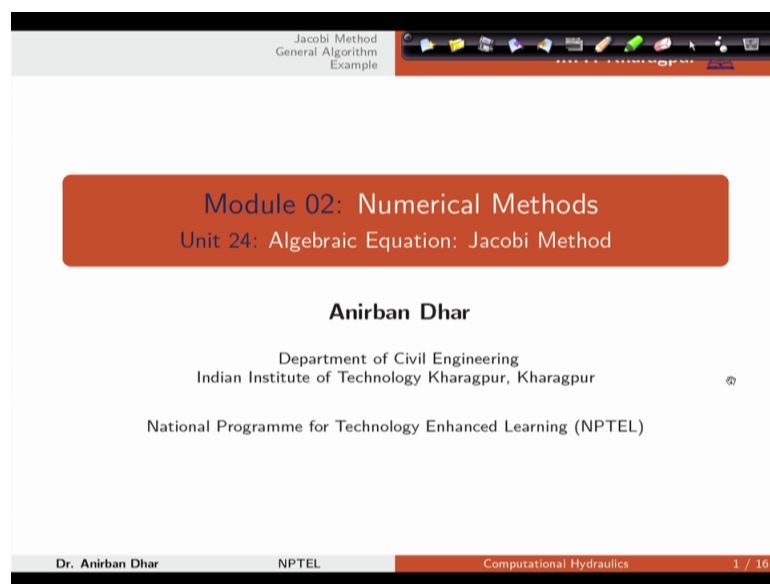


Computational Hydraulics
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Lecture 28
Algebraic Equation: Jacobi's Method

Welcome to lecture number 28 of the course computational hydraulics. We are in module 2, numerical methods. And in this particular unit number 24, I will be talking about algebraic equation, Jacobi method.

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Jacobi method, essentially this is iterative technique to solve any algebraic system with constant coefficient. We can utilize either direct approach. We have seen that Gauss elimination, LU decomposition or tridiagonal matrix algorithm can be utilized to solve our direct our matrix to get our phi value. But in Jacobi's method or Jacobi method we will talk about iterative technique to solve the problem starting from a guess value.

Learning objective, at the end of this unit students will be able to apply Jacobi method for iterative solution of our algebraic system.

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Jacobi Method
General Algorithm
Example

Learning Objective

- To apply Jacobi Method for iterative solution.

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Matrix form, full matrix. If we consider full matrix then we have N by N , N cross 1 , N cross 1 . In this case we are defining the algorithm for general purpose matrix. That means it's having full coefficients. That means all nonzero coefficients or maximum number of nonzero coefficients available within the system. ϕ is our desired variable.

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Jacobi Method
General Algorithm
Example

Matrix Form Full Matrix

$$A \phi = r$$
$$\begin{pmatrix} \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \\ \times & \times & \times & \dots & \times & \times & \times \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{pmatrix}$$

$N \times N$ $N \times 1$ $N \times 1$

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Now in this case what are the basic steps? In case of LU decomposition we have divided or decomposed the matrix into LU format. That means one lower and one upper triangular matrix. In that case this decomposition was valid for multiplication. But in this case we are essentially considering L plus D plus U, where L is strictly lower triangular matrix, this is

strictly diagonal matrix and this is strictly upper triangular matrix. This has got no connection with our LU decomposition discussed in our direct solution approach.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

The coefficient matrix A can be written as

$$A = L + D + U$$

where L , D , U are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

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Now in this case overall calculation can be presented like this. As we have decomposed our A , our general structure was $A\phi = r$. So in place of A we can replace L plus D plus U . That means lower, diagonal, upper.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

The coefficient matrix A can be written as

$$A = L + D + U$$

where L , D , U are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

Overall calculation can be presented as

$$(L + D + U)\phi = r$$

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Now for this one we can write it like this. In this case we have divided it into diagonal parts and off diagonal parts. So if we say that this is our matrix so we will divide it into one lower part, one diagonal part and one again upper triangular part. These are strictly. That means this

diagonal term is not included either in our upper or lower triangular. This is only applicable for diagonal matrix.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

The coefficient matrix A can be written as

$$A = L + D + U$$

where L , D , U are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

Overall calculation can be presented as

$$(L + D + U)\phi = r$$

Iterative form can be written as

$$D\phi^{(p)} + (L + U)\phi^{(p-1)} = r$$

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Now let us decompose this phi like this. Phi can be multiplied with our D. And L plus U this is multiplied with previous iteration value P. P stands for the iteration counter. So if we start from zero which is a guess level, this equation or expression is valid from P greater than equal to 1. That is why P equals to 1 that means this is for first iteration. This L plus U into phi P minus 1. That means this value will be evaluated for guess value.

