the structure and we have to find out what is the displacement in any particular direction.

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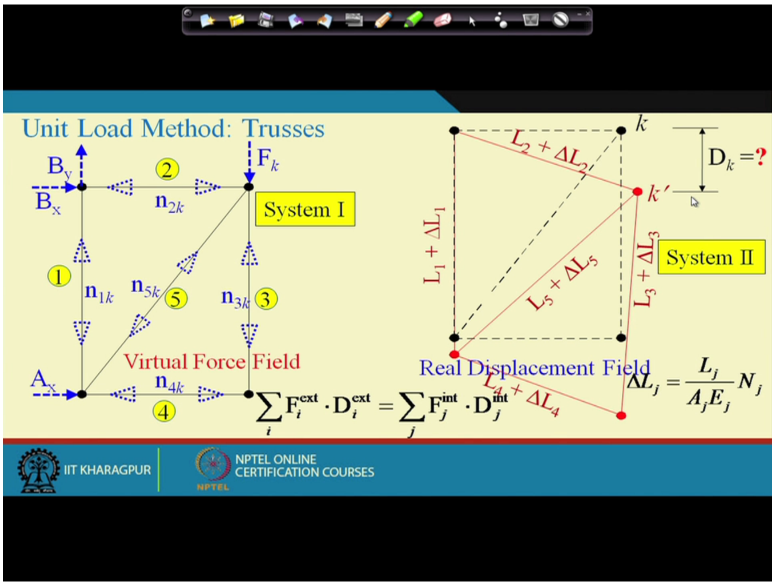

What we want to have to do is we have to apply a virtual force of 1 unit load in the direction. Suppose this is the displacement we want in vertical direction, right?

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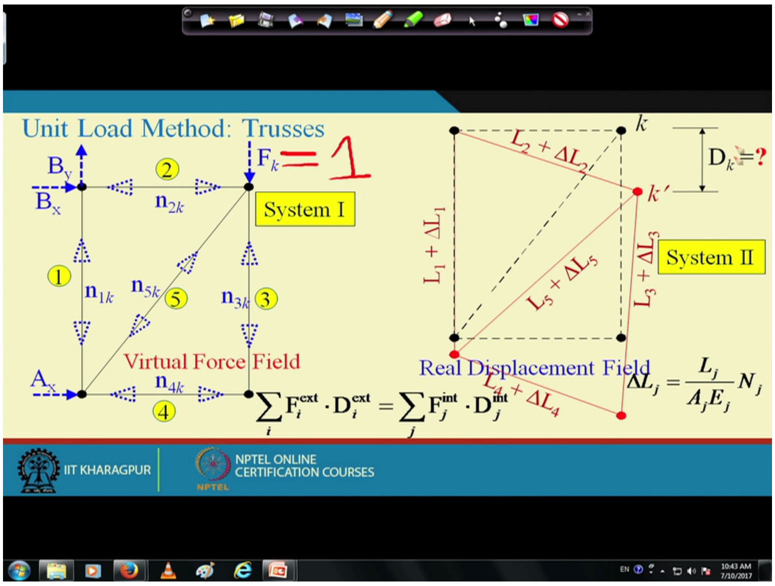

Therefore the force needs to be applied in vertical direction, okay. We apply a vertical force 1 unit load in the direction which we want to determine a displacement. Now this virtual load will generate some internal forces, okay. And then we also know the internal (for) elongation of this member by this.

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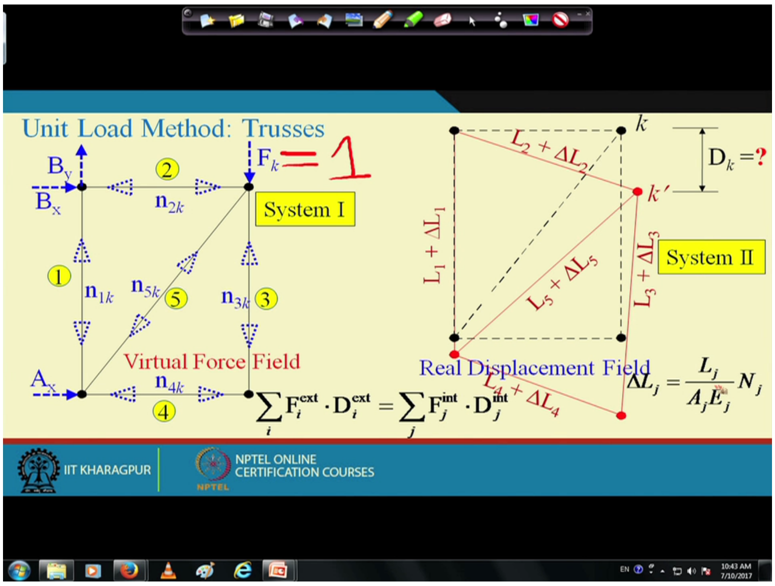

And then (app) take this as (in) F external F internal this is D external and these are all D internal.

