Computational Hydraulics
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Lecture 9
Partial Differential Equation: BVP

Welcome to the lecture number 9 of the course computational hydraulics. We are in model number 2 numerical methods. And in this particular class we will be covering unit 5, partial differential equation with boundary value problems.

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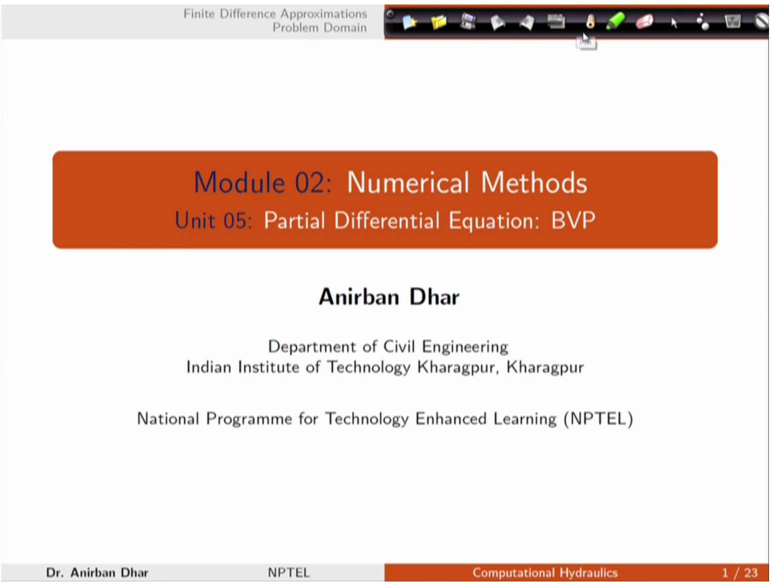

What are the learning objective for this particular unit? First objective is to discretize the derivative of single valued multidimensional functions using finite difference approximations. And second 1 is to derive the algebraic form using discretized partial differential equation and boundary conditions.

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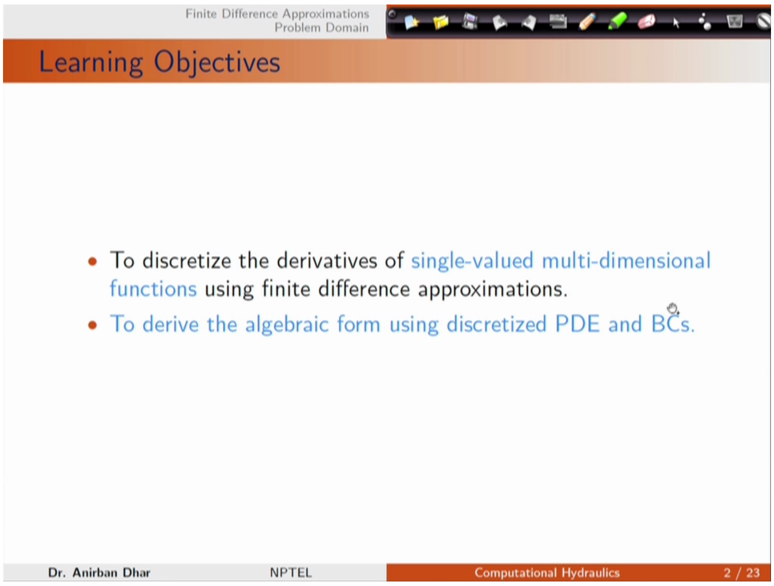

Let us consider a surface in 3 dimension and with x and y this phi surface is varying. So phi is a function of x and y only.

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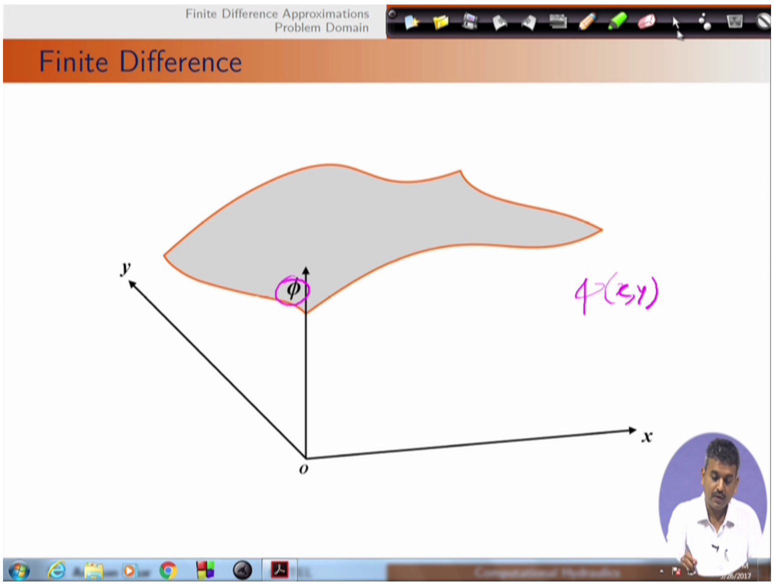

If we discretize this using finite grid size then with this rectangular domain for this i, i minus 1, i plus 1, j, j minus 1,j plus 1, j minus 1. We can get internal general points and for those points we can define our partial derivatives and corresponding finite difference approximations.

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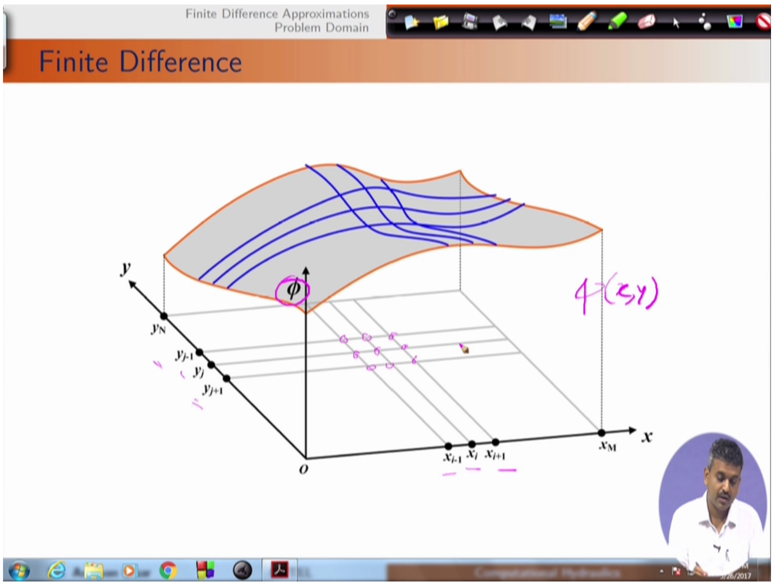

So for this internal points if we extend it to the surface we will get corresponding function values. So obviously these functions values are at discrete points. And intermediate points we do not have information.

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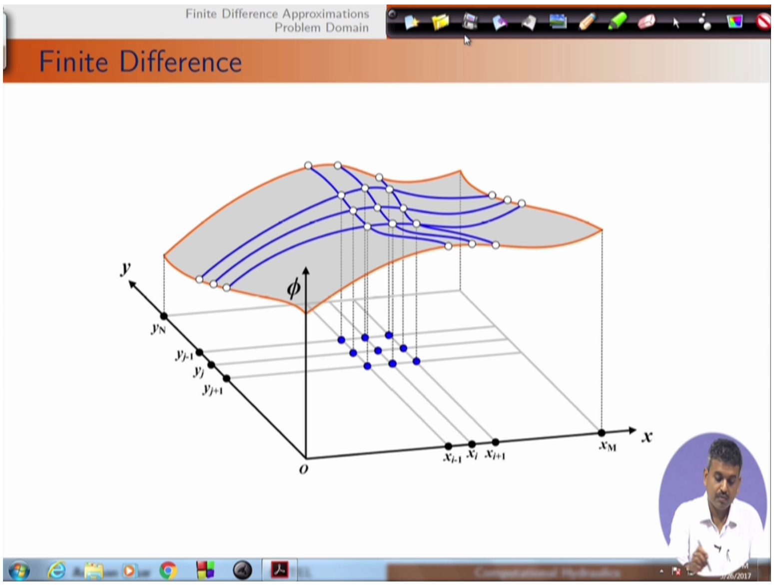

So with this setup we can use the Taylor series expansion. In Taylor series expansion for that dependent variable phi x and y these two are independent variables. So with increment del x and del y in two directions we can write this phi xy plus delta x into del phi by del x, delta y into del phi by del y, plus second order term for this 1.

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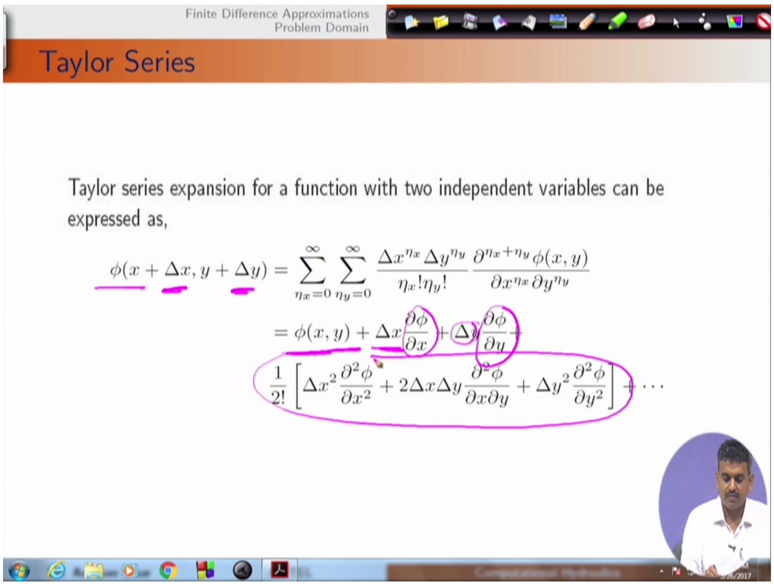

We can utilize this information to get the approximation of partial derivatives. So for the two dimensional domain let us say it is starting from x0 to xM. That means we have M numbers of segments and m plus 1 number of grid points or node points in x direction.

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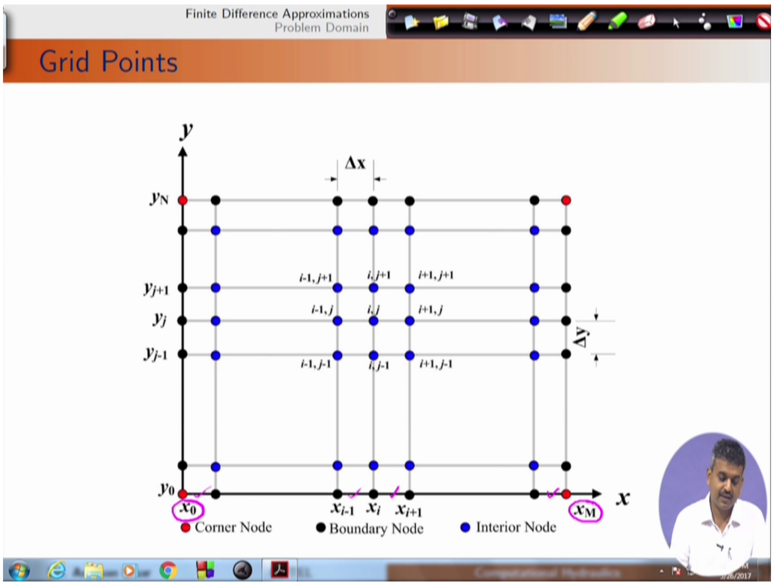

N plus 1 number of node points or grid points in y direction. Again we have N number of segments in y direction.

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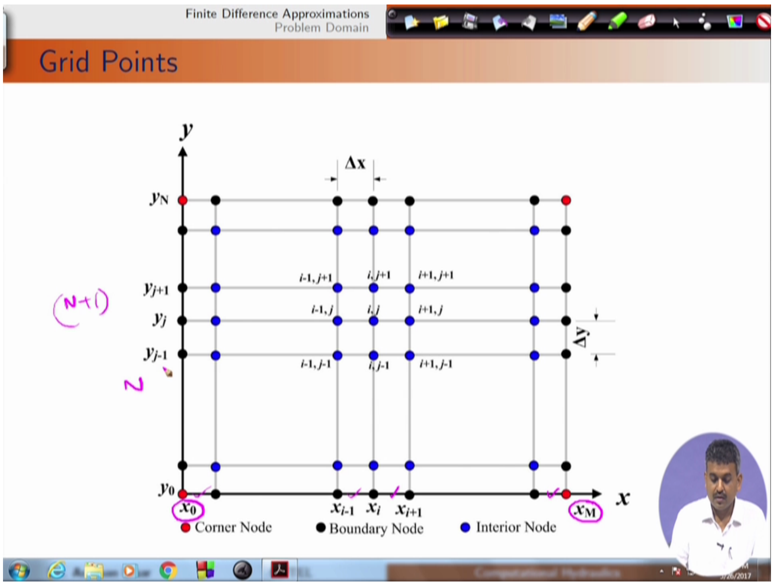

So del x and del y these are constant values for this particular configuration. IJ is the central point for any general discretization stencil. So with this configuration we can start discretization of partial derivatives.