That means P equals to zero level. Now in this case four things are fixed. That means D matrix is constant, L matrix is constant, U matrix this is constant and r is also constant. The only varying thing is phi. Every iteration phi is varied. We are not changing this L U. That means like our direct approach we have decomposed our matrix into different forms and we have changed those values.

But in this case we are not changing the values directly. We are keeping this structure intact and we are just dividing it into lower, diagonal and upper matrix, in additive form.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

The coefficient matrix A can be written as

$$A = L + D + U$$

where L , D , U are strictly lower triangular, diagonal, strictly upper triangular matrices respectively.

Overall calculation can be presented as

$$(L + D + U)\phi = r$$

Iterative form can be written as

$$D\phi^{(p)} + (L + U)\phi^{(p-1)} = r$$

Handwritten notes: $p=0$, $p \geq 1$

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So iterative form, because we are not disturbing this diagonal, lower, upper value and r value. So I am using the blue color. So that means we are not changing these values. And this ϕ value at previous iteration P minus 1, this is available. If we start with zero this should be given.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

Iterative form can be written as

$$D\phi^{(p)} + (L + U)\phi^{(p-1)} = r$$

Handwritten note: $\phi^{(0)} = \text{given}$

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Now in iterative form if we further write this, so we transfer it on the right hand side. Now this is necessarily inverting the diagonal matrix and multiplying it with the first one is our previous iteration value. And this is our right hand side vector multiplied by this diagonal term inverse.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

Iterative form can be written as

$$\mathbf{D}\phi^{(p)} + (\mathbf{L} + \mathbf{U})\phi^{(p-1)} = \mathbf{r}$$
$$\mathbf{D}\phi^{(p)} = -(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{r}$$

Final form can be written as

$$\phi^{(p)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{D}^{-1}\mathbf{r}$$

where p is the iteration counter ($p \geq 1$).

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Now this iterative process, this is valid for P greater than equals to 1, because we cannot apply this for zeroth level which is the guess level for any iteration.

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

Iterative form can be written as

$$\mathbf{D}\phi^{(p)} + (\mathbf{L} + \mathbf{U})\phi^{(p-1)} = \mathbf{r}$$
$$\mathbf{D}\phi^{(p)} = -(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{r}$$

Final form can be written as

$$\phi^{(p)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{D}^{-1}\mathbf{r}$$

where p is the iteration counter ($p \geq 1$).

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Now iteration starts with a guess value ϕ_0 .

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Jacobi Method
General Algorithm
Example

Basic Steps Jacobi Method

Iterative form can be written as

$$\mathbf{D}\phi^{(p)} + (\mathbf{L} + \mathbf{U})\phi^{(p-1)} = \mathbf{r}$$
$$\mathbf{D}\phi^{(p)} = -(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{r}$$

Final form can be written as

$$\phi^{(p)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\phi^{(p-1)} + \mathbf{D}^{-1}\mathbf{r}$$

where p is the iteration counter ($p \geq 1$).

Iteration starts with a guess value $\phi^{(0)}$

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Now in this process if you want to explain the whole thing, blue values are undisturbed values, r values are also undisturbed values, like over direct approach where we are changing different rows to get the desired solution.

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Jacobi Method
General Algorithm
Example

Jacobi Method Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix}$$

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So in this case the coefficient matrix A can be written as L which is strictly lower triangular matrix. Notice that we are not including the diagonal term here. This is strictly lower. The coefficients are exactly same.

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Jacobi Method
General Algorithm
Example

Jacobi Method Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix}$$

Coefficient matrix **A** can be written as

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where

Strictly Lower Triangular Matrix

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix}$$

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Next level is we are dividing it with diagonal term. So that means only diagonal term is there. We have 5 by 5 matrix. So 5 terms.

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Jacobi Method
General Algorithm
Example

Jacobi Method Example

Diagonal Matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

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Our next level we have strictly upper triangular. This also is without diagonal term because we are not including diagonal term in this process.

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Jacobi Method
General Algorithm
Example

Jacobi Method Example

Diagonal Matrix

$$D = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

Strictly Upper Triangular Matrix

$$U = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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So iteration, this starts with ϕ_0 . Now ϕ_0 in this case this is a column vector. Now this is $\phi_1, \phi_2, \phi_3, \phi_4$, this is valid for zeroth level. These values are given values. Now iteration 1, we are calculating the iteration 1 values and this is we are just transferring off diagonal coefficients including the off diagonal variables. So for 1 this is a_{1j} and ϕ_j at zero. That means whatever value is available at the previous time level we are utilizing that for calculation of this off diagonal thing. So if I divide it by a_{11} , I should get guess for from this iteration 1.

