

Computational Hydraulics
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Lecture 9
Partial Differential Equation: BVP

Welcome to the lecture number 9 of the course computational hydraulics. We are in model number 2 numerical methods. And in this particular class we will be covering unit 5, partial differential equation with boundary value problems.

(Refer Slide Time 00:30)

Finite Difference Approximations
Problem Domain

Module 02: Numerical Methods
Unit 05: Partial Differential Equation: BVP

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Dr. Anirban Dhar NPTEL Computational Hydraulics 1 / 23

What are the learning objective for this particular unit? First objective is to discretize the derivative of single valued multidimensional functions using finite difference approximations. And second 1 is to derive the algebraic form using discretized partial differential equation and boundary conditions.

(Refer Slide Time 01:15)

Finite Difference Approximations
Problem Domain

Learning Objectives

- To discretize the derivatives of **single-valued multi-dimensional functions** using finite difference approximations.
- To derive the algebraic form using discretized PDE and BCs.

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Let us consider a surface in 3 dimension and with x and y this ϕ surface is varying. So ϕ is a function of x and y only.

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Finite Difference Approximations
Problem Domain

Finite Difference

$\phi(x,y)$

ϕ

x

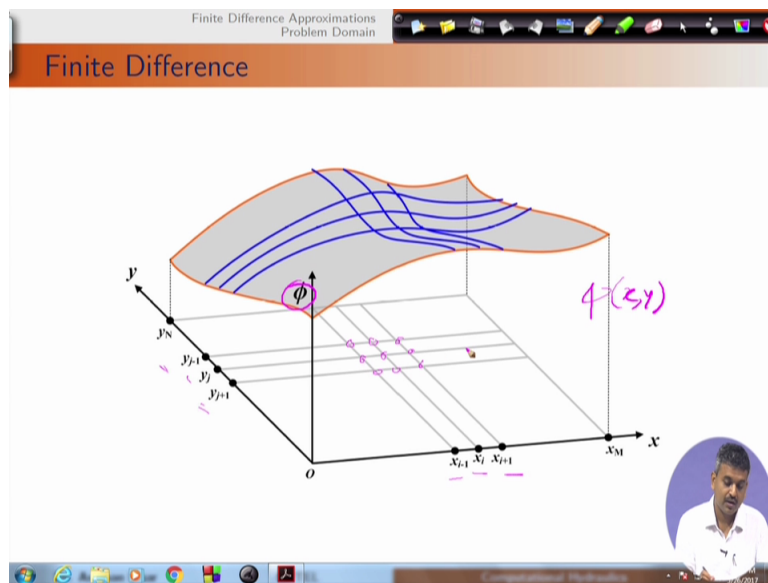
y

o

Computational Hydraulics 7/26/2017

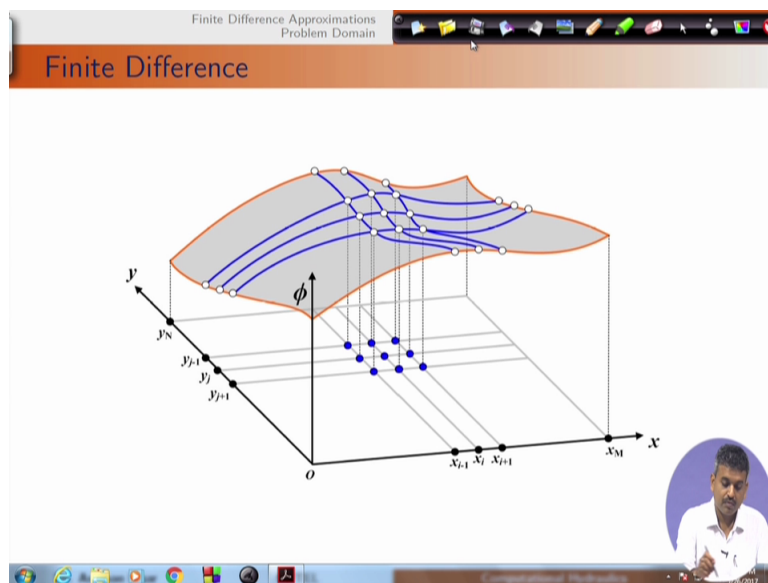
If we discretize this using finite grid size then with this rectangular domain for this i , i minus 1, i plus 1, j , j minus 1, j plus 1, j minus 1. We can get internal general points and for those points we can define our partial derivatives and corresponding finite difference approximations.

(Refer Slide Time 02:18)



So for this internal points if we extend it to the surface we will get corresponding function values. So obviously these functions values are at discrete points. And intermediate points we do not have information.

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So with this setup we can use the Taylor series expansion. In Taylor series expansion for that dependent variable phi x and y these two are independent variables. So with increment del x and del y in two directions we can write this phi xy plus delta x into del phi by del x, delta y into del phi by del y, plus second order term for this 1.