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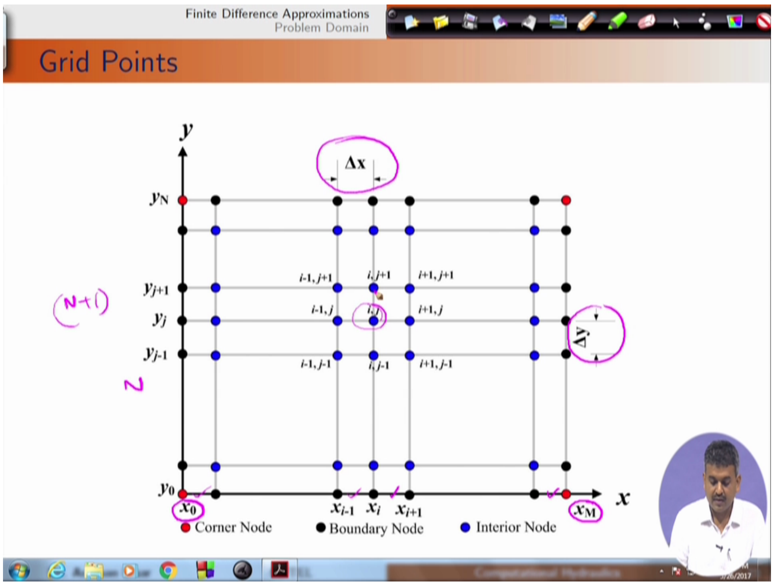

In this case we have black dots are boundary nodes, red dots are corner nodes and blue dots are interior nodes. We have seen in our ordinary differential equation discretization that we need to specify the boundary conditions at boundary nodes and governing equation for interior nodes.

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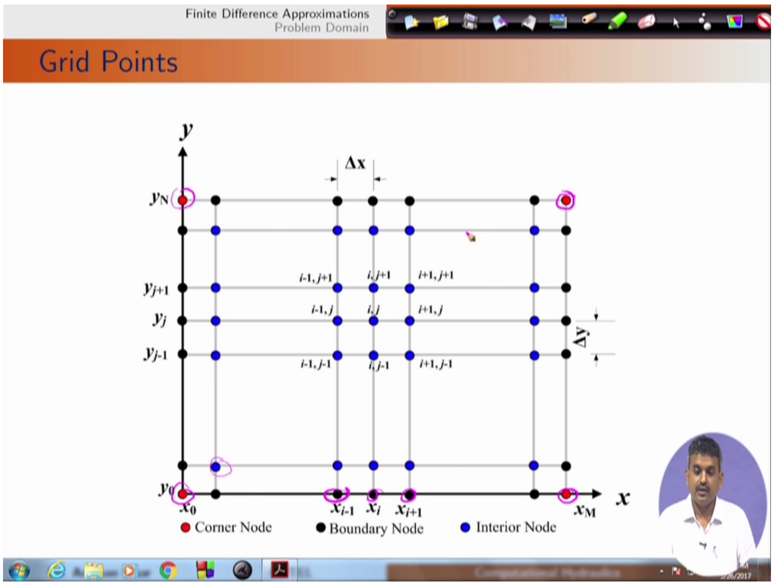

So first approximation is forward difference with respect to x. So in our discretization we have seen for single valued with single dependent variable if we take the forward difference obviously this is phi i plus 1 minus phi i divided by del x. In this case we are using the same concept and we are extending it for partial derivative.

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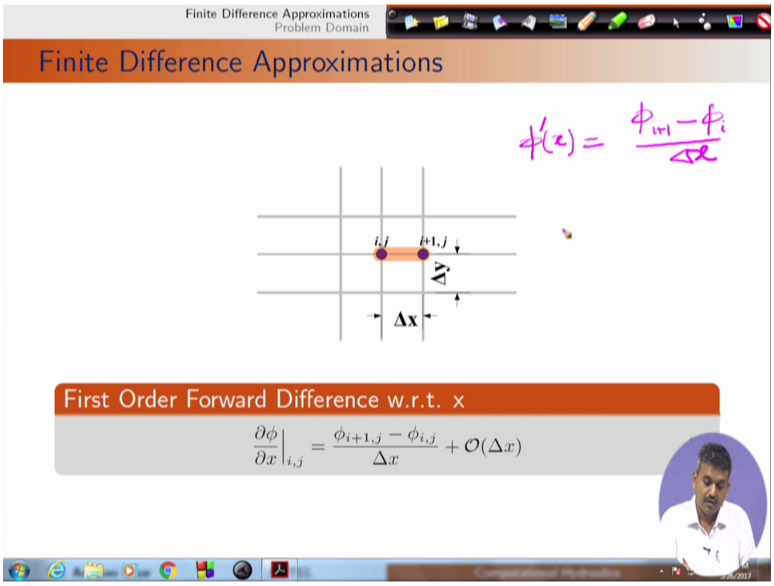

And in case of partial derivative we need to consider the second dependent variable independent variable and there is no change in that independent variable because this is derivative with respect to a particular independent variable x. That's why we are increasing the index for x.

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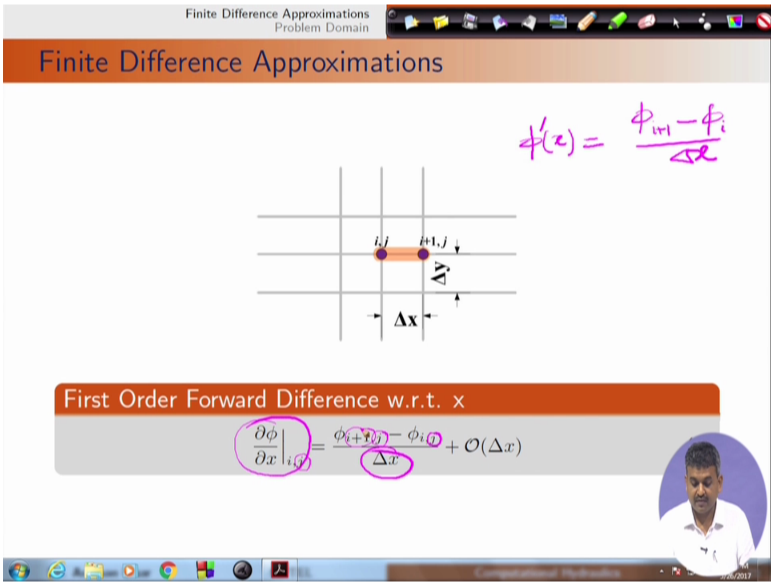

And in this case we have first order accuracy like our ordinary differential equation approximation and finite difference.

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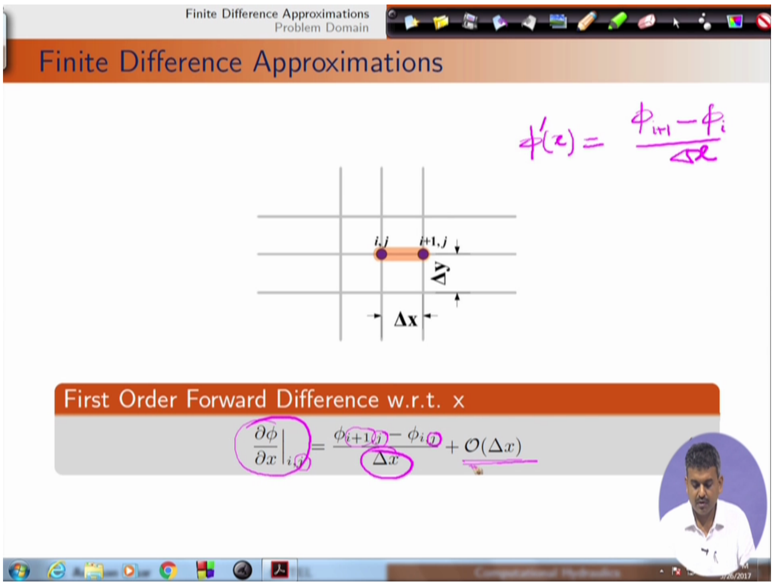

Similar thing is for first order backward difference with respect to x. We need to consider i, i minus 1 point. Here again there is no change in the index for j but there is change in the i and i minus 1.

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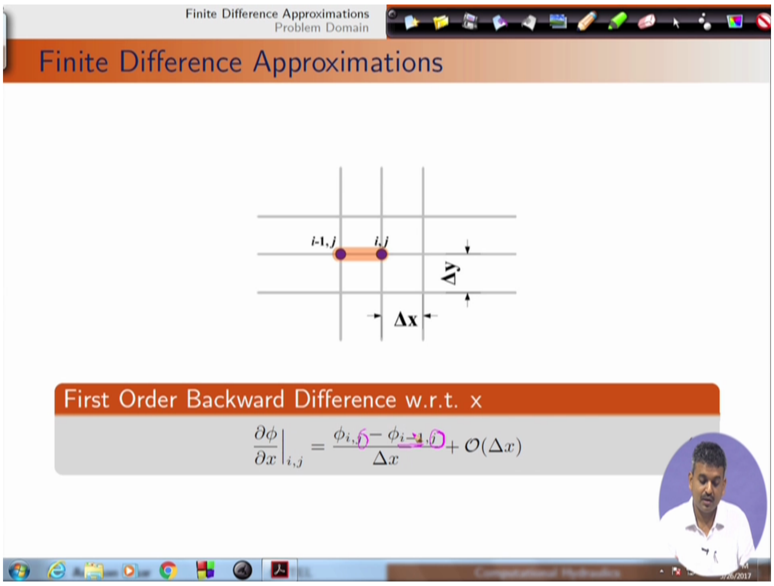

Further this second order center difference. Second order center difference for first order. This is first order derivative, this is second order accuracy. So i plus 1 and phi i minus 1. Two points and this is our point ij.

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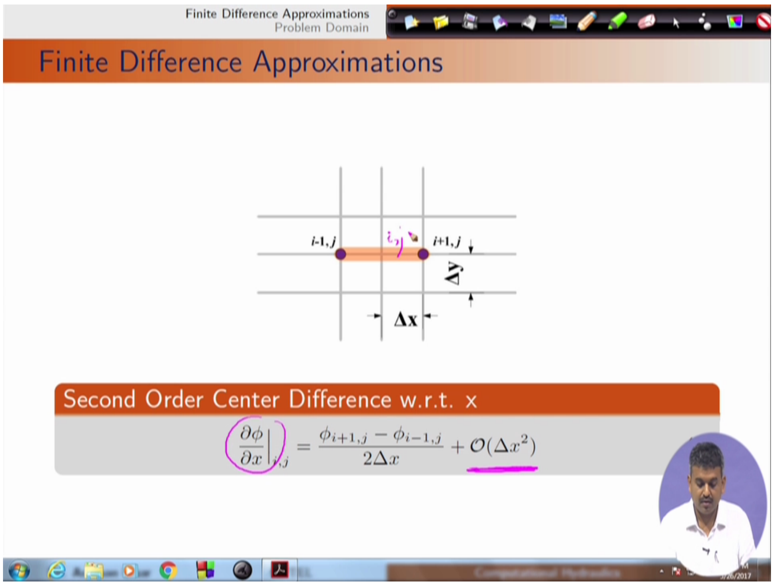

We can again discretize the derivatives in another direction. The first order forward difference with respect to y, again this is with del y accuracy.