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Jacobi Method
General Algorithm
Example

Jacobi Method Example

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_3^{(0)} \quad \phi_4^{(0)} \quad \phi_5^{(0)}]^T$$

Iteration 1:

Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1, \\ j \neq 1}}^5 a_{1j} \phi_j^{(0)} \right]$$

Row 2:

$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1, \\ j \neq 2}}^5 a_{2j} \phi_j^{(0)} \right]$$

Row 3:

$$\phi_3^{(1)} = \frac{1}{a_{33}} \left[r_3 - \sum_{\substack{j=1, \\ j \neq 3}}^5 a_{3j} \phi_j^{(0)} \right]$$

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Now with this case I am not calculating anything. For a particular iteration I have calculated this. Now this is valid for row 1. Now for row again I am calculating without including the

diagonal term. So in this case a_{11} is the coefficient of the diagonal term. In this case a_{22} is the coefficient of diagonal term. So without including j equals to 2, j starting from 1 to 5, we are calculating this using our old value or guess value.

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Jacobi Method
General Algorithm
Example

Jacobi Method
Example

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_3^{(0)} \quad \phi_4^{(0)} \quad \phi_5^{(0)}]^T$$

Iteration 1:

Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1, \\ j \neq 1}}^5 a_{1j} \phi_j^{(0)} \right] \quad a_{11}$$

Row 2:

$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1, \\ j \neq 2}}^5 a_{2j} \phi_j^{(0)} \right] \quad a_{22}$$

Row 3:

$$\phi_3^{(1)} = \frac{1}{a_{33}} \left[r_3 - \sum_{\substack{j=1, \\ j \neq 3}}^5 a_{3j} \phi_j^{(0)} \right] \quad a_{33}$$

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Row 3, again a_{33} and without including j equals to 3. That means without including the diagonal term we have again calculated this ϕ_3 .

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Jacobi Method
General Algorithm
Example

Jacobi Method
Example

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_3^{(0)} \quad \phi_4^{(0)} \quad \phi_5^{(0)}]^T$$

Iteration 1:

Row 1:

$$\phi_1^{(1)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1, \\ j \neq 1}}^5 a_{1j} \phi_j^{(0)} \right] \quad a_{11}$$

Row 2:

$$\phi_2^{(1)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1, \\ j \neq 2}}^5 a_{2j} \phi_j^{(0)} \right] \quad a_{22}$$

Row 3:

$$\phi_3^{(1)} = \frac{1}{a_{33}} \left[r_3 - \sum_{\substack{j=1, \\ j \neq 3}}^5 a_{3j} \phi_j^{(0)} \right] \quad a_{33}$$

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Now in this process we can again calculate a_{44} by omitting this fourth term which is diagonal term. Then fifth one, by omitting that fifth one. Now in iteration 2 we will utilize this updated value from one.

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Jacobi Method
General Algorithm
Example

Jacobi Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{\substack{j=1, \\ j \neq 4}}^5 a_{4j} \phi_j^{(0)} \right]$$

Row 5:

$$\phi_5^{(1)} = \frac{1}{a_{55}} \left[r_5 - \sum_{\substack{j=1, \\ j \neq 5}}^5 a_{5j} \phi_j^{(0)} \right]$$

Iteration 2:
Row 1:

$$\phi_1^{(2)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1, \\ j \neq 1}}^5 a_{1j} \phi_j^{(1)} \right]$$

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So whatever value we have got from our iteration 1, this stands for iteration level 1 because iteration level zero was our guess level or initially provided value. So iteration level 2 value calculated based on iteration level 1.

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Jacobi Method
General Algorithm
Example

Jacobi Method
Example

Row 4:

$$\phi_4^{(1)} = \frac{1}{a_{44}} \left[r_4 - \sum_{\substack{j=1, \\ j \neq 4}}^5 a_{4j} \phi_j^{(0)} \right]$$

Row 5:

$$\phi_5^{(1)} = \frac{1}{a_{55}} \left[r_5 - \sum_{\substack{j=1, \\ j \neq 5}}^5 a_{5j} \phi_j^{(0)} \right]$$

Iteration 2:
Row 1:

$$\phi_1^{(2)} = \frac{1}{a_{11}} \left[r_1 - \sum_{\substack{j=1, \\ j \neq 1}}^5 a_{1j} \phi_j^{(1)} \right]$$

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Now in this case row 2, row 3, row 4, row 5, these values can be calculated based on our this straight forward process where we calculate the values on previous iteration level value. We are not considering time level here, we are simply considering the iteration level.