(Refer Slide Time 03:44)

Finite Difference Approximations
Problem Domain

Taylor Series

Taylor series expansion for a function with two independent variables can be expressed as,

$$\begin{aligned} \phi(x + \Delta x, y + \Delta y) &= \sum_{\eta_x=0}^{\infty} \sum_{\eta_y=0}^{\infty} \frac{\Delta x^{\eta_x} \Delta y^{\eta_y}}{\eta_x! \eta_y!} \frac{\partial^{\eta_x + \eta_y} \phi(x, y)}{\partial x^{\eta_x} \partial y^{\eta_y}} \\ &= \phi(x, y) + \Delta x \frac{\partial \phi}{\partial x} + \Delta y \frac{\partial \phi}{\partial y} + \frac{1}{2!} \left[\Delta x^2 \frac{\partial^2 \phi}{\partial x^2} + 2\Delta x \Delta y \frac{\partial^2 \phi}{\partial x \partial y} + \Delta y^2 \frac{\partial^2 \phi}{\partial y^2} \right] + \dots \end{aligned}$$

We can utilize this information to get the approximation of partial derivatives. So for the two dimensional domain let us say it is starting from x_0 to x_M . That means we have M numbers of segments and m plus 1 number of grid points or node points in x direction.

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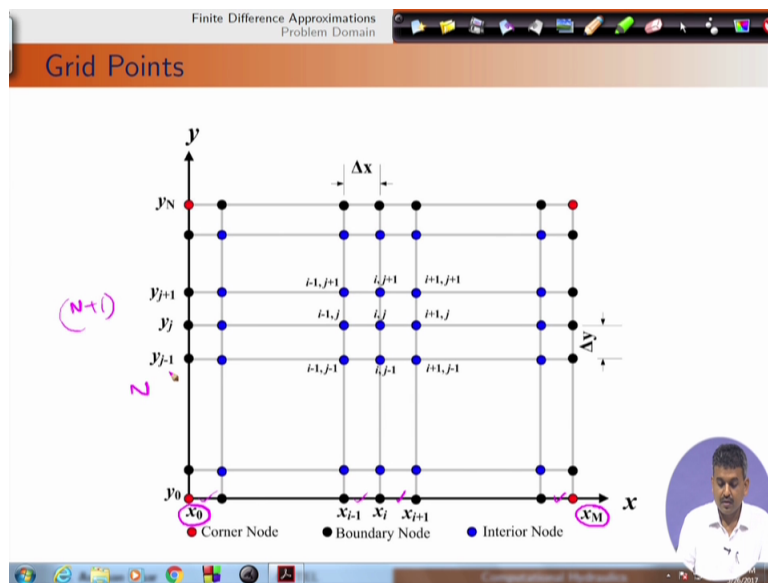
Finite Difference Approximations
Problem Domain

Grid Points

● Corner Node
 ● Boundary Node
 ● Interior Node

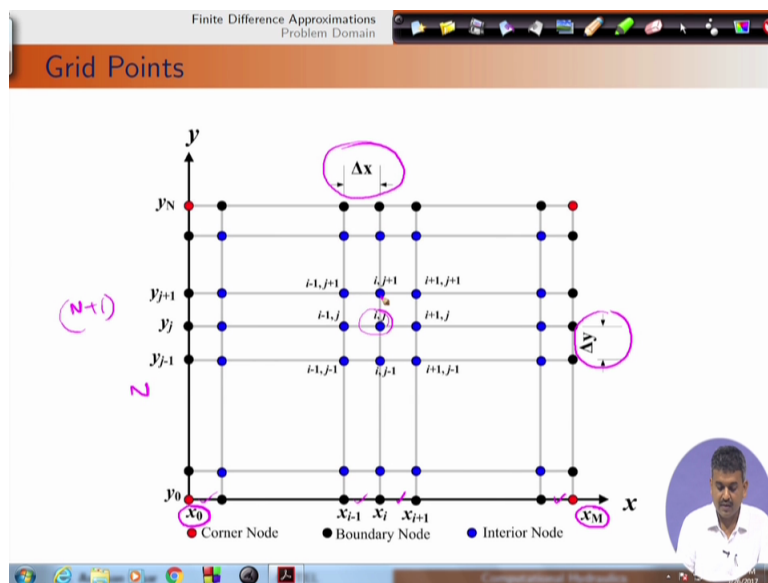
N plus 1 number of node points or grid points in y direction. Again we have N number of segments in y direction.

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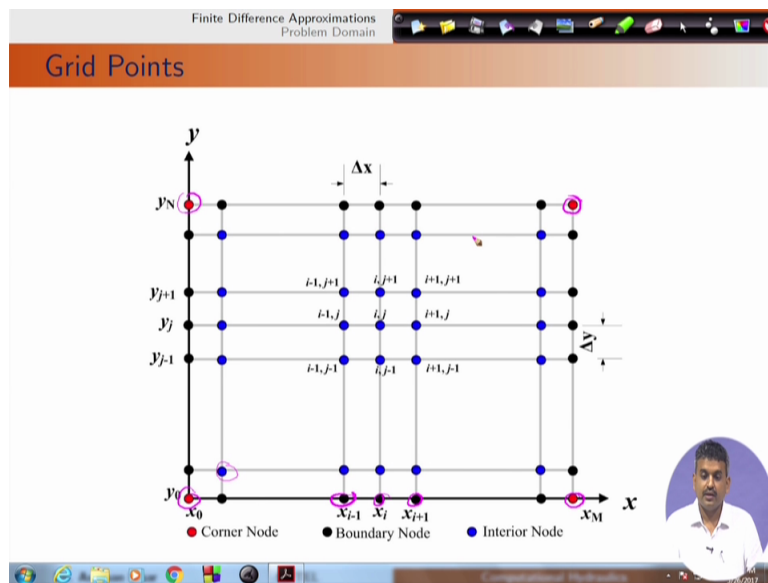
So Δx and Δy these are constant values for this particular configuration. i, j is the central point for any general discretization stencil. So with this configuration we can start discretization of partial derivatives.