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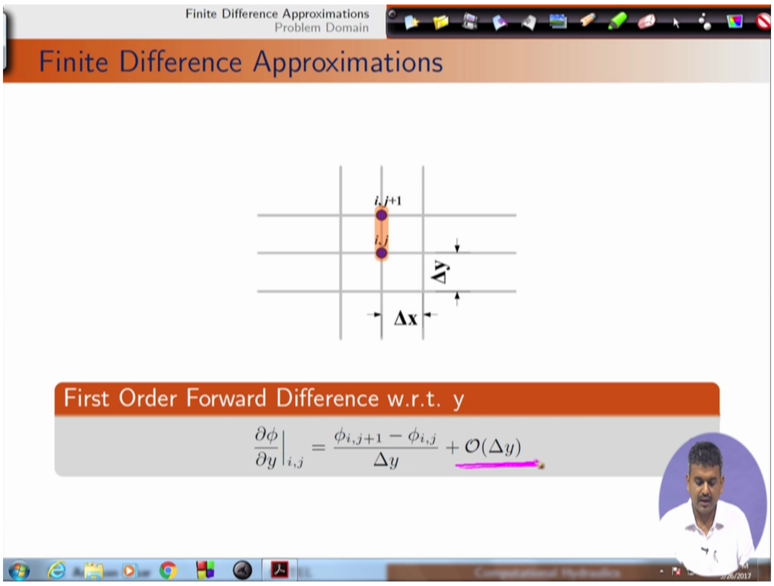

If we consider backward difference with respect to y this is again with del y and this is with del y accuracy first order accurate.

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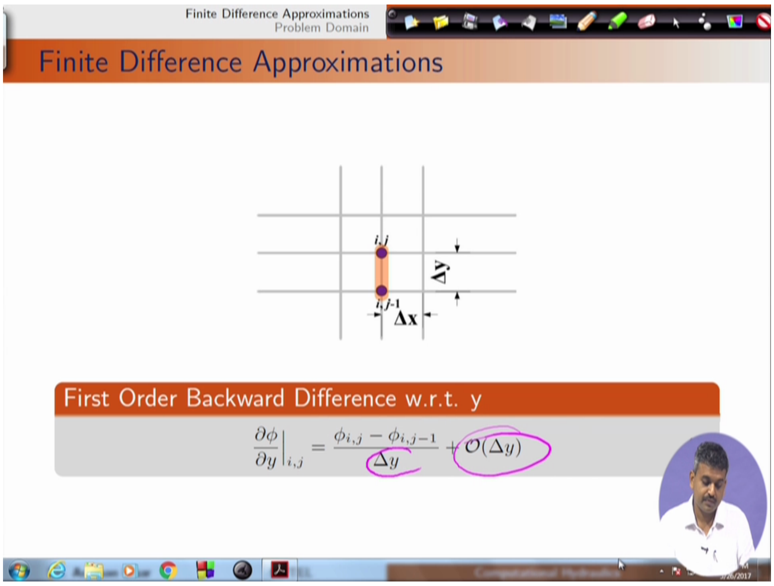

And with second order center difference we have change in the index for j, j plus 1 and j minus 1. This is again second order accurate method.

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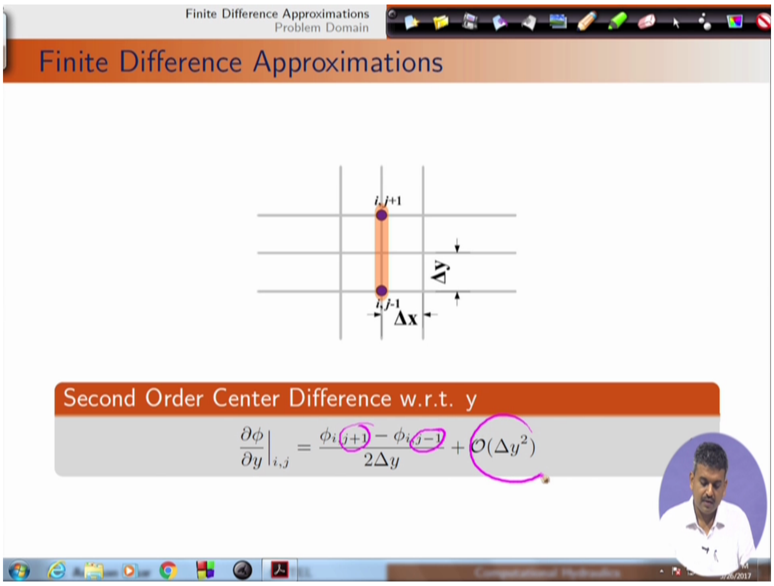

We have two extreme points we have i and j.

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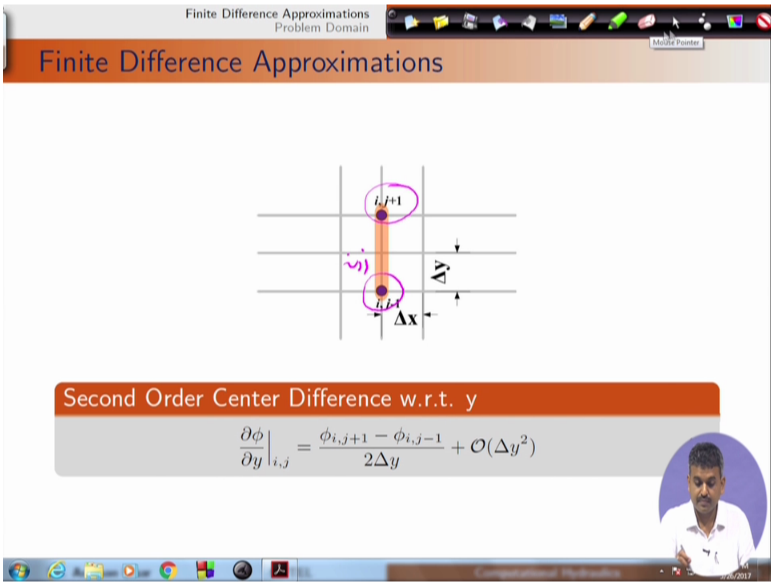

Finite difference approximation this is for second order center difference with respect to x. This is second order derivative we have phi i minus 1,phiI,phi i plus 1, this is del x square this is similar to our single variable case. And again this is del x square accuracy. Interesting point is that for this second order derivative we need three points.

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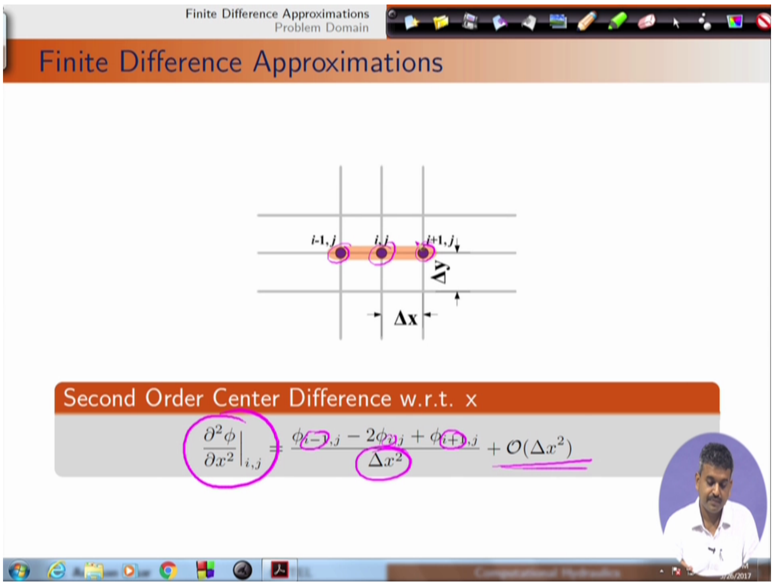

If we consider second order center difference with respect to y, so in this case also this is change in i minus 1, i plus 1, j, j plus 1, j minus 1 and j. This is del y square overall accuracy of del y square.

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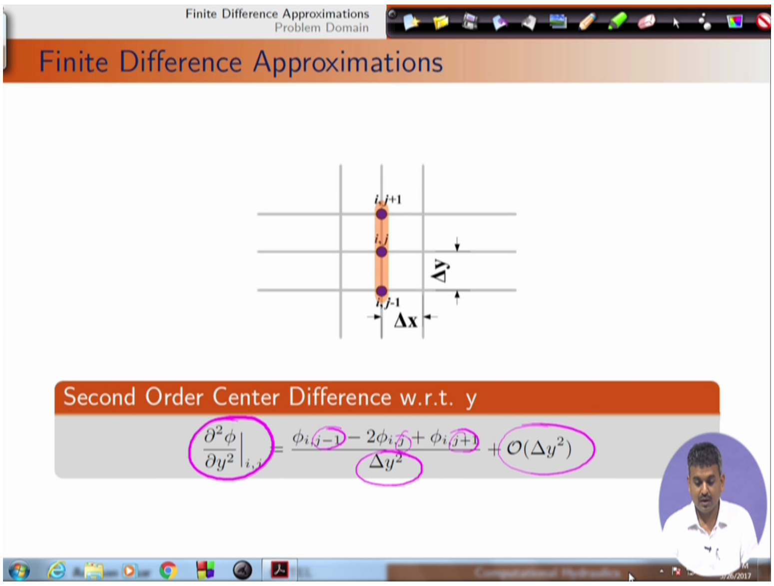

And this is mixed difference with respect to x and y. We need to consider the extreme points. We are considering the mix derivative at ij. However we need to consider the points in diagonal direction.

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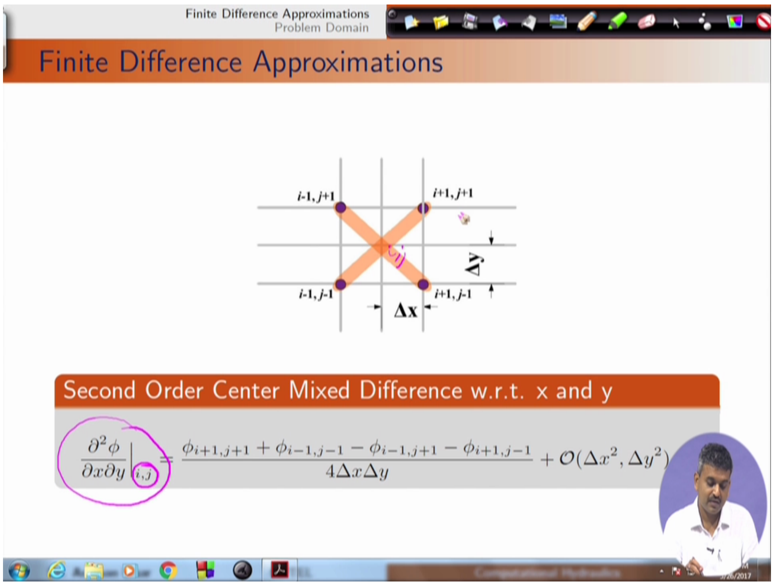

So this is i plus 1, j plus 1, which is positive. Then i minus 1, j minus 1, this 1 and i minus 1, j plus 1, this is i plus 1, j minus 1.

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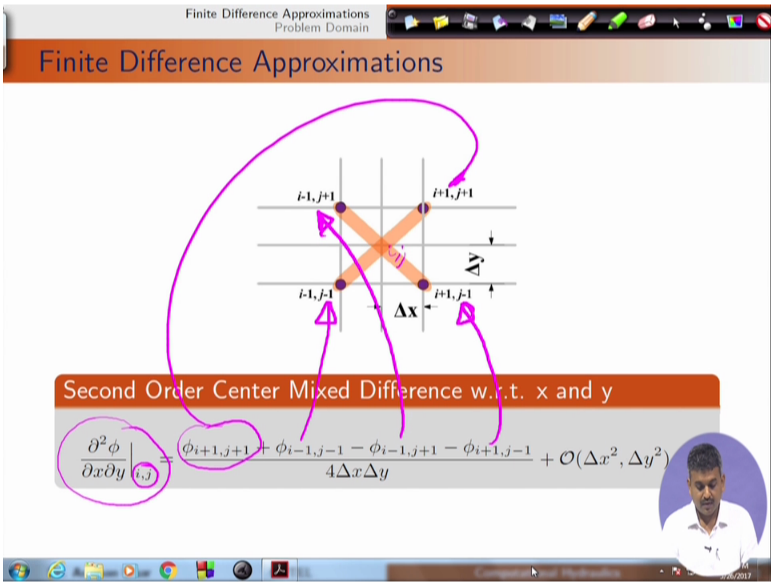

This two are with positive signs, this diagonally these two are with negative signs. Divided by 4 delta x delta y and this is our delta x square delta y square accuracy.