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Jacobi Method
General Algorithm
Example

Row 2:

$$\phi_2^{(2)} = \frac{1}{a_{22}} \left[r_2 - \sum_{\substack{j=1, \\ j \neq 2}}^5 a_{2j} \phi_j^{(1)} \right]$$

Row 3:

$$\phi_3^{(2)} = \frac{1}{a_{33}} \left[r_3 - \sum_{\substack{j=1, \\ j \neq 3}}^5 a_{3j} \phi_j^{(1)} \right]$$

Row 4:

$$\phi_4^{(2)} = \frac{1}{a_{44}} \left[r_4 - \sum_{\substack{j=1, \\ j \neq 4}}^5 a_{4j} \phi_j^{(1)} \right]$$

Row 5:

$$\phi_5^{(2)} = \frac{1}{a_{55}} \left[r_5 - \sum_{j=1}^5 a_{5j} \phi_j^{(1)} \right]$$

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So if we talk about the general algorithm, again if we have N numbers of variables. So phi 1, 2 to N minus 1, N. Again this one we can calculate ith row at pth iteration. That depends on r_i minus the summation of j equals to 1 to N and by omitting ith term only. And a_ij, blue terms because we are not changing these coefficients or right hand terms. So that's why a_ij, phi_j p minus 1. And this is valid for all rows. And this is applicable for P greater than equals to 1. So we are dividing it by a ii.

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Jacobi Method
General Algorithm

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \dots \quad \phi_{N-1}^{(0)} \quad \phi_N^{(0)}]^T$$

$$\phi_i^{(p)} = \frac{1}{a_{ii}} \left[r_i - \sum_{\substack{j=1, \\ j \neq i}}^N a_{ij} \phi_j^{(p-1)} \right], \quad \forall i \in \{1, \dots, N\}, p \geq 1$$

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Now in this process we can get the value for pth iteration level. Or by adding and subtracting this phi_i p minus 1 we can get that, let us say that we are adding phi_i p minus 1. We have omitted here. Let us add that value within the system. So if we add this value we have this

efficient a ii, phi_i, i p minus 1. All values are calculated as p minus 1 and divided by a ii. So old value, this is old value plus this is right hand side minus left hand side. Because we have calculated all values here.

So this is at p minus 1 level and a ii. So we can divide it by the diagonal term. So we can say that in this case the Jacobi iterative technique can be generalized like this.

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Jacobi Method
General Algorithm
Example

Jacobi Method General Algorithm

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \dots \quad \phi_{N-1}^{(0)} \quad \phi_N^{(0)}]^T$$

$$\phi_i^{(p)} = \frac{1}{a_{ii}} \left[r_i - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij} \phi_j^{(p-1)} \right], \quad \forall i \in \{1, \dots, N\}, p \geq 1$$

By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij} \phi_j^{(p-1)} - a_{ii} \phi_i^{(p-1)} - \sum_{j=i+1}^N a_{ij} \phi_j^{(p-1)} \right], \quad \forall i, p \geq 1$$

Handwritten red annotations: $\phi_i^{(p-1)} + \frac{RHS - LHS^{(p-1)}}{a_{ii}}$

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That if we have any row i throw, updated level value that should depend on the previous time level value plus residual. The residual means you are right hand side minus your left hand side. This is your right hand side minus left hand side. This is residual for ith row divided by a ii. So we can get a simplified form of our Jacobi iteration or iterative technique.

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Jacobi Method
General Algorithm

Iteration starts with the guess value $\phi^{(0)}$

$$\phi^{(0)} = [\phi_1^{(0)} \quad \phi_2^{(0)} \quad \dots \quad \phi_{N-1}^{(0)} \quad \phi_N^{(0)}]^T$$

$$\phi_i^{(p)} = \frac{1}{a_{ii}} \left[r_i - \sum_{\substack{j=1, \\ j \neq i}}^N a_{ij} \phi_j^{(p-1)} \right], \quad \forall i \in \{1, \dots, N\}, p \geq 1$$

By adding and subtracting $\phi_i^{(p-1)}$ in right hand side

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{1}{a_{ii}} \left[r_i - \sum_{j=1}^{i-1} a_{ij} \phi_j^{(p-1)} - a_{ii} \phi_i^{(p-1)} - \sum_{j=i+1}^N a_{ij} \phi_j^{(p-1)} \right], \quad \forall i$$

In compact form

$$\phi_i^{(p)} = \phi_i^{(p-1)} + \frac{Res_i}{a_{ii}}, \quad \forall i, p \geq 1$$

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Now residual error is important because every iteration there will be error associated with that. Now what should be the stopping criteria? Stopping criteria should be based on this residual error. Now residual error we can utilize to calculate this maximum absolute error. So this is maximum for all i in 1 to N . We can calculate the maximum value of this residual error for different rows and we can get the maximum values. That should be below the desired error. This can be one criteria.