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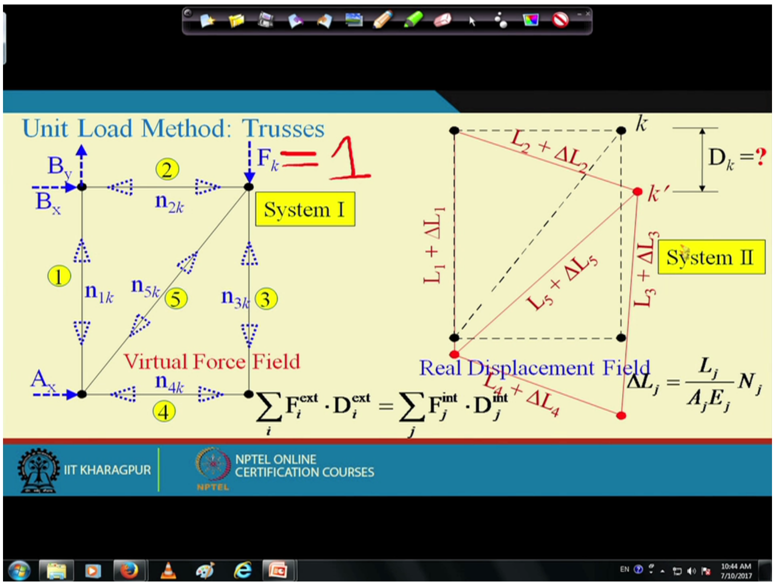

Substitute this equation and get final expression like this. And this is called unit load method.

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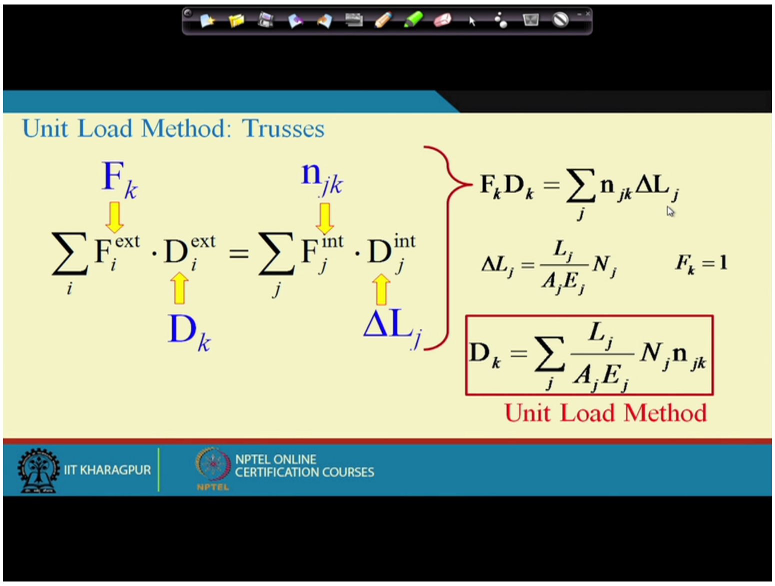

So we have just (demons) introduced unit load method. Probably when you demonstrate unit load method through some examples the concepts will be clear and that we will be doing in the next class. So next class what we will do we will determine the deflection of truss using unit load method, okay.

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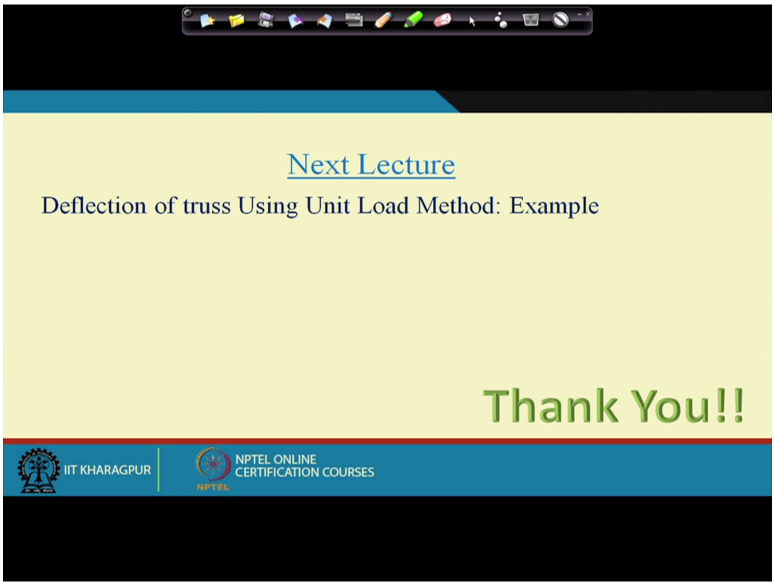

Then I stop here. See you in the next class. Thank you.