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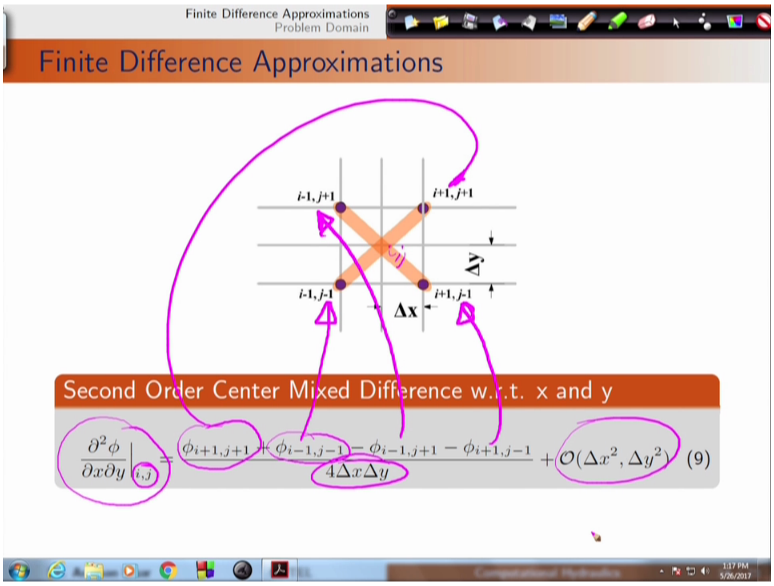

So in this case we have two independent variable and we are considering variation for both the variables. That's why we need to show this (accu)order of accuracy in term of both the independent variable.

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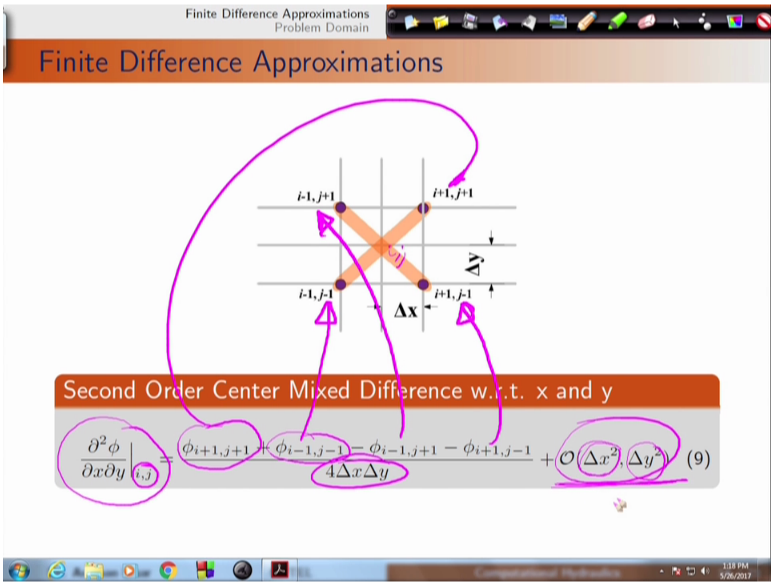

This is a general form of differential equation with a general variable phi. We have already discussed this in our earlier lecture and in this case phi is some general variable lambda and upsilon these are problem dependent parameters. And this gamma phi is a tensor if phi0 or f phi0 other forces, Sphi is source sink term for this one.

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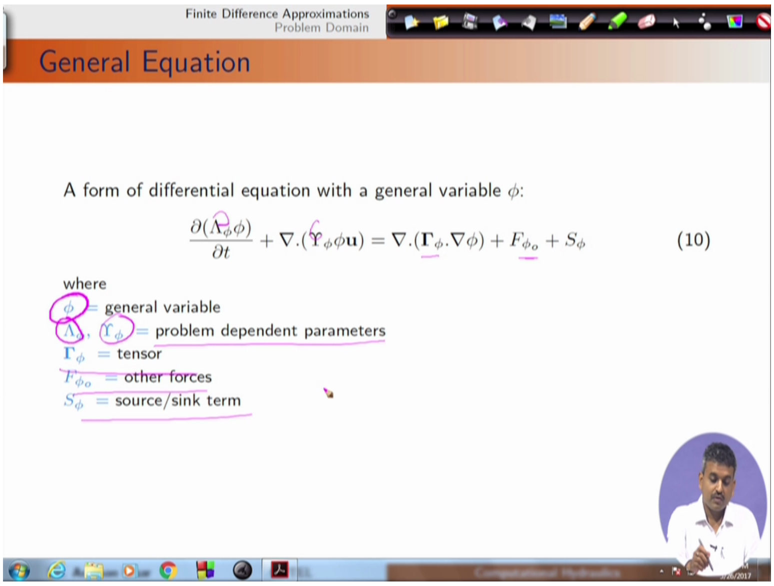

So if we approximate this equation and we utilize it for defining partial differential equations as boundary value problem then we need to neglect this term zero, this advective term as zero. This is also zero.

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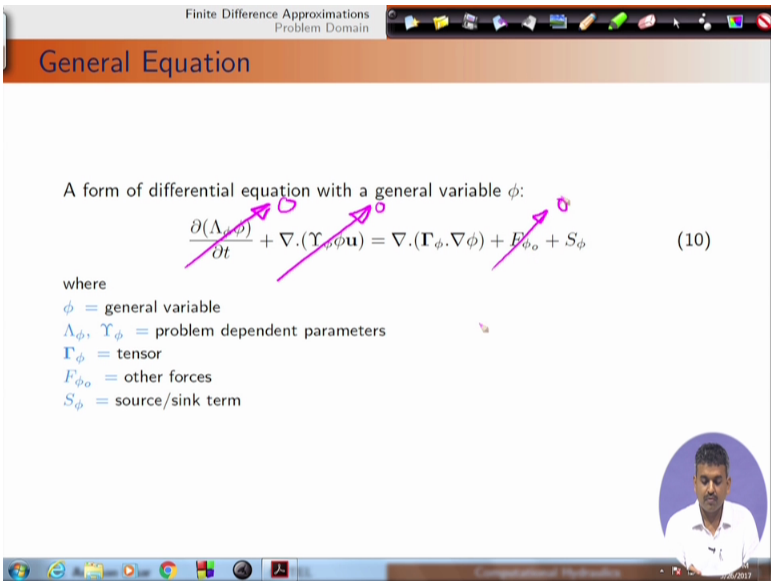

We will consider only these two terms. In this case del operator we are considering only variation of x and variation of y.These are unique vector in x and y direction. So this is del operator. So with this information we can use a simplified governing equation to define the boundary value problem.

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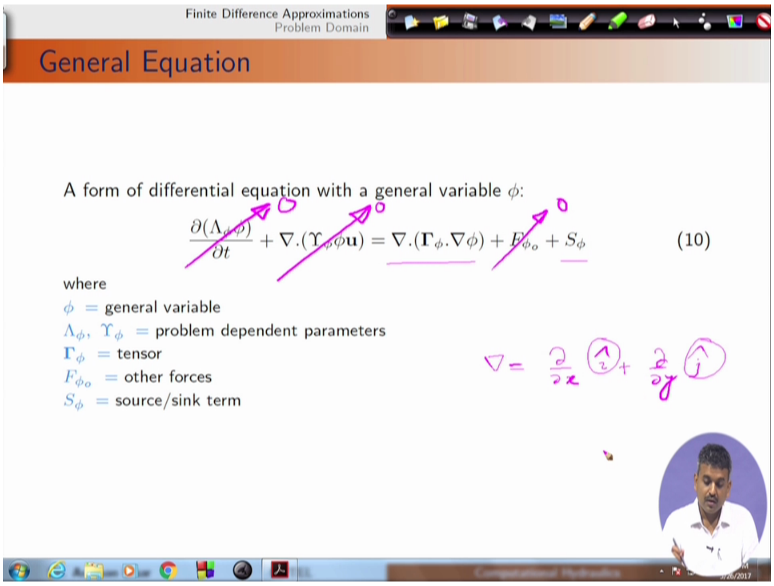

In this case without any cross term we have defined this del2 phi del x2. So obviously phi in previous case it's a two dimensional tensor with gamma x, gamma y, and cross terms are zero.

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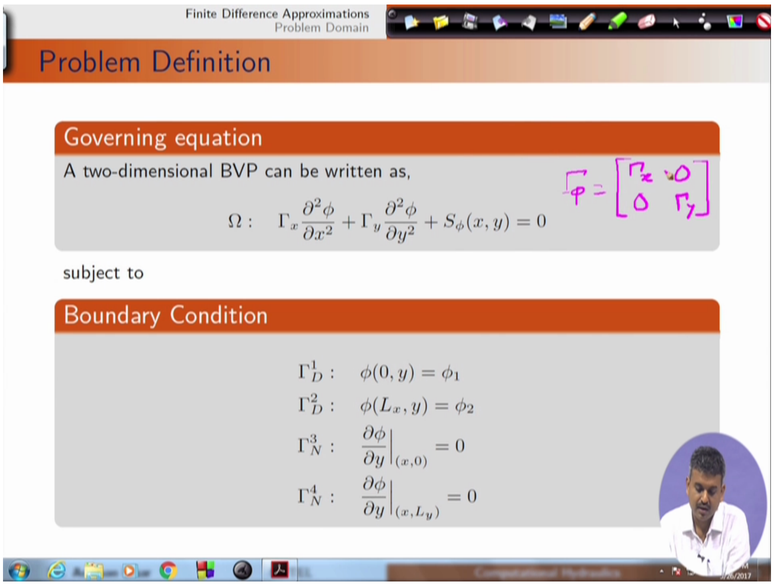

So if we simplify this and consider that gamma x, gamma y these two are constant then we can write in this format that means gamma y and x these are not varying with x and y.So Sphi is some source sink term.

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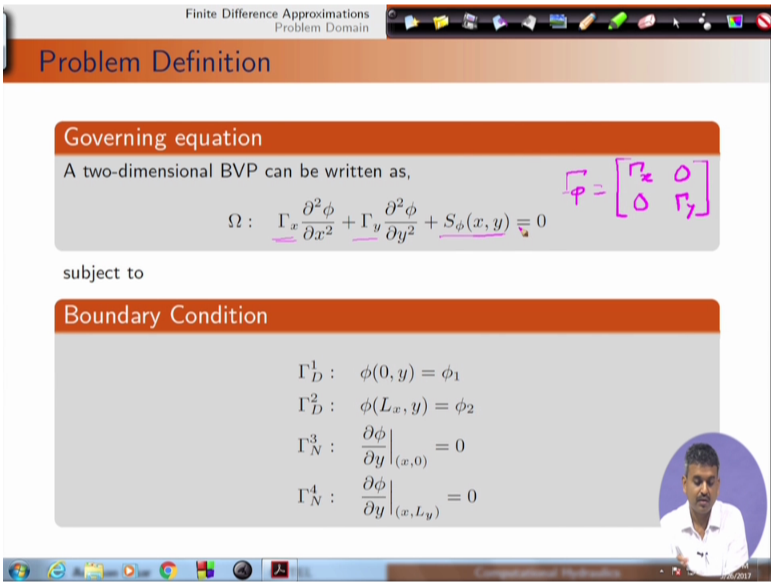

We can define boundary condition for a rectangular domain. So we have rectangular domain, this is lx for that rectangular domain ly for this y direction and this is zero zero point.

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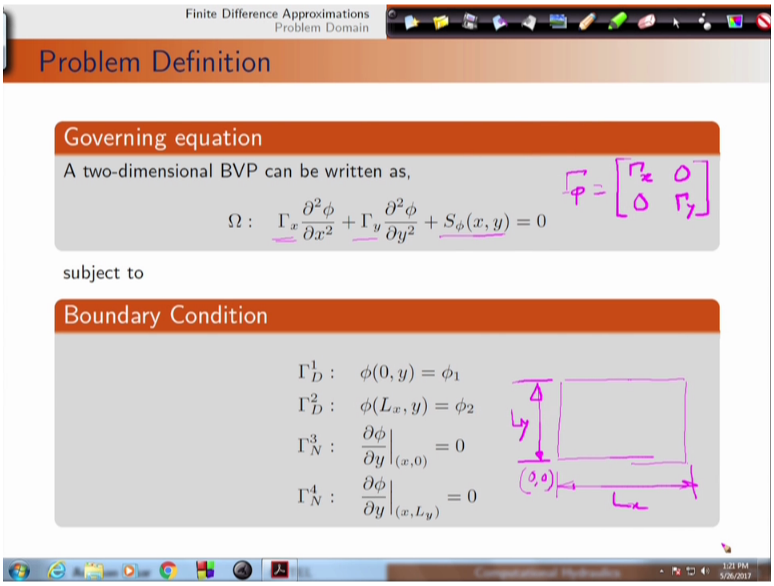

So with this information we have defined these four boundary conditions gamma d1 that means this left boundary at x is equal to zero and y this is Dirichlet kind of boundary or specified boundary, phi1.