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Jacobi Method
Stopping Criterion

Residual Error in a particular iteration can be expressed as

$$e^{(p)} = A\phi^{(p)} - r$$

Maximum Absolute Error:

$$\max_{i \in \{1, \dots, N\}} |e_i^{(p)}| \leq \epsilon_{max}$$

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And other one can be root mean square error that we have difference, we can take the square, sum it up, divide it by N . And we can take the squareroot of that. And this should be below the desired level. So we can use either this one or this one or we can use the combined one that

maximum value of root mean square error. This can be the maximum absolute error or RMSE value.

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Jacobi Method
General Algorithm
Example

Jacobi Method

Stopping Criterion

Residual Error in a particular iteration can be expressed as

$$e^{(p)} = A\phi^{(p)} - r$$

✓ Maximum Absolute Error: (MAE)

$$\max_{i=1, \dots, N} |e_i^{(p)}| \leq \epsilon_{max}$$

✓ Root Mean Square Error: (RMSE)

$$\sqrt{\frac{1}{N} \sum_{i=1}^N (e_i^{(p)})^2} \leq \epsilon_{max}$$

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Now these values can be utilized for our coding purpose in scilab. Then comes this diagonal dominance. For convergence of iterative schemes we need these diagonal dominance. That means if the modulus value of diagonal term is a ii. If it is equal to sum of absolute value of off diagonal terms and there exist one L for which it is greater, then we can call this as diagonal dominance.

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Jacobi Method
General Algorithm
Example

Convergence Criteria

Diagonal Dominance

Diagonal Dominance:

$$|a_{ii}| = \sum_{\substack{j=1 \\ j \neq i}}^N |a_{ij}|$$

Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^N |a_{ij}|$$

Strict Diagonal Dominance:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^N |a_{ij}|$$

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And if we have weak diagonal dominance then there can be multiple number of rows which can be which can satisfy this particular criteria.

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Jacobi Method
General Algorithm
Example

Convergence Criteria

Diagonal Dominance

Diagonal Dominance:

$$|a_{ii}| = \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}|$$

$$\exists l : |a_{ll}| > \sum_{\substack{j=1, \\ j \neq l}}^N |a_{lj}|$$

Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}|$$

Strict Diagonal Dominance:

$$|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}|$$

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And for strict diagonal dominance, this strictly greater than condition should be (satis) satisfied for all rows. This is valid for all rows and this is also valid for all rows. That means for all i this condition should be satisfied. So we can call it as weak diagonal dominance or strict diagonal dominance.

(Refer Slide Time 21:19)

Jacobi Method
General Algorithm
Example

Convergence Criteria

Diagonal Dominance

Diagonal Dominance:

$$|a_{ii}| = \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}|$$

$$\exists l : |a_{ll}| > \sum_{\substack{j=1, \\ j \neq l}}^N |a_{lj}|$$

Weak Diagonal Dominance:

$$|a_{ii}| \geq \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}| \quad \forall i$$

Strict Diagonal Dominance:

$$|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^N |a_{ij}| \quad \forall i$$

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Now in iterative scheme diagonal dominance is important. Diagonal dominance criteria should be satisfied otherwise we will not get convergence in case of iterative schemes.

Now let us consider example. We already know that we have utilized this problem for our Gauss elimination, LU decomposition and TDMA or tridiagonal matrix algorithm. Now in this case interesting part is that if we consider first row, this first row is greater than mod 0. Because all values are zero, has got no meaning. So we can say that this is greater than zero.

(Refer Slide Time 22:21)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$$

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Then for second row the diagonal term is this one, a ii. So a22 term is mod 2. This is equals to mod 1 plus mod 1. So this is equal.

(Refer Slide Time 22:41)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$$

Handwritten notes: $11 > 0$, $12 = 11 + 11$

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Next row, this is 3 and next condition is 1 plus minus 1. That means 3 is greater than 2. That is why this diagonal dominance is there. It is not strict because one case we have equality

sign. Then this is for fourth row. Again we have this equality condition. Equality condition 1 plus mod 1. And 5th row again we have diagonal dominance in this case.