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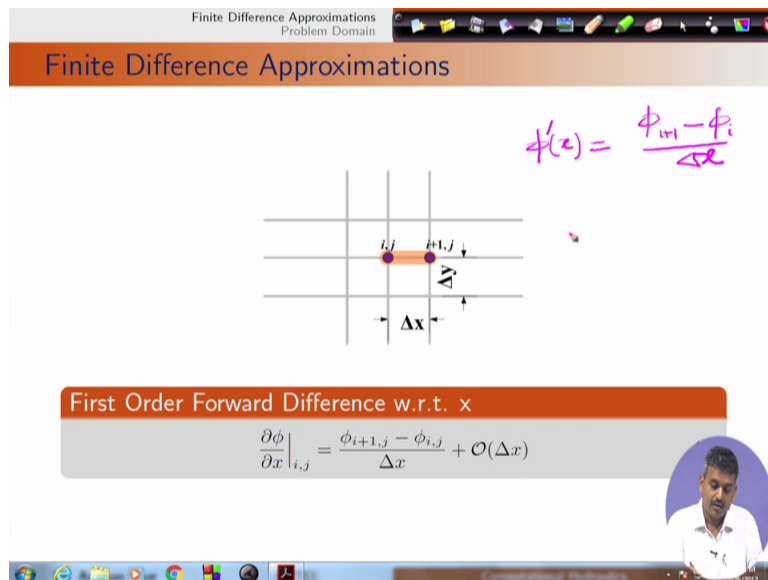
In this case we have black dots are boundary nodes, red dots are corner nodes and blue dots are interior nodes. We have seen in our ordinary differential equation discretization that we need to specify the boundary conditions at boundary nodes and governing equation for interior nodes.

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So first approximation is forward difference with respect to x . So in our discretization we have seen for single valued with single dependent variable if we take the forward difference obviously this is $\phi_{i+1} - \phi_i$ divided by Δx . In this case we are using the same concept and we are extending it for partial derivative.

(Refer Slide Time 06:53)



And in case of partial derivative we need to consider the second dependent variable independent variable and there is no change in that independent variable because this is derivative with respect to a particular independent variable x . That's why we are increasing the index for x .

(Refer Slide Time 07:24)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$$\phi'(x) = \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

First Order Forward Difference w.r.t. x

$$\left. \frac{\partial \phi}{\partial x} \right|_{(i,j)} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + \mathcal{O}(\Delta x)$$

Computer Graphics

And in this case we have first order accuracy like our ordinary differential equation approximation and finite difference.

(Refer Slide Time 07:31)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$$\phi'(x) = \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

First Order Forward Difference w.r.t. x

$$\left. \frac{\partial \phi}{\partial x} \right|_{(i,j)} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + \mathcal{O}(\Delta x)$$

Computer Graphics

Similar thing is for first order backward difference with respect to x. We need to consider i, i minus 1 point. Here again there is no change in the index for j but there is change in the i and i minus 1.

(Refer Slide Time 08:09)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$i-1,j$ i,j

Δx Δy

First Order Backward Difference w.r.t. x

$$\left. \frac{\partial \phi}{\partial x} \right|_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + \mathcal{O}(\Delta x)$$

Further this second order center difference. Second order center difference for first order. This is first order derivative, this is second order accuracy. So $i+1$ and $i-1$. Two points and this is our point i,j .

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Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$i-1,j$ i,j $i+1,j$

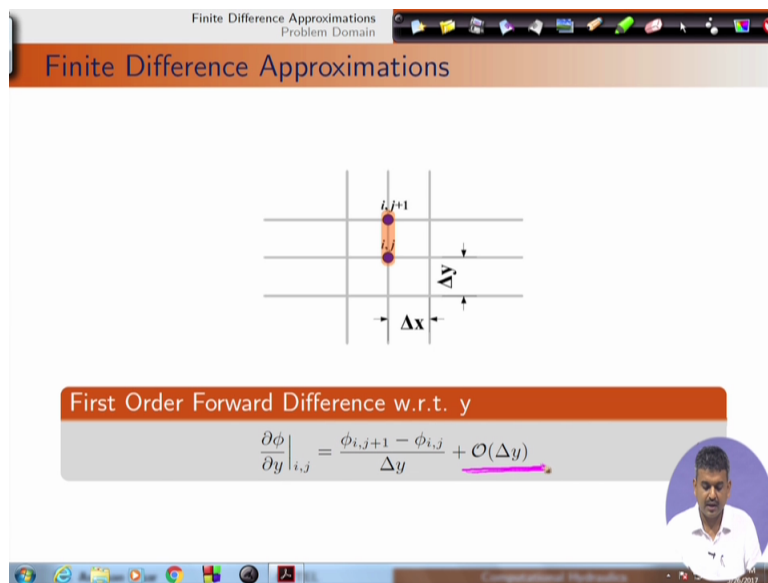
Δx Δy

Second Order Center Difference w.r.t. x

$$\left. \frac{\partial \phi}{\partial x} \right|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

We can again discretize the derivatives in another direction. The first order forward difference with respect to y , again this is with Δy accuracy.