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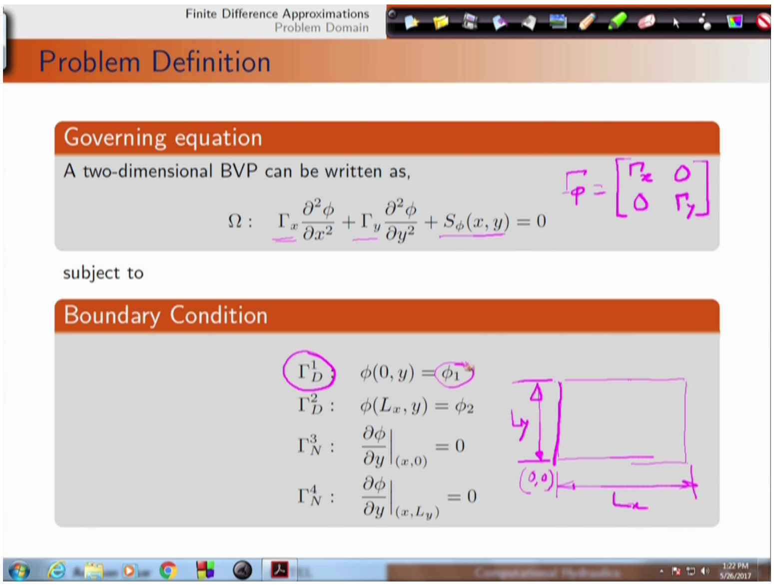

And gamma 2d this is again Dirichlet boundary with x is equal to lx and for all y we have this phi two value. This is phi1.

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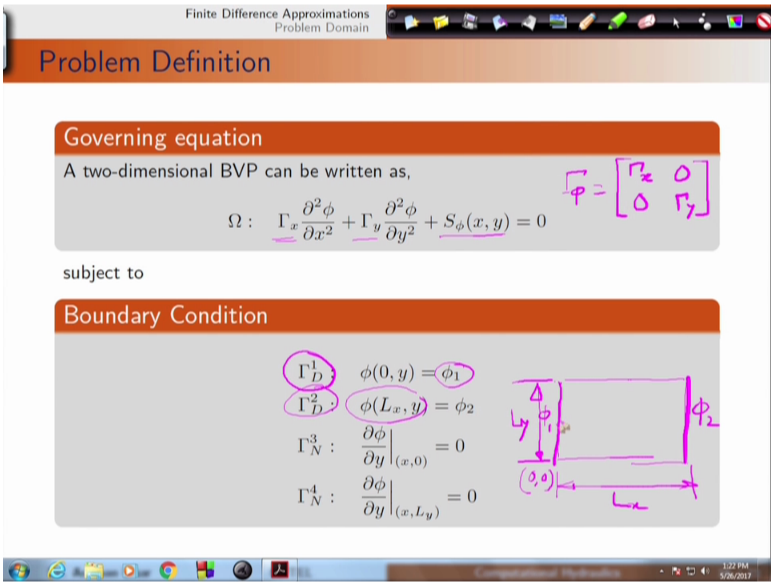

And for gamma three, this is gamma n3, this is actually phi n. In y direction there is no variation. And top we have del phi by del y equals to zero. So we can see that for this boundary value problem values are either specified for all boundaries or they are written in terms of boundary conditions.

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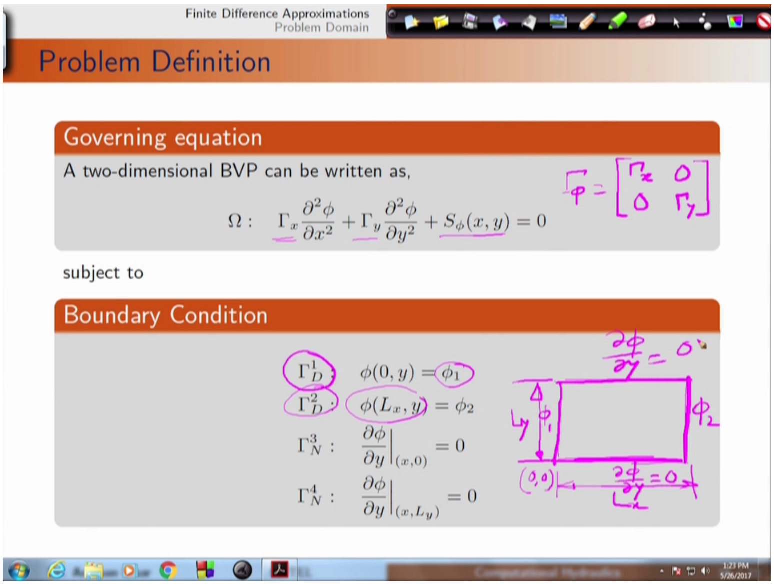

So domain discretization, this is lx this is y Dirichlet, Dirichlet boundary, Neumann, Neumann boundary.

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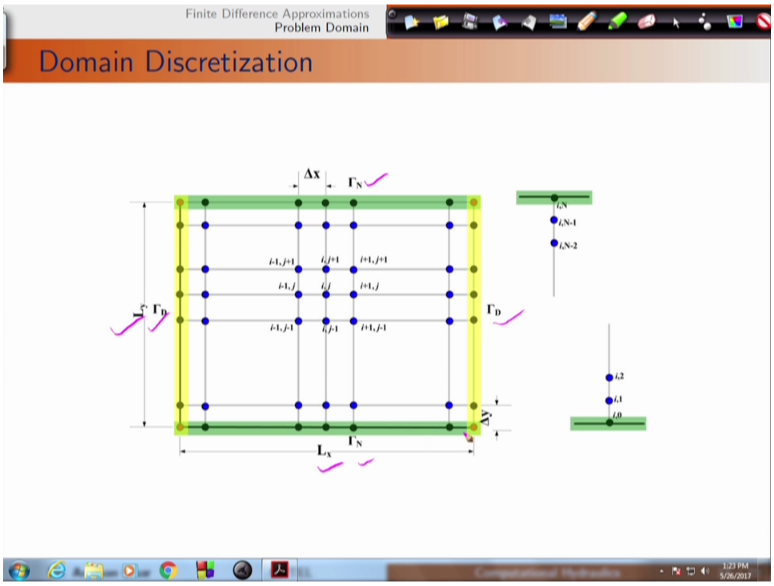

Important point is for this corner points. For corner points either we can consider it in this gamma n domain or in gamma d domain. In this case we can consider it in gamma d domain because in this case let us say that value is specified for these points these four points 1, two, three and four.

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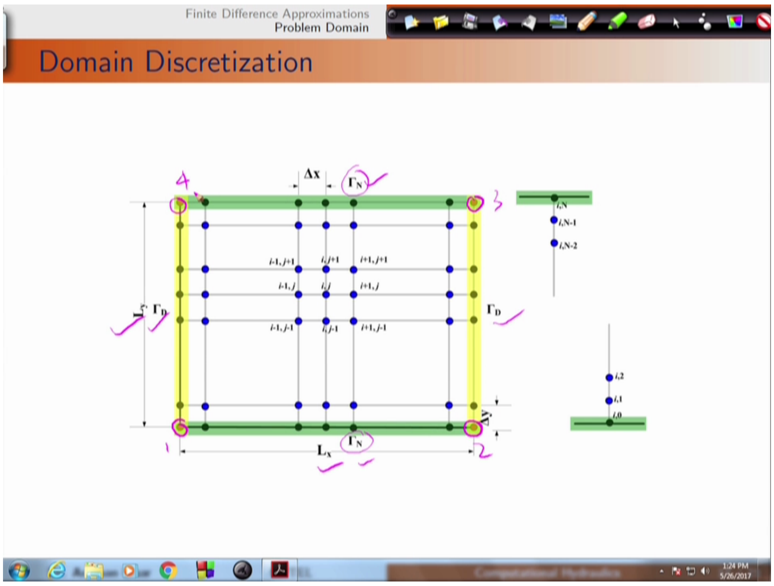

For this green portion that is Neumann boundary we can define the boundary condition based on three points or two points depending on the desired accuracy.

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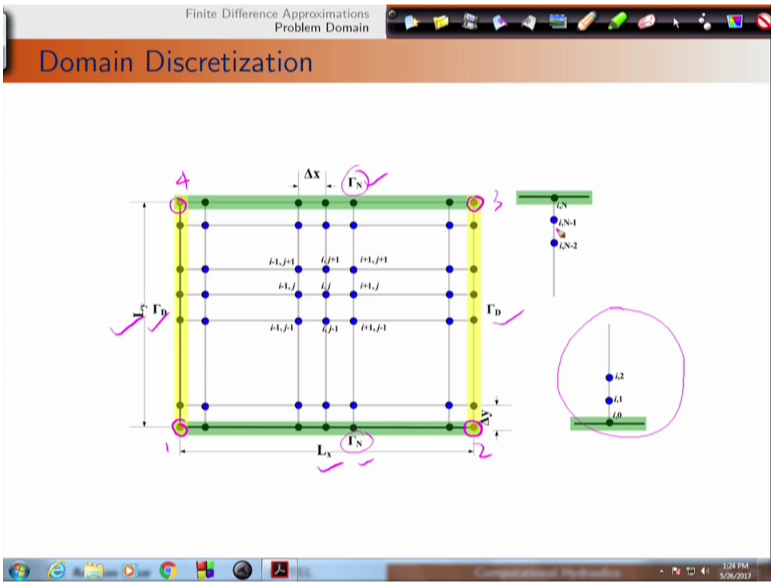

So let us consider the discretization of governing equation. In this case we have discretized the governing equation with second order accurate scheme gamma x, gamma y into this.

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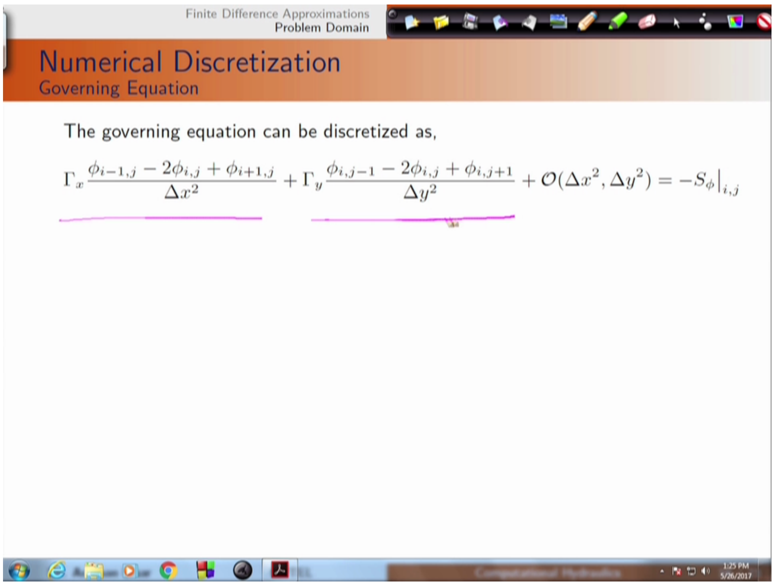

This is second order accurate. Minus Sphi ij which is specified value source sink term. We have transferred it into right hand side.

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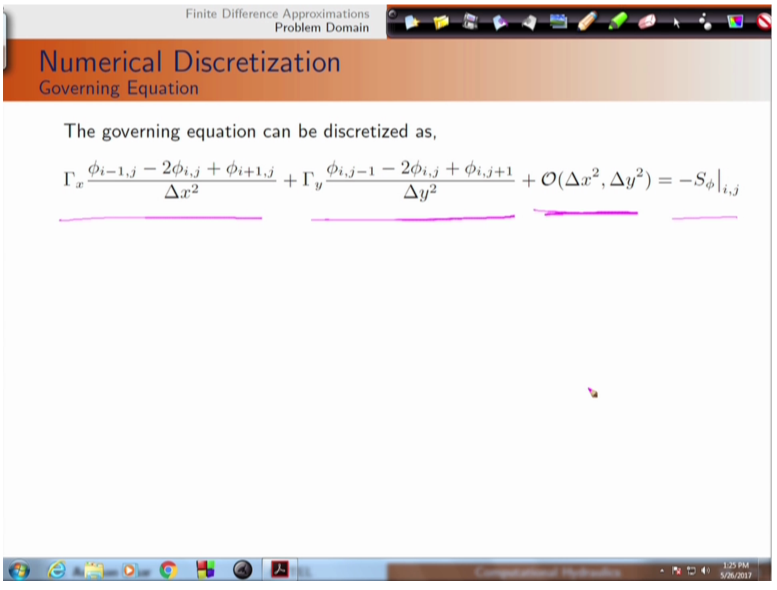

So this equation can be arranged as following. So with this, this i j minus 1. That means if we have any general ij structure then this is ij minus 1, this ij plus 1, this is i minus 1 j, this is i plus 1 j.