(Refer Slide Time 23:35)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$$

Handwritten notes:

- $|1| > 0$
- $|2| = |1| + |1|$
- $|3| > |1| + |1| - |1|$
- $|2| = |1| + |1|$
- $|1| > 0$

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So we can see that for 3 rows 1, 2, 3, we have diagonal dominance or strict dominance. And for two rows we have equality sign available.

(Refer Slide Time 23:55)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$$

Handwritten notes:

- $|1| > 0$
- $|2| = |1| + |1|$
- $|3| > |1| + |1| - |1|$
- $|2| = |1| + |1|$
- $|1| > 0$

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Now in this case if we write our conditions then we can write our code in scilab. So let us use it for this one.

(Refer Slide Time 24:20)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 11 \\ 28 \\ 9 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$$

scribble

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We can use the structure here. The clc again this clear screen, this is clearstructure or clear memory. This count is for number counter for iteration. Rmse is root mean square error and phi. Phi is the variable. Now in Jacobi iteration we need it A. A is the matrix. R then phi o. Phi o is the initial guessthat needs to be supplied and epsilon max. This is maximum allowable error which needs to be specified for this case.

(Refer Slide Time 25:22)

```
1 clc;
2 clear;
3 function [count,rmse,phi] = jacobi(A,r,phi0, eps_max)
4 //Linear Equation: A*x=r
5
6
7 [nr_A,nc_A]=size(A) //Size of A
8 [nr_r,nc_r]=size(r) //Size of r
9
10 if nc_A <> nr_A then
11     error('A is Not a Square Matrix')
12     abort;
13 end
14 if nc_A <> nr_r then
15     error('Not Compatible Matrices')
16     abort;
17 end
18 n=nc_A
19 count = 0
20 rmse=1
21 while rmse > eps_max
22     rmse=0
23     for i=1:n
24         resi=r(i)
25         for j=1:n
26             resi = resi - A(i,j)*phi0(j)
27         end
28         phi(i)=phi0(i)+resi/A(i,i)
29         rmse=rmse+(resi/A(i,i)).^2
30     end
31     phi0=phi
32     rmse=sqrt(rmse/n)
33     count = count + 1;
34 end
```

Now counter we already know this part from our Gauss elimination, LU decomposition and other course. So we are starting with count equals to zero. So counter is zero. Initially I am specifying rmse equals to 1. Why? Because otherwise this while loop will not be executed here. So if rmse is greater than epsilon max then only this will be executed. So let us say that

rmse equals to 1 then automatically the process will start within this. And at the beginning itself, I am specifying rmse equals to zero.

(Refer Slide Time 26:13)

```

1 clear
2 function [count,rmse,phi] = jacobi(A,r,phio, eps_max)
3 //Linear Equation: A*phi=r
4 [nr_A,nc_A]=size(A) //Size of A
5 [nr_r,nc_r]=size(r) //Size of r
6
7 if nc_A <> nr_A then
8     error('A is Not a Square Matrix')
9     abort;
10 end
11 if nc_A <> nr_r then
12     error('Not Compatible Matrices')
13     abort;
14 end
15 n=nc_A
16 count = 0
17 rmse=1
18
19 while rmse > eps_max
20     rmse=0
21     for i=1:n
22         resi=r(i)
23         for j=1:n
24             resi = resi - A(i,j)*phio(j)
25         end
26         phi(i)=phio(i)+resi/A(i,i)
27         rmse=rmse+(resi/A(i,i)).^2
28     end
29     phio=phi
30     rmse=sqrt(rmse/n)
31     count = count + 1;
32 end
33 endfunction
34
35 //A=[1 0 0 0 0
36 // 2 1 0 0
37 // 0 1 3 -1 0
38 // 0 0 1 2 1
39 // 0 0 0 0 1];
40
41 //r=[1
42 //12
43 //11
44 //20
45 //9];
46
47 A=[1 2 0 0 0

```

So I am initializing everything. Now in this process I am running it for all rows. So i starting from 1 to N. This residual let us say r_i , this is right hand side minus for all j. So $resi$ equals to $resi$ i minus a_{ij} into $phio$ that is initial guess minus j for j th element, we are calculating this. Now this is nothing but right hand side minus left hand side and we can divide it by the diagonal term a_{ii} . So $rmse$ is nothing but the difference in old value and new value. So old value and new value difference is $resi$ divided by a_{ii} . So I am taking square of that.