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Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$i,j+1$
 i,j
 Δy
 Δx

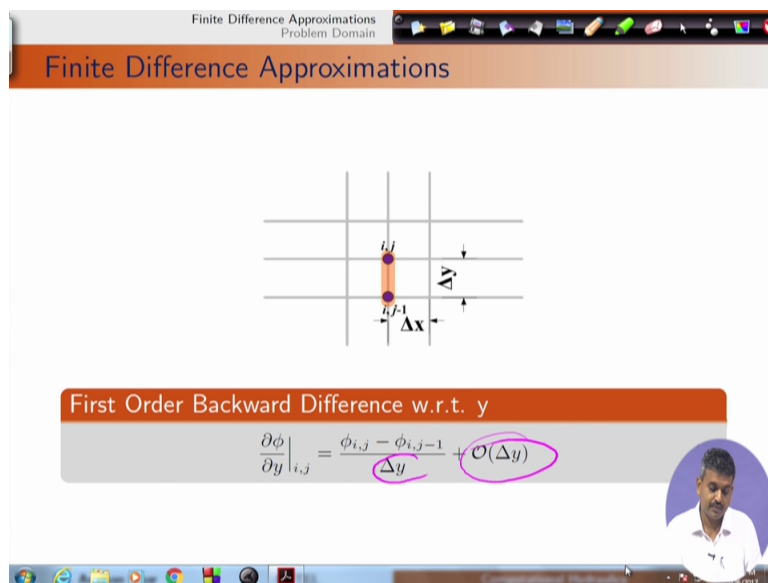
First Order Forward Difference w.r.t. y

$$\frac{\partial \phi}{\partial y} \Big|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} + \mathcal{O}(\Delta y)$$

Computer Graphics

If we consider backward difference with respect to y this is again with Δy and this is with Δy accuracy first order accurate.

(Refer Slide Time 09:25)



Finite Difference Approximations
Problem Domain

Finite Difference Approximations

i,j
 $i,j-1$
 Δy
 Δx

First Order Backward Difference w.r.t. y

$$\frac{\partial \phi}{\partial y} \Big|_{i,j} = \frac{\phi_{i,j} - \phi_{i,j-1}}{\Delta y} + \mathcal{O}(\Delta y)$$

Computer Graphics

And with second order center difference we have change in the index for j , j plus 1 and j minus 1. This is again second order accurate method.

(Refer Slide Time 09:41)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$i,j+1$
 $i,j-1$
 Δy
 Δx

Second Order Center Difference w.r.t. y

$$\frac{\partial \phi}{\partial y} \Big|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y} + \mathcal{O}(\Delta y^2)$$

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We have two extreme points we have i and j .

(Refer Slide Time 09:51)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

$i,j+1$
 i,j
 $i,j-1$
 Δy
 Δx

Second Order Center Difference w.r.t. y

$$\frac{\partial \phi}{\partial y} \Big|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y} + \mathcal{O}(\Delta y^2)$$

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Finite difference approximation this is for second order center difference with respect to x . This is second order derivative we have $\phi_{i,j-1}, \phi_{i,j}, \phi_{i,j+1}$, this is Δx square this is similar to our single variable case. And again this is Δx square accuracy. Interesting point is that for this second order derivative we need three points.

(Refer Slide Time 10:42)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Diagram illustrating the second-order center difference approximation with respect to x . The grid shows points $(i-1, j)$, (i, j) , and $(i+1, j)$ along the x -axis. The spacing between points is Δx .

Second Order Center Difference w.r.t. x

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j} = \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

If we consider second order center difference with respect to y , so in this case also this is change in i minus 1, i plus 1, j , j plus 1, j minus 1 and j . This is Δy square overall accuracy of Δy square.

(Refer Slide Time 11:10)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Diagram illustrating the second-order center difference approximation with respect to y . The grid shows points $(i, j-1)$, (i, j) , and $(i, j+1)$ along the y -axis. The spacing between points is Δy .

Second Order Center Difference w.r.t. y

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j} = \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

And this is mixed difference with respect to x and y . We need to consider the extreme points. We are considering the mix derivative at ij . However we need to consider the points in diagonal direction.

(Refer Slide Time 11:47)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Second Order Center Mixed Difference w.r.t. x and y

$$\frac{\partial^2 \phi}{\partial x \partial y} \Big|_{(i,j)} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2)$$

So this is i plus 1, j plus 1, which is positive. Then i minus 1, j minus 1, this 1 and i minus 1, j plus 1, this is i plus 1, j minus 1.

(Refer Slide Time 12:09)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Second Order Center Mixed Difference w.r.t. x and y

$$\frac{\partial^2 \phi}{\partial x \partial y} \Big|_{(i,j)} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2)$$

This two are with positive signs, this diagonally these two are with negative signs. Divided by 4 delta x delta y and this is our delta x square delta y square accuracy.