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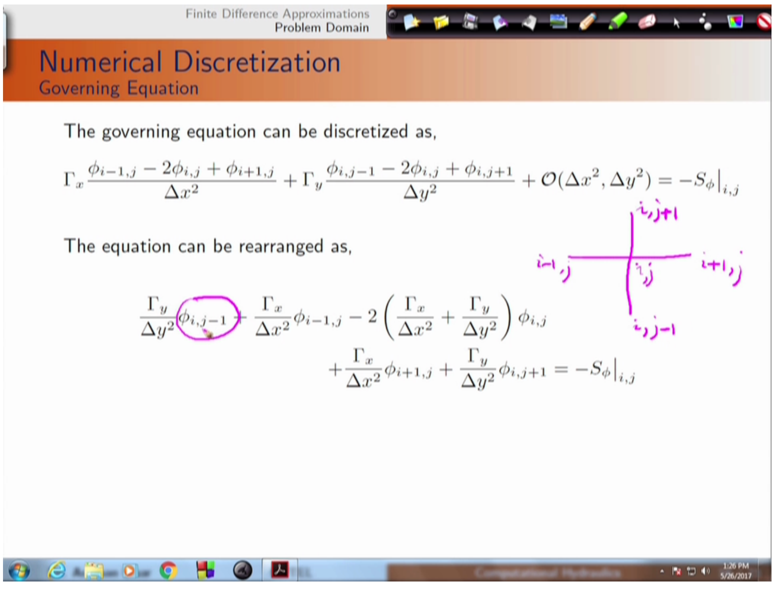

In this case this coefficient we are starting with ij minus 1 this is first, then i minus 1 j this is second, ij this is third, i plus 1 j this is fourth and fifth one is ij plus 1.So we have considered the coefficient for all these points.

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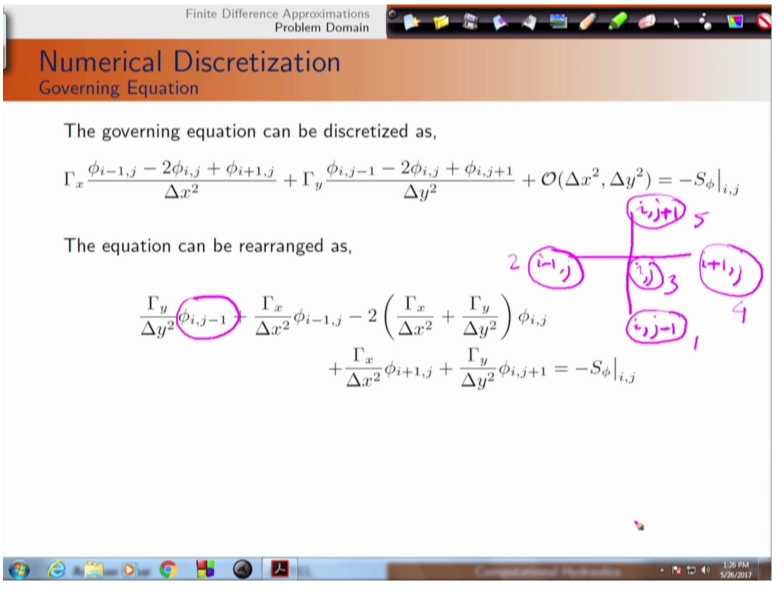

Now if we simplify this by using this notation that alpha x and alpha y can be written as gamma x by delta x square and gamma y by delta y square.

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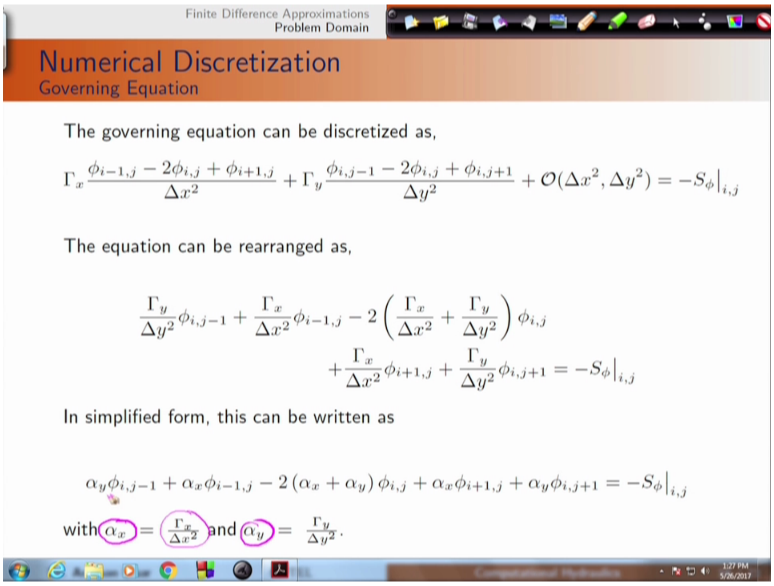

So we can write this in simple form but the problem is we cannot construct the algebraic matrix forms because we have double index notation present. In 1 dimension single index is possible to form this matrix easily.

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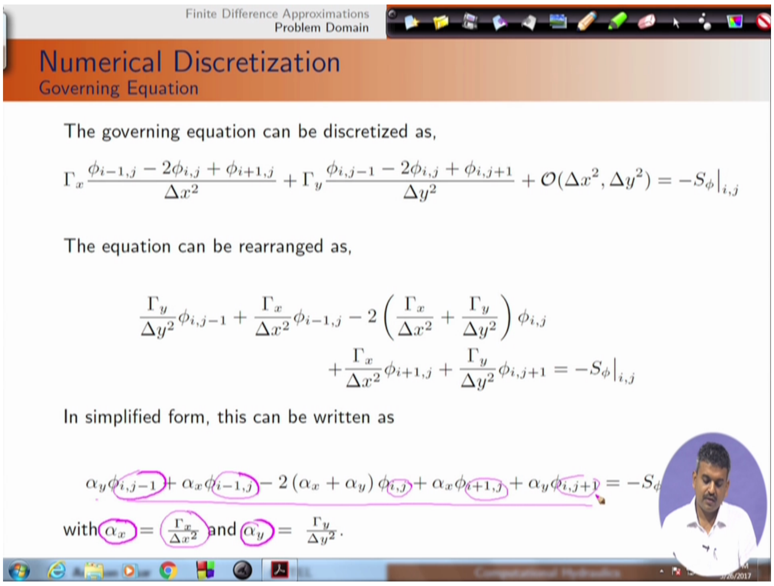

But individual points will have individual governing equations or boundary condition equations. These individual points will have individual equations. So these points are itself in ij format. So we cannot construct the matrix directly.

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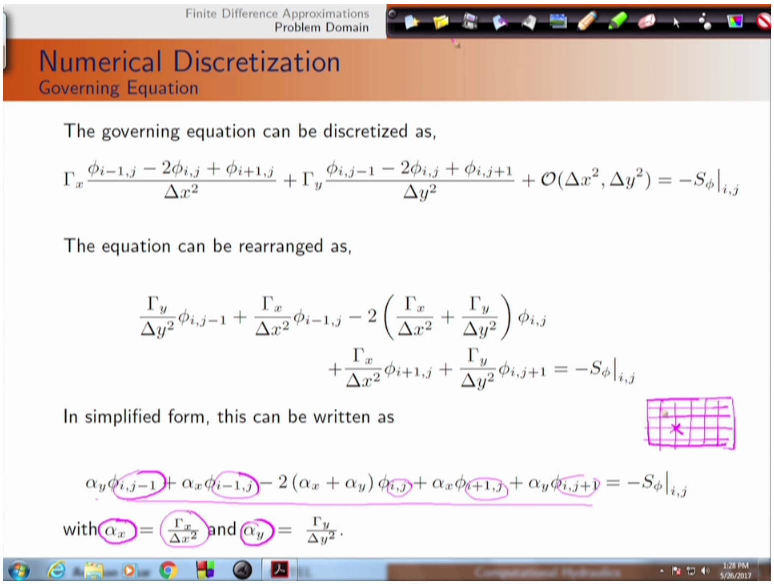

So what we can do, we can introduce single index notation. Single index L can be written as i j m plus 1. This is ij. This point can be represented as L.

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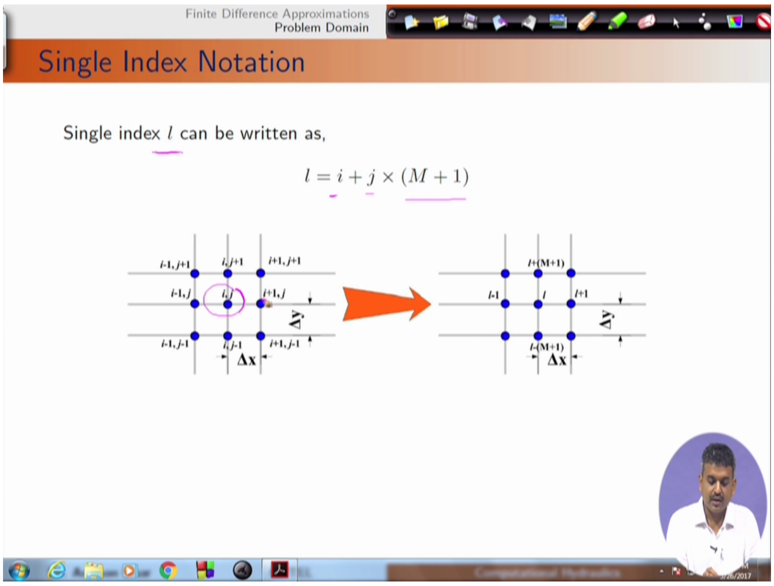

So L minus 1 is basically i minus j point, L plus 1 is i plus 1 j, L minus m plus 1 this is ij minus 1 and L plus m plus 1 is ij plus 1 point.

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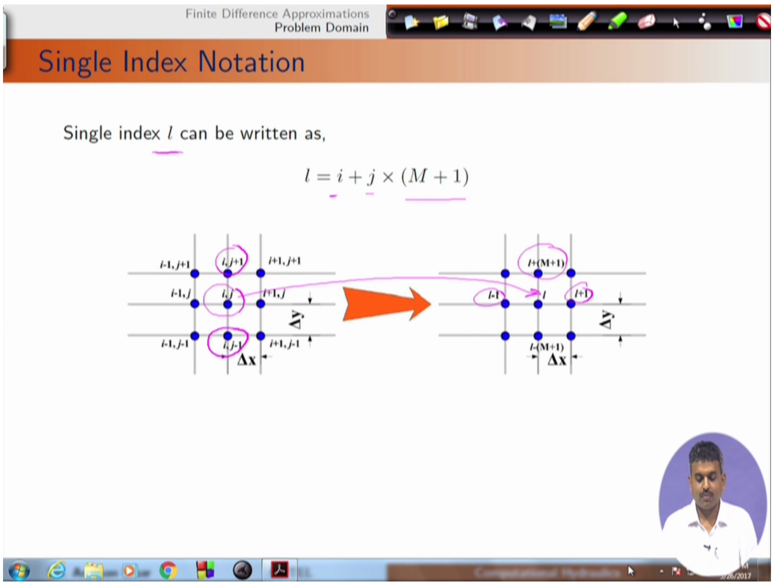

We are starting from point 0 to m, that's why we have m plus 1 number of points. So if we take the next level in y direction so obviously there will be difference of m plus 1 number of nodes.