(Refer Slide Time 27:33)

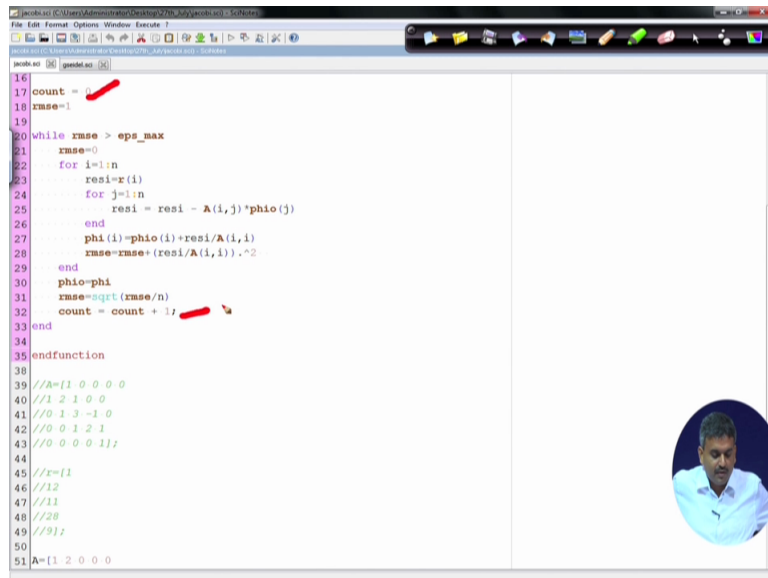
```

16 count = 0
17 rmse=1
18
19 while rmse > eps_max
20     rmse=0
21     for i=1:n
22         resi=r(i)
23         for j=1:n
24             resi = resi - A(i,j)*phio(j)
25         end
26         phi(i)=phio(i)+resi/A(i,i)
27         rmse=rmse+(resi/A(i,i)).^2
28     end
29     phio=phi
30     rmse=sqrt(rmse/n)
31     count = count + 1;
32 end
33 endfunction
34
35 //A=[1 0 0 0 0
36 // 2 1 0 0
37 // 0 1 3 -1 0
38 // 0 0 1 2 1
39 // 0 0 0 0 1];
40
41 //r=[1
42 //12
43 //11
44 //20
45 //9];
46
47 A=[1 2 0 0 0

```


So after this now I am specifying this old value as new value. So in place of old value I am changing this thing. So and finally in this process I have calculated rmse for all rows and finally its square root sqrt of rmse divided by n. So taking some of that we are dividing it by n. And then we are taking it as square root. And count is equal to count plus 1 that every iteration. So we have started with counter zero and this process we are counting.

(Refer Slide Time 28:27)



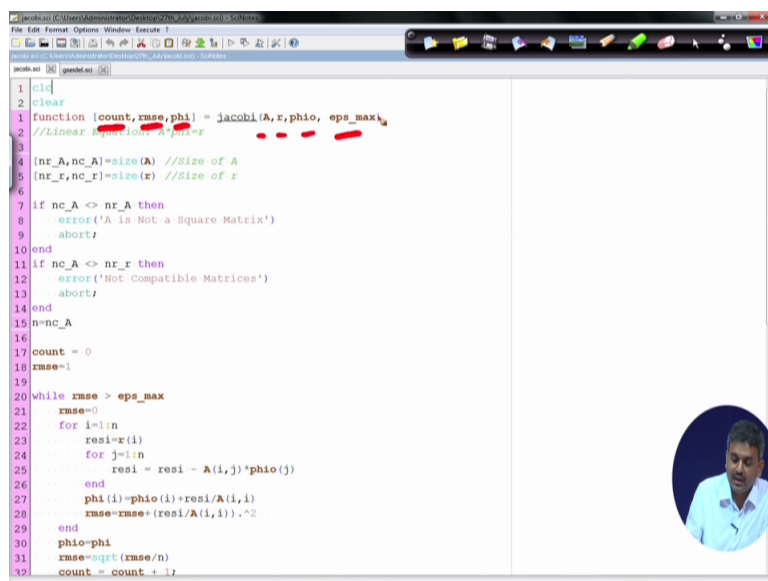
```

16 count = 0;
17 rmse=1;
18
19 while rmse > eps_max
20     rmse=0;
21     for i=1:n
22         resi=r(i);
23         for j=1:n
24             resi = resi - A(i,j)*phio(j);
25         end
26         phi(i)=phio(i)+resi/A(i,i);
27         rmse=rmse+(resi/A(i,i)).^2;
28     end
29     phio=phi;
30     rmse=sqrt(rmse/n);
31     count = count + 1;
32 end
33 endfunction
34
35 //A=[1 0 0 0 0
36 //1 2 1 0 0
37 //0 1 3 -1 0
38 //0 0 1 2 1
39 //0 0 0 0 1];
40 //r=[
41 //12
42 //11
43 //28
44 //9];
45 A=[1 2 0 0 0

```

So count equals to count plus 1. So we have calculated rmse, we have calculated count. Now let us use this as function for calculation of our case. So function is count rmse phi and a, r, phi o, epsilon max, these are the values.