(Refer Slide Time 12:31)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Second Order Center Mixed Difference w.r.t. x and y

$$\frac{\partial^2 \phi}{\partial x \partial y} \Big|_{i,j} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2) \quad (9)$$

So in this case we have two independent variable and we are considering variation for both the variables. That's why we need to show this (accu)order of accuracy in term of both the independent variable.

(Refer Slide Time 12:56)

Finite Difference Approximations
Problem Domain

Finite Difference Approximations

Second Order Center Mixed Difference w.r.t. x and y

$$\frac{\partial^2 \phi}{\partial x \partial y} \Big|_{i,j} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y} + \mathcal{O}(\Delta x^2, \Delta y^2) \quad (9)$$

This is a general form of differential equation with a general variable phi. We have already discussed this in our earlier lecture and in this case phi is some general variable lambda and epsilon these are problem dependent parameters. And this gamma phi is a tensor if phi0 or f phi0 other forces, Sphi is source sink term for this one.

(Refer Slide Time 13:50)

Finite Difference Approximations
Problem Domain

General Equation

A form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \cdot \nabla \phi) + F_{\phi_o} + S_\phi \quad (10)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_o} = other forces
- S_ϕ = source/sink term

So if we approximate this equation and we utilize it for defining partial differential equations as boundary value problem then we need to neglect this term zero, this advective term as zero. This is also zero.

(Refer Slide Time 14:16)

Finite Difference Approximations
Problem Domain

General Equation

A form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \cdot \nabla \phi) + F_{\phi_o} + S_\phi \quad (10)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_o} = other forces
- S_ϕ = source/sink term

We will consider only these two terms. In this case del operator we are considering only variation of x and variation of y. These are unique vector in x and y direction. So this is del operator. So with this information we can use a simplified governing equation to define the boundary value problem.

(Refer Slide Time 14:49)

Finite Difference Approximations
Problem Domain

General Equation

A form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \cdot \nabla \phi) + F_{\phi_o} + S_\phi \quad (10)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_o} = other forces
- S_ϕ = source/sink term

$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}$

In this case without any cross term we have defined this $\text{del}^2 \phi / \text{del} x^2$. So obviously ϕ in previous case it's a two dimensional tensor with γ_x, γ_y , and cross terms are zero.

(Refer Slide Time 15:40)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\Gamma_D^1 : \phi(0, y) = \phi_1$$

$$\Gamma_D^2 : \phi(L_x, y) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$$

$\Gamma = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_y \end{bmatrix}$

So if we simplify this and consider that γ_x, γ_y these two are constant then we can write in this format that means γ_y and x these are not varying with x and y . So S_ϕ is some source sink term.

(Refer Slide Time 15:59)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\Gamma_D^1 : \phi(0, y) = \phi_1$$

$$\Gamma_D^2 : \phi(L_x, y) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$$

We can define boundary condition for a rectangular domain. So we have rectangular domain, this is l_x for that rectangular domain l_y for this y direction and this is zero zero point.

(Refer Slide Time 16:32)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\Gamma_D^1 : \phi(0, y) = \phi_1$$

$$\Gamma_D^2 : \phi(L_x, y) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$$

So with this information we have defined these four boundary conditions gamma d1 that means this left boundary at x is equal to zero and y this is Dirichlet kind of boundary or specified boundary, ϕ_1 .

(Refer Slide Time 16:54)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation
A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\Gamma_D^1 : \phi(0, y) = \phi_1$$

$$\Gamma_D^2 : \phi(L_x, y) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$$

And gamma 2d this is again Dirichlet boundary with x is equal to lx and for all y we have this phi two value. This is phi1.

(Refer Slide Time 17:14)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation
A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$$\Gamma_D^1 : \phi(0, y) = \phi_1$$

$$\Gamma_D^2 : \phi(L_x, y) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$$

And for gamma three, this is gamma n3, this is actually phi n. In y direction there is no variation. And top we have del phi by del y equals to zero. So we can see that for this boundary value problem values are either specified for all boundaries or they are written in terms of boundary conditions.

(Refer Slide Time 17:40)

Finite Difference Approximations
Problem Domain

Problem Definition

Governing equation

A two-dimensional BVP can be written as,

$$\Omega : \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y) = 0$$

subject to

Boundary Condition

$\Gamma_D^1 : \phi(0, y) = \phi_1$
 $\Gamma_D^2 : \phi(L_x, y) = \phi_2$
 $\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x,0)} = 0$
 $\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x,L_y)} = 0$

So domain discretization, this is L_x this is L_y Dirichlet, Dirichlet boundary, Neumann, Neumann boundary.