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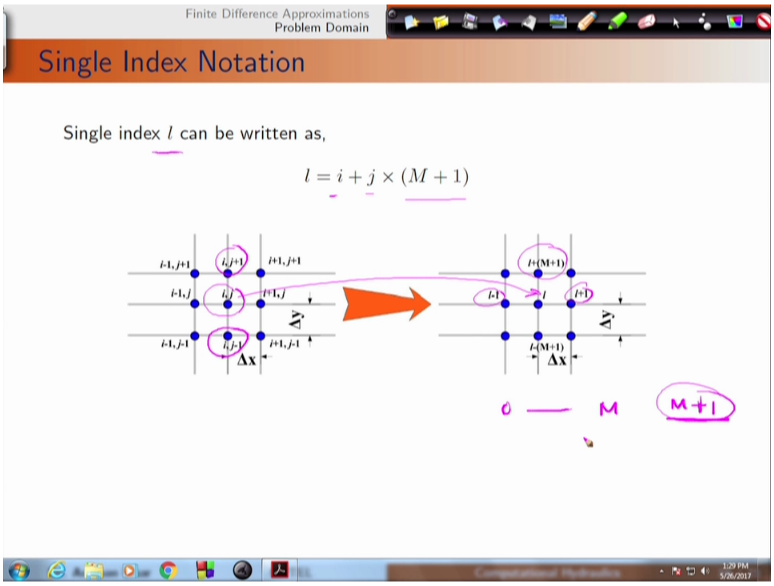

So with this information we can construct our grid system with 0 as starting point, m here and finally this will give the maximum number of points.

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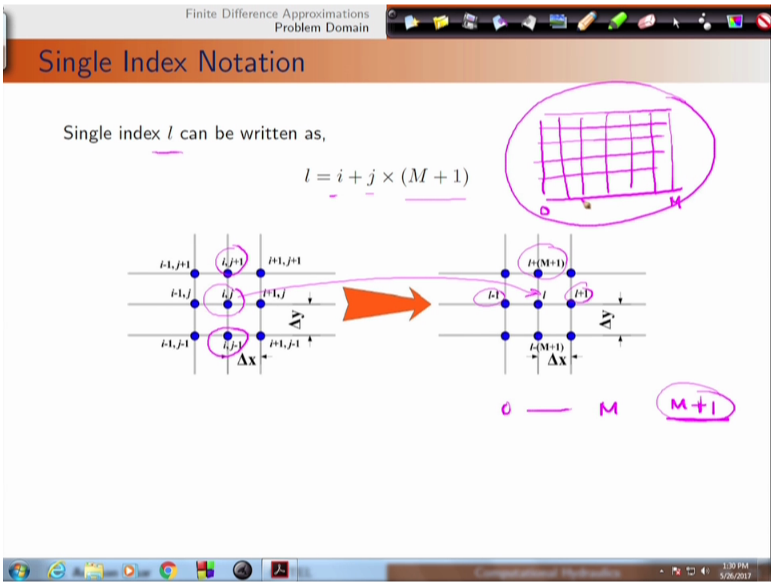

Wehave started from this point then we will move towards this.

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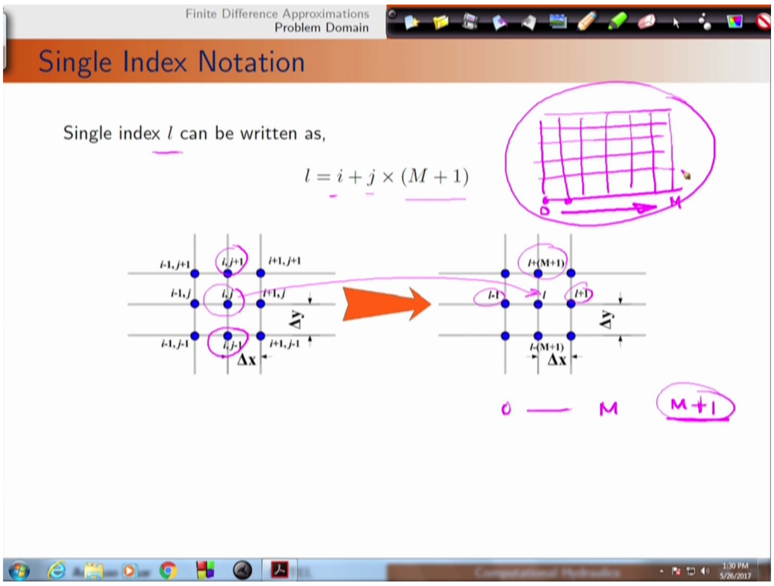

Again the next level will start from here this point.

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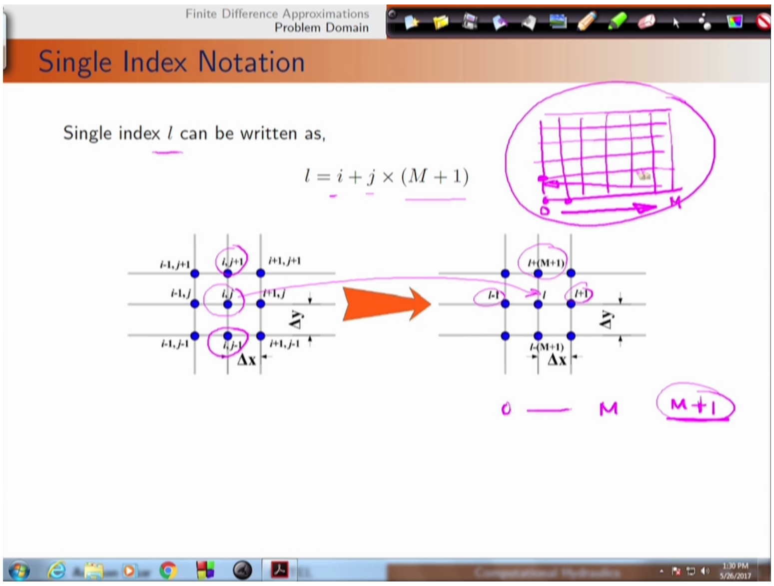

Again we will move in this direction again we will come back to this point and move to this direction.

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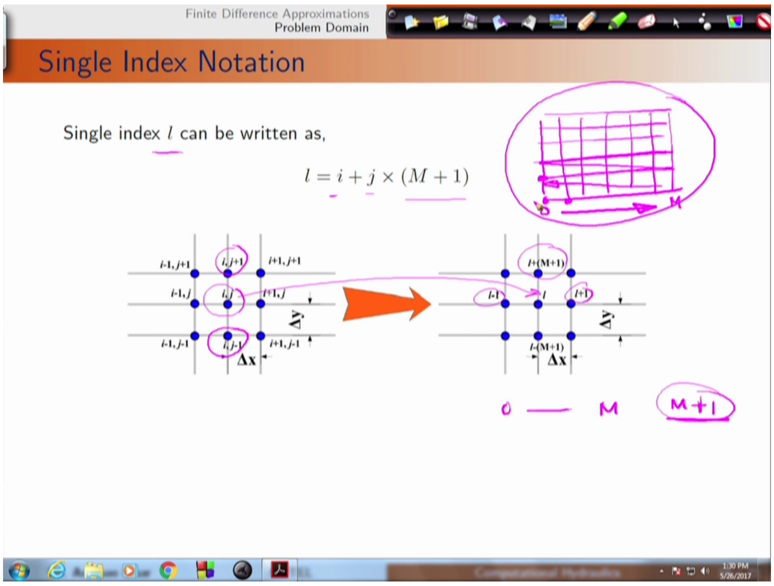

So i we are starting with j. For j equals to zero level we have m plus 1 number of points starting from zero to M. Then we will have m plus 1, like that we can define our nodal points with single index notation so that we can easily form the final matrix for solution.

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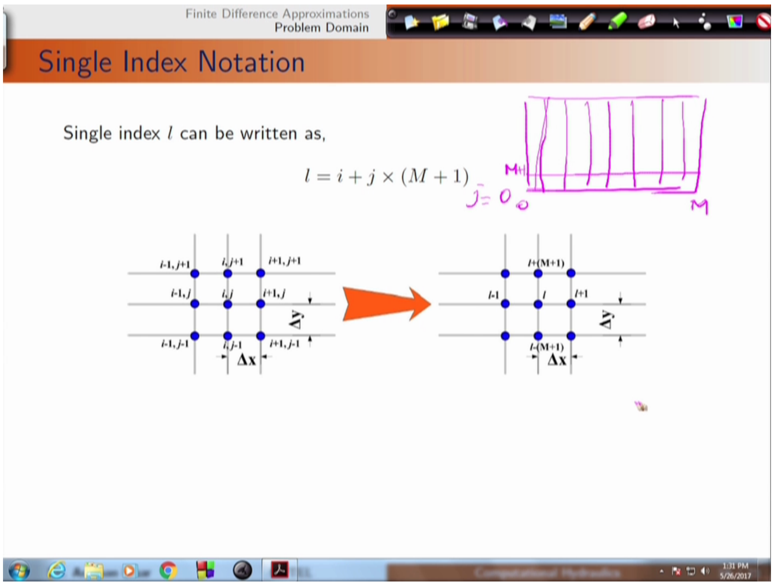

With single index notation the equation can be written as, in this case we have this kind of stencils that we have L, L minus 1, L plus 1. So we have five points.

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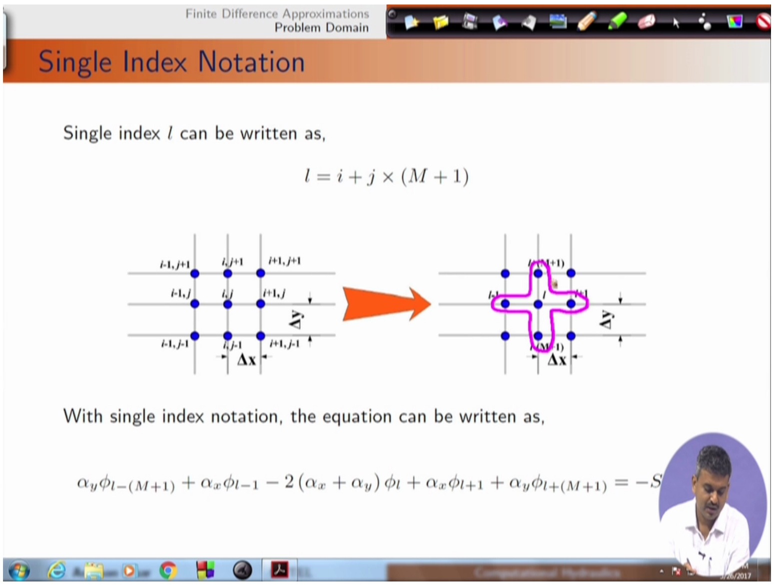

So m plus 1, L minus 1, L, L plus 1, L plus m plus 1. And this S phi is basically defined for ij or S phi we can write it in terms of L. So we can uses thisequation for interior points or blue points. For boundary points we need to define the boundary conditions.

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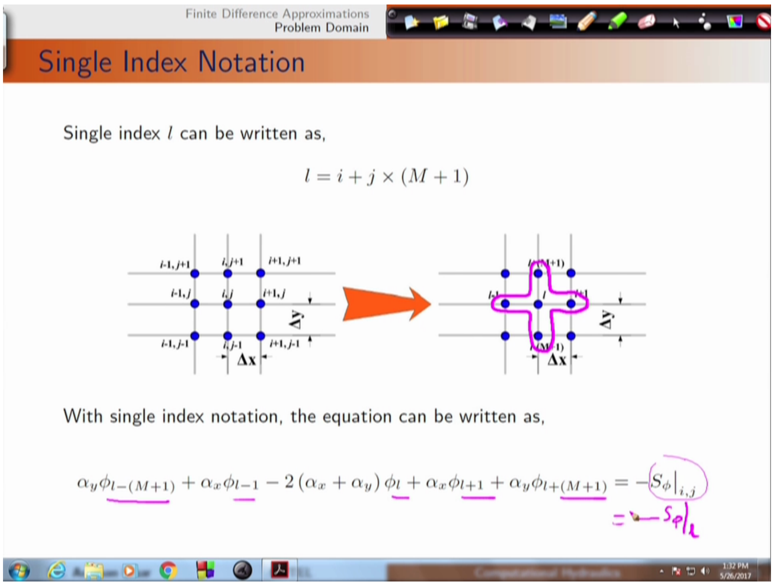

For Dirichlet boundary things are clear because we can directly specify the boundary conditions without any error. But Neumann boundary we need to consider the second order discretization because we have second order accurate scheme here for governing equation.