(Refer Slide Time 28:57)



```

1 clc;
2 clear;
3 function [count,rmse,phi] = jacobi(A,r,phio, eps_max)
4 //linear equation A*x=r
5
6 [nr_A,nc_A]=size(A) //Size of A
7 [nr_r,nc_r]=size(r) //Size of r
8
9 if nc_A <> nr_A then
10     error('A is Not a Square Matrix');
11     abort;
12 end
13 if nc_A <> nr_r then
14     error('Not Compatible Matrices');
15     abort;
16 end
17 n=nc_A;
18 count = 0;
19 rmse=1;
20 while rmse > eps_max
21     rmse=0;
22     for i=1:n
23         resi=r(i);
24         for j=1:n
25             resi = resi - A(i,j)*phio(j);
26         end
27         phi(i)=phio(i)+resi/A(i,i);
28         rmse=rmse+(resi/A(i,i)).^2;
29     end
30     phio=phi;
31     rmse=sqrt(rmse/n);
32     count = count + 1;

```

Now in this process let us say that we have our original tridiagonal matrix available. So 121, this is 131. Now in this case if we calculate this. In this case let us say this is our definition regarding A, this is r vector, this is phi o. We are starting from zero-zero value. We don't know what is the value. So epsilon max, we have specified 1×10 to the power minus 6. So 1×10^{-6} means 1 into 10 to the power minus 6. So we are calling this Jacobi function with count, rmse, phi and we are providing a, r, phi o in terms of zero-zero and epsilon max as 1×10^{-6} .

(Refer Slide Time 30:26)

```

23     resi=r(i)
24     for j=1:n
25         resi = resi - A(i,j)'phi(j)
26     end
27     phi(i)=phi(i)+resi/A(i,i)
28     rmse=rmse+(resi/A(i,i)).^2
29 end
30 phi=phi
31 rmse=sqrt(rmse/n)
32 count = count + 1;
33 end
34
35 endfunction
36
37 A=[1 0 0 0 0
40 2 -1 0 0
41 0 1 3 -1 0
42 0 0 1 2 1
43 0 0 0 0 1];
44
45 r=[1
46 12
47 11
48 28
49 9];
50
51 phi={0
52 0
53 0
54 0
55 0};
56 eps_max=1e-6;
57 [count,rmse,phi] = Jacobi(A,r,phi, eps_max)
58

```

So if we run this program by selecting it and then executing it. So you can see that we are getting the exact desired value 1, 3, 5, 7, 9. And in this case this rmse is very less. It is close to zero 10 to the power this is minus 16 and count 5. That means with 5 iteration only we are getting this solution.

(Refer Slide Time 31:15)

```

> eps_max=1e-6;
> eps_max = eps_max + 1;
> end
> endfunction
-->
--> A=[1 0 0 0 0
> 1 2 1 0 0
> 0 1 3 -1 0
> 0 0 1 2 1
> 0 0 0 0 1];
-->
--> r=[1
> 12
> 11
> 28
> 9];
-->
--> phi(0)=0
> 0
> 0
> 0
> 0];
--> eps_max=1e-6;
--> [phi(1), phi(2), phi(3)] = jacobi(A, r, phi(0), eps_max);
phi =
     1
     2
     3
     4
     5
  
```

So with this we have another example. This is again 1, 2, 3, minus 5, like that.

(Refer Slide Time 31:50)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

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So this is 1, 2, 3, minus 5, 0, 0, minus 3. So in this case clearly it's visible, this is our diagonal term, this is not greater than the off diagonal term. So already we know that the solution is, 1, 2, 3, 4, 5. But let us see what value we are getting out of our calculation process.


(Refer Slide Time 33:49)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$


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In this case also this is 4 to 6 which is greater than 3. 3 is our diagonal term. In this case also 10 which is 7 and 20. 20 is greater than 10. Again 2, 9 all the cases we have strict violation.


(Refer Slide Time 34:15)

Jacobi Method
General Algorithm
Example

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 3 & -5 & 0 & 0 \\ 0 & -4 & 3 & -2 & 0 \\ 0 & 0 & -7 & -10 & 13 \\ 0 & 0 & 0 & -9 & 2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -7 \\ 4 \\ -26 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$


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So this iterative method is not applicable for this kind of matrixes. Thank you.