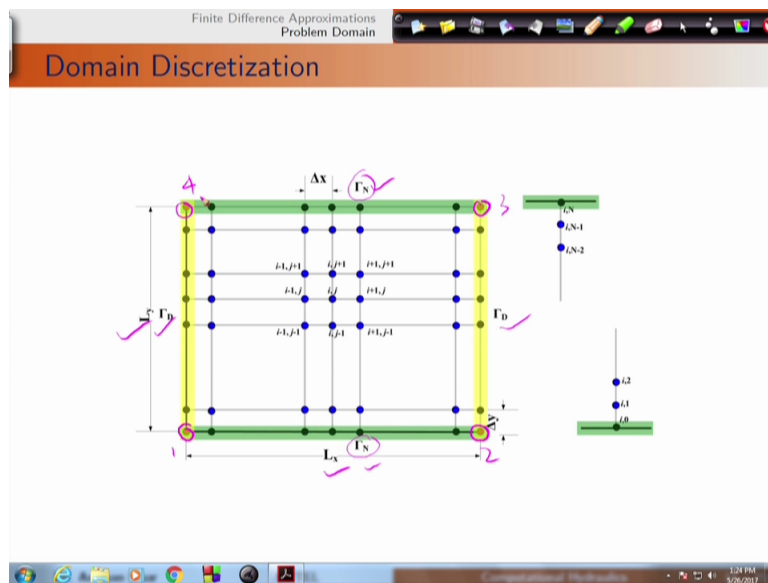
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Finite Difference Approximations
Problem Domain

Domain Discretization

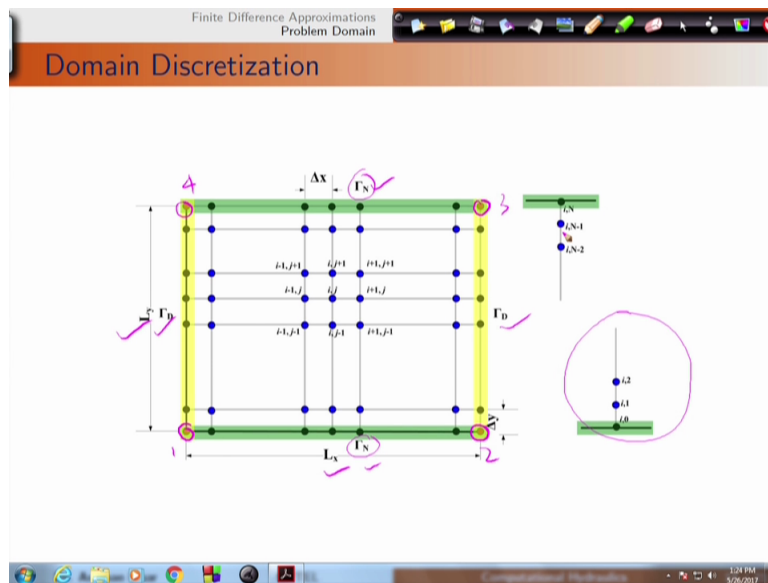
Important point is for this corner points. For corner points either we can consider it in this gamma n domain or in gamma d domain. In this case we can consider it in gamma d domain because in this case let us say that value is specified for these points these four points 1, two, three and four.

(Refer Slide Time 18:57)



For this green portion that is Neumann boundary we can define the boundary condition based on three points or two points depending on the desired accuracy.

(Refer Slide Time 19:18)



So let us consider the discretization of governing equation. In this case we have discretized the governing equation with second order accurate scheme γ_x, γ_y into this.

(Refer Slide Time 19:45)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

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This is second order accurate. Minus $S_\phi|_{i,j}$ which is specified value source sink term. We have transferred it into right hand side.

(Refer Slide Time 19:54)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

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So this equation can be arranged as following. So with this, this i, j minus 1. That means if we have any general i, j structure then this is i, j minus 1, this i, j plus 1, this is i minus 1 j , this is i plus 1 j .

(Refer Slide Time 20:54)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2} \phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2} \phi_{i-1,j} - 2 \left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2} \right) \phi_{i,j} + \frac{\Gamma_x}{\Delta x^2} \phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2} \phi_{i,j+1} = -S_\phi|_{i,j}$$

In this case this coefficient we are starting with ij minus 1 this is first, then i minus 1 j this is second, ij this is third, i plus 1 j this is fourth and fifth one is ij plus 1. So we have considered the coefficient for all these points.

(Refer Slide Time 21:21)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2} \phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2} \phi_{i-1,j} - 2 \left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2} \right) \phi_{i,j} + \frac{\Gamma_x}{\Delta x^2} \phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2} \phi_{i,j+1} = -S_\phi|_{i,j}$$

Now if we simplify this by using this notation that α_x and α_y can be written as γ_x by Δx square and γ_y by Δy square.

(Refer Slide Time 21:50)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2} \phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2} \phi_{i-1,j} - 2 \left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2} \right) \phi_{i,j} + \frac{\Gamma_x}{\Delta x^2} \phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2} \phi_{i,j+1} = -S_\phi|_{i,j}$$

In simplified form, this can be written as

$$\alpha_y \phi_{i,j-1} + \alpha_x \phi_{i-1,j} - 2(\alpha_x + \alpha_y) \phi_{i,j} + \alpha_x \phi_{i+1,j} + \alpha_y \phi_{i,j+1} = -S_\phi|_{i,j}$$

with $\alpha_x = \frac{\Gamma_x}{\Delta x^2}$ and $\alpha_y = \frac{\Gamma_y}{\Delta y^2}$.

So we can write this in simple form but the problem is we cannot construct the algebraic matrix forms because we have double index notation present. In 1 dimension single index is possible to form this matrix easily.