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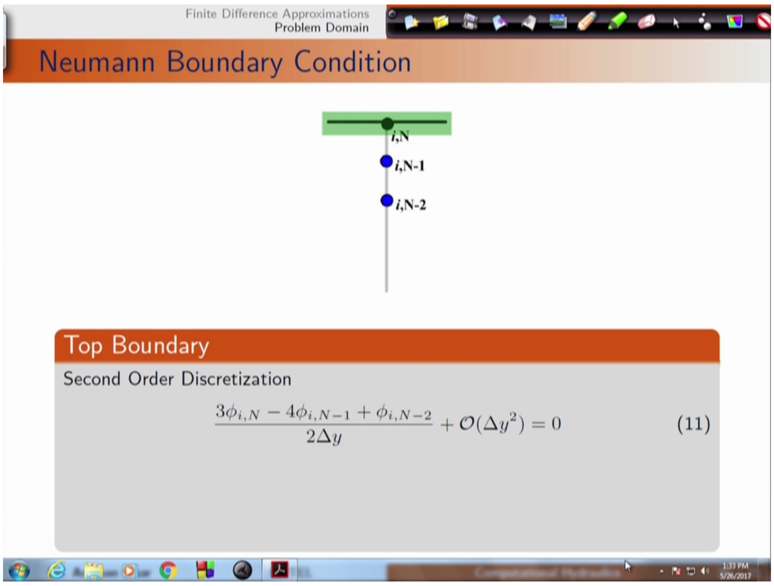

So we can include n, n minus 1, n minus 2 points. And this is second order accurate.

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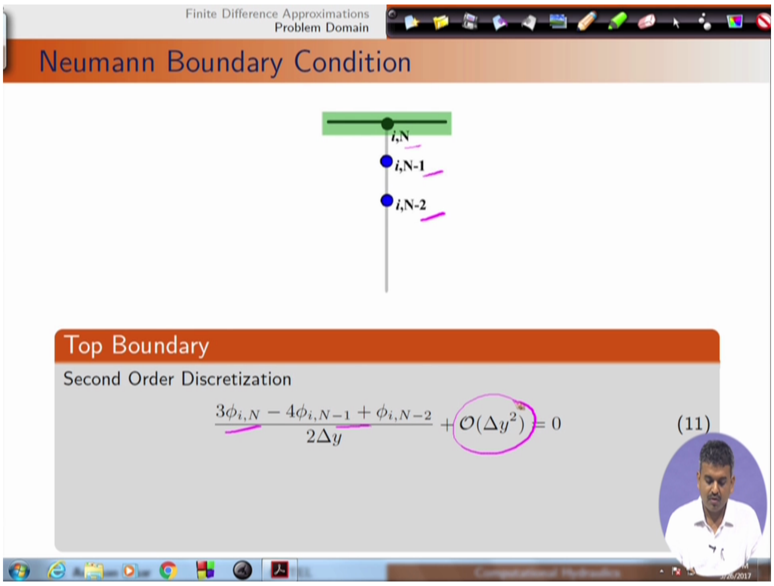

With this information we can use the single index notation to represent the thing. And we can directly use this one as equation for matrix form.

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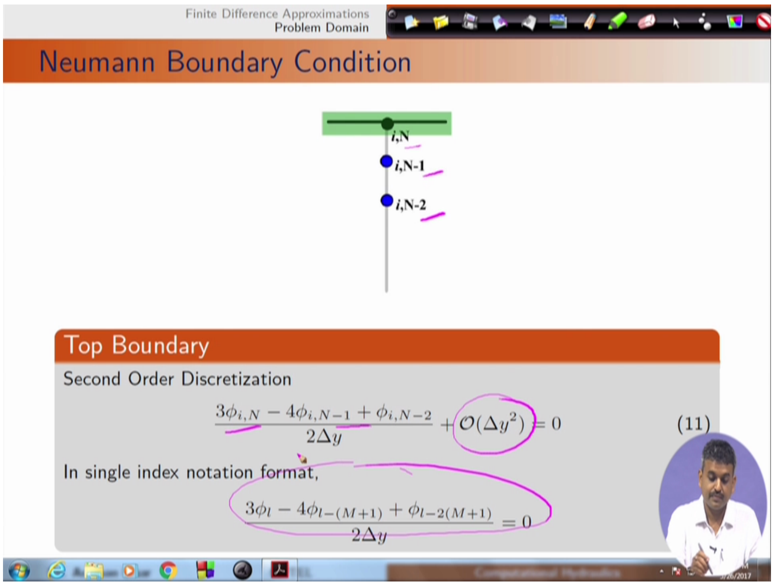

Similarly for this is for the top boundary, we can use it for bottom boundary zero, 1, 2 points and we can use the single index notation to represent the boundary condition.

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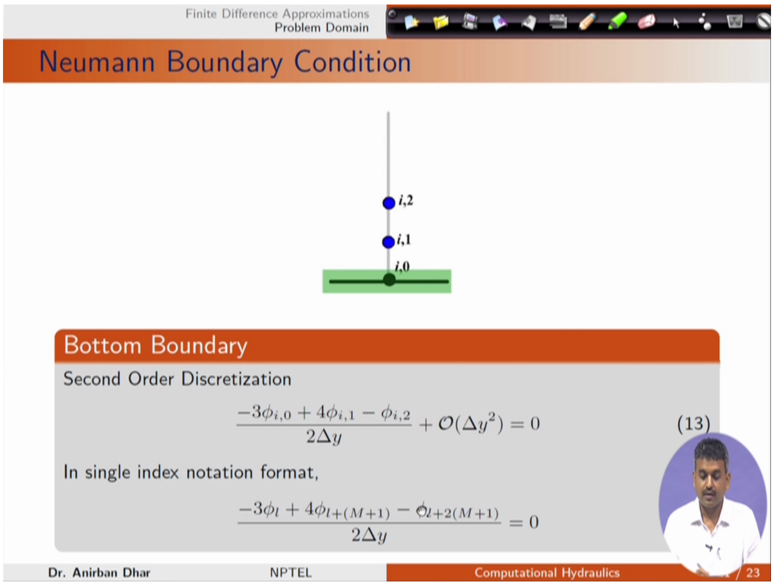

Again we are getting some equation for those boundary points. Now with this governing equation in discretize form and boundary conditions we can form the matrix for solution.

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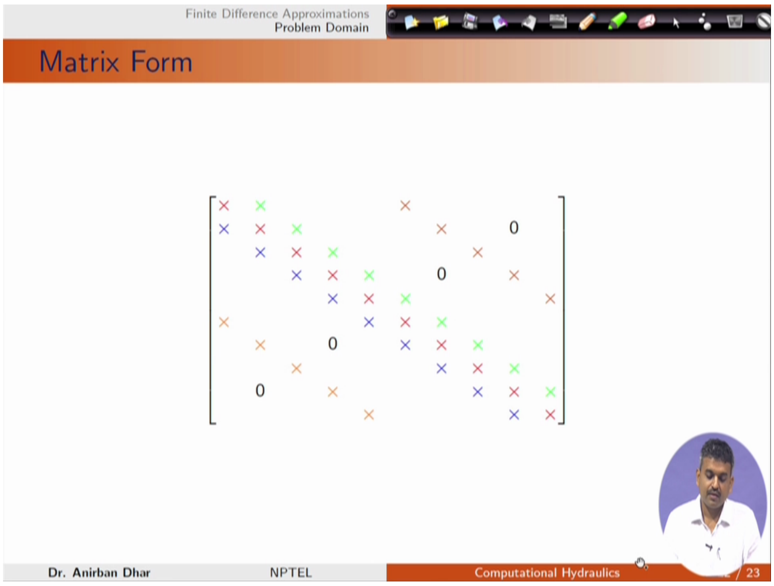

Interestingly for interior points we will have five coefficients. For L, L minus 1, L plus 1 and this is L plus m plus 1. This is for L minus m plus 1.

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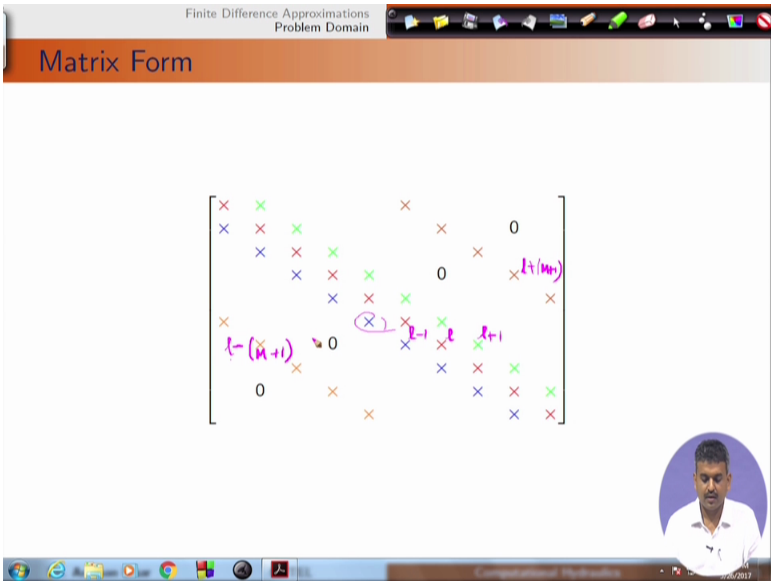

This is valid for interior points but if we have boundary points then we need to consider another structure point here which will consider the L minus two m plus 1and on this side also L plus 2 into m plus 1 to consider the three points.

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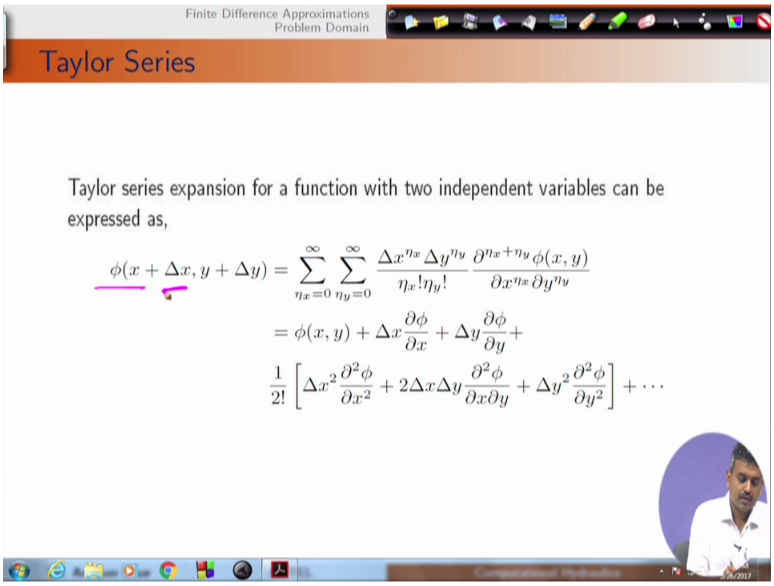

Now we can solve this A matrix with phi and define for single index notation. And we will have something bL for right hand side. So with this we can solve this phi Lby inverting the A matrix. And this will give solution for the desired problem.

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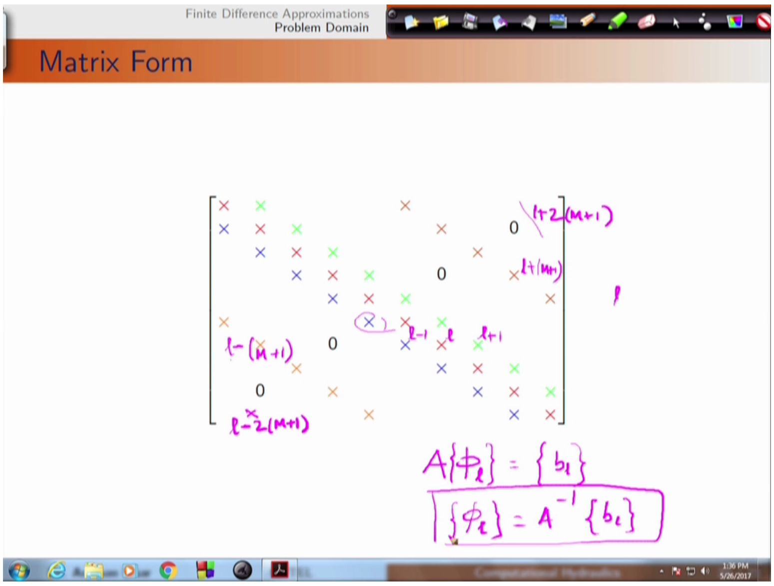

And next lecture class we will be discussing the time derivative and partial differential equation. Thank you.