(Refer Slide Time 22:30)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation

The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2} \phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2} \phi_{i-1,j} - 2 \left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2} \right) \phi_{i,j} + \frac{\Gamma_x}{\Delta x^2} \phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2} \phi_{i,j+1} = -S_\phi|_{i,j}$$

In simplified form, this can be written as

$$\alpha_y \phi_{i,j-1} + \alpha_x \phi_{i-1,j} - 2(\alpha_x + \alpha_y) \phi_{i,j} + \alpha_x \phi_{i+1,j} + \alpha_y \phi_{i,j+1} = -S_\phi|_{i,j}$$

with $\alpha_x = \frac{\Gamma_x}{\Delta x^2}$ and $\alpha_y = \frac{\Gamma_y}{\Delta y^2}$.

But individual points will have individual governing equations or boundary condition equations. These individual points will have individual equations. So these points are itself in ij format. So we cannot construct the matrix directly.

(Refer Slide Time 23:00)

Finite Difference Approximations
Problem Domain

Numerical Discretization

Governing Equation


The governing equation can be discretized as,

$$\Gamma_x \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} + \mathcal{O}(\Delta x^2, \Delta y^2) = -S_\phi|_{i,j}$$

The equation can be rearranged as,

$$\frac{\Gamma_y}{\Delta y^2} \phi_{i,j-1} + \frac{\Gamma_x}{\Delta x^2} \phi_{i-1,j} - 2 \left(\frac{\Gamma_x}{\Delta x^2} + \frac{\Gamma_y}{\Delta y^2} \right) \phi_{i,j} + \frac{\Gamma_x}{\Delta x^2} \phi_{i+1,j} + \frac{\Gamma_y}{\Delta y^2} \phi_{i,j+1} = -S_\phi|_{i,j}$$

In simplified form, this can be written as



$$\alpha_y \phi_{i,j-1} + \alpha_x \phi_{i-1,j} - 2(\alpha_x + \alpha_y) \phi_{i,j} + \alpha_x \phi_{i+1,j} + \alpha_y \phi_{i,j+1} = -S_\phi|_{i,j}$$

with $\alpha_x = \frac{\Gamma_x}{\Delta x^2}$ and $\alpha_y = \frac{\Gamma_y}{\Delta y^2}$.

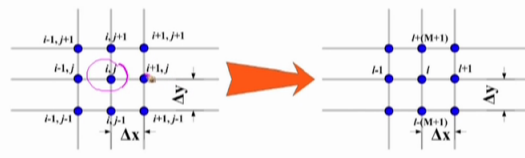

So what we can do, we can introduce single index notation. Single index L can be written as i j m plus 1. This is ij. This point can be represented as L.

(Refer Slide Time 23:21)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$



So L minus 1 is basically i minus j point, L plus 1 is i plus 1 j, L minus m plus 1 this is ij minus 1 and L plus m plus 1 is ij plus 1 point.

(Refer Slide Time 23:49)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

The diagram shows a 2D grid of nodes. The horizontal axis is labeled Δx and the vertical axis is labeled Δy . Nodes are labeled with coordinates (i,j) . An orange arrow points from a node (i,j) to a single index l . The equation $l = i + j \times (M + 1)$ is shown. A small video inset of a man is in the bottom right corner.

We are starting from point 0 to m , that's why we have $m + 1$ number of points. So if we take the next level in y direction so obviously there will be difference of $m + 1$ number of nodes.

(Refer Slide Time 24:16)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

The diagram shows a 2D grid of nodes. The horizontal axis is labeled Δx and the vertical axis is labeled Δy . Nodes are labeled with coordinates (i,j) . An orange arrow points from a node (i,j) to a single index l . The equation $l = i + j \times (M + 1)$ is shown. A small video inset of a man is in the bottom right corner. Handwritten notes $0 - M (M+1)$ are at the bottom.

So with this information we can construct our grid system with 0 as starting point, m here and finally this will give the maximum number of points.

(Refer Slide Time 24:47)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

0 — M (M+1)

We have started from this point then we will move towards this.

(Refer Slide Time 24:55)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

0 — M (M+1)

Again the next level will start from here this point.

(Refer Slide Time 25:00)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

$0 \quad \dots \quad M \quad (M+1)$

Again we will move in this direction again we will come back to this point and move to this direction.

(Refer Slide Time 25:11)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

$0 \quad \dots \quad M \quad (M+1)$

So i we are starting with j . For j equals to zero level we have m plus 1 number of points starting from zero to M . Then we will have m plus 1, like that we can define our nodal points with single index notation so that we can easily form the final matrix for solution.

(Refer Slide Time 25:54)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation the equation can be written as, in this case we have this kind of stencils that we have L , L minus 1, L plus 1. So we have five points.

(Refer Slide Time 26:33)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation, the equation can be written as,

$$\alpha_y \phi_{l-(M+1)} + \alpha_x \phi_{l-1} - 2(\alpha_x + \alpha_y) \phi_l + \alpha_x \phi_{l+1} + \alpha_y \phi_{l+(M+1)} = -S$$

So m plus 1, L minus 1, L , L plus 1, L plus m plus 1. And this S phi is basically defined for ij or S phi we can write it in terms of L . So we can use this equation for interior points or blue points. For boundary points we need to define the boundary conditions.

(Refer Slide Time 27:13)

Finite Difference Approximations
Problem Domain

Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation, the equation can be written as,

$$\alpha_y \phi_{l-(M+1)} + \alpha_x \phi_{l-1} - 2(\alpha_x + \alpha_y) \phi_l + \alpha_x \phi_{l+1} + \alpha_y \phi_{l+(M+1)} = -S \phi_{i,j}$$

$= -S \phi_l$

Windows taskbar: 1:12 PM 3/26/2017

For Dirichlet boundary things are clear because we can directly specify the boundary conditions without any error. But Neumann boundary we need to consider the second order discretization because we have second order accurate scheme here for governing equation.

(Refer Slide Time 27:43)

Finite Difference Approximations
Problem Domain

Neumann Boundary Condition

Top Boundary

Second Order Discretization

$$\frac{3\phi_{i,N} - 4\phi_{i,N-1} + \phi_{i,N-2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \quad (11)$$

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So we can include n, n minus 1, n minus 2 points. And this is second order accurate.

(Refer Slide Time 27:53)

The slide is titled "Finite Difference Approximations Problem Domain" and "Neumann Boundary Condition". It features a diagram of a vertical line representing a domain with a green horizontal bar at the top. Points are labeled i,N , $i,N-1$, and $i,N-2$ from top to bottom. Below the diagram, a box titled "Top Boundary" contains the text "Second Order Discretization" and the equation
$$\frac{3\phi_{i,N} - 4\phi_{i,N-1} + \phi_{i,N-2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \quad (11)$$
 A small circular inset shows a man speaking.

With this information we can use the single index notation to represent the thing. And we can directly use this one as equation for matrix form.

(Refer Slide Time 28:13)

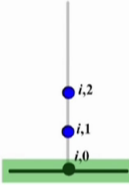
The slide is identical to the previous one, but the equation in the "Top Boundary" box is now written in single index notation:
$$\frac{3\phi_l - 4\phi_{l-(M+1)} + \phi_{l-2(M+1)}}{2\Delta y} = 0$$
 The text "In single index notation format," is written above the equation. A small circular inset shows a man speaking.

Similarly for this is for the top boundary, we can use it for bottom boundary zero, 1, 2 points and we can use the single index notation to represent the boundary condition.

(Refer Slide Time 28:35)

Finite Difference Approximations
Problem Domain

Neumann Boundary Condition



Bottom Boundary

Second Order Discretization

$$\frac{-3\phi_{i,0} + 4\phi_{i,1} - \phi_{i,2}}{2\Delta y} + \mathcal{O}(\Delta y^2) = 0 \quad (13)$$

In single index notation format,

$$\frac{-3\phi_l + 4\phi_{l+(M+1)} - \phi_{l+2(M+1)}}{2\Delta y} = 0$$

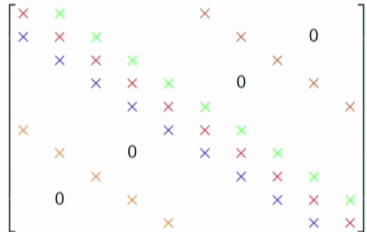
Dr. Anirban Dhar NPTEL Computational Hydraulics 23

Again we are getting some equation for those boundary points. Now with this governing equation in discretize form and boundary conditions we can form the matrix for solution.

(Refer Slide Time 29:00)

Finite Difference Approximations
Problem Domain

Matrix Form



Dr. Anirban Dhar NPTEL Computational Hydraulics 23

Interestingly for interior points we will have five coefficients. For L , L minus 1, L plus 1 and this is L plus m plus 1. This is for L minus m plus 1.

(Refer Slide Time 29:42)

The slide shows a sparse matrix representing a finite difference approximation. The matrix is mostly zero, with non-zero entries (marked with 'x') forming a band structure. Handwritten annotations in purple and blue include:

- $t - (M+1)$ on the left side.
- $t - 1$ and $t + 1$ near the center.
- $t + (M+1)$ on the right side.

 The matrix is enclosed in large square brackets.

This is valid for interior points but if we have boundary points then we need to consider another structure point here which will consider the L minus two m plus 1 and on this side also L plus 2 into m plus 1 to consider the three points.

(Refer Slide Time 30:21)

The slide is titled "Taylor Series" and contains the following text and equation:

Taylor series expansion for a function with two independent variables can be expressed as,

$$\begin{aligned} \phi(x + \Delta x, y + \Delta y) &= \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \frac{\Delta x^{n_x} \Delta y^{n_y}}{n_x! n_y!} \frac{\partial^{n_x+n_y} \phi(x, y)}{\partial x^{n_x} \partial y^{n_y}} \\ &= \phi(x, y) + \Delta x \frac{\partial \phi}{\partial x} + \Delta y \frac{\partial \phi}{\partial y} + \\ &\quad \frac{1}{2!} \left[\Delta x^2 \frac{\partial^2 \phi}{\partial x^2} + 2\Delta x \Delta y \frac{\partial^2 \phi}{\partial x \partial y} + \Delta y^2 \frac{\partial^2 \phi}{\partial y^2} \right] + \dots \end{aligned}$$

Now we can solve this A matrix with phi and define for single index notation. And we will have something bL for right hand side. So with this we can solve this phi L by inverting the A matrix. And this will give solution for the desired problem.

