Computational Hydraulics
ProfessorAnirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 26
Algebraic Equation: LU Decomposition Method

Welcome to this lecture number 26 of the course computational hydraulics and we are in module 2, numerical methods. And in this particular lecture we will discuss unit 22, algebraic equation and we will try to discuss LU decomposition method for decomposition of matrixes to get the solutionusing direct numerical approach.

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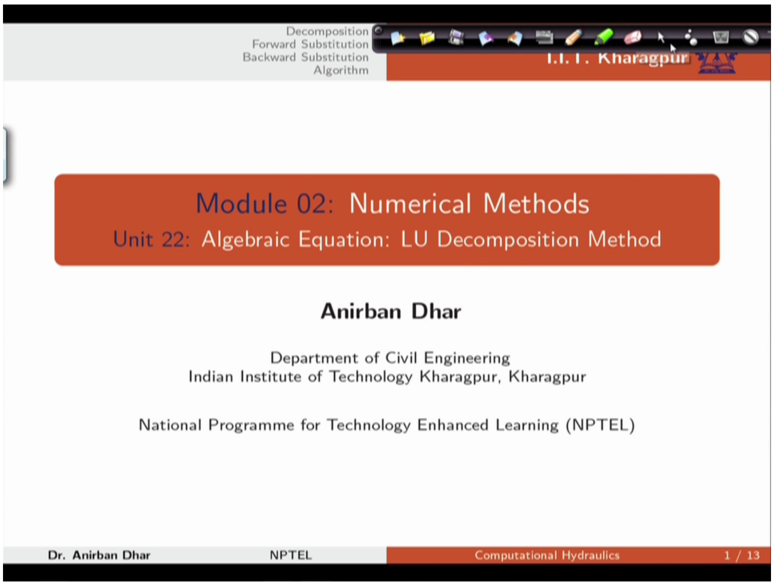

So what is the learning objective? At the end of this particular unit students will be able to apply LU decomposition method for direct solution.

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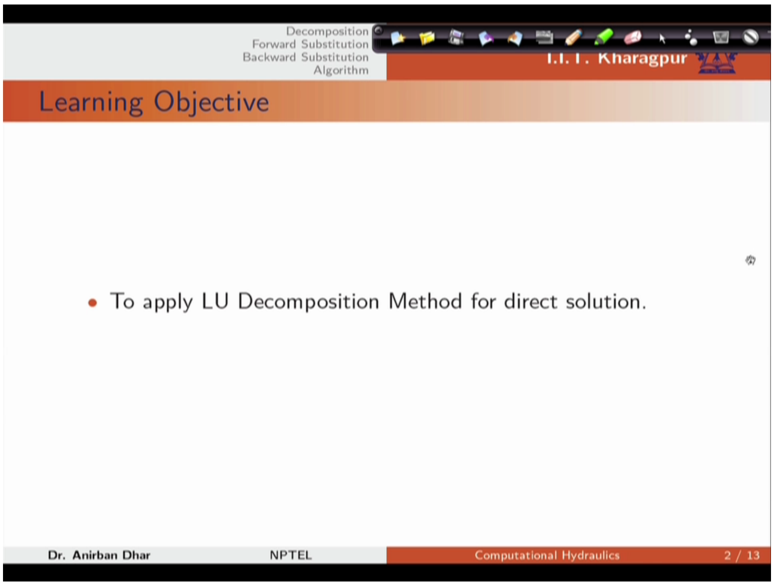

So we already know what is our matrix form? We have Aas constant coefficient matrix which is a square matrix. Phiis our variable vector and r is right hand side vector. In this case we haveagain a square matrix format, N cross N and N cross 1, then again N cross 1 here. So these are compatible matrixes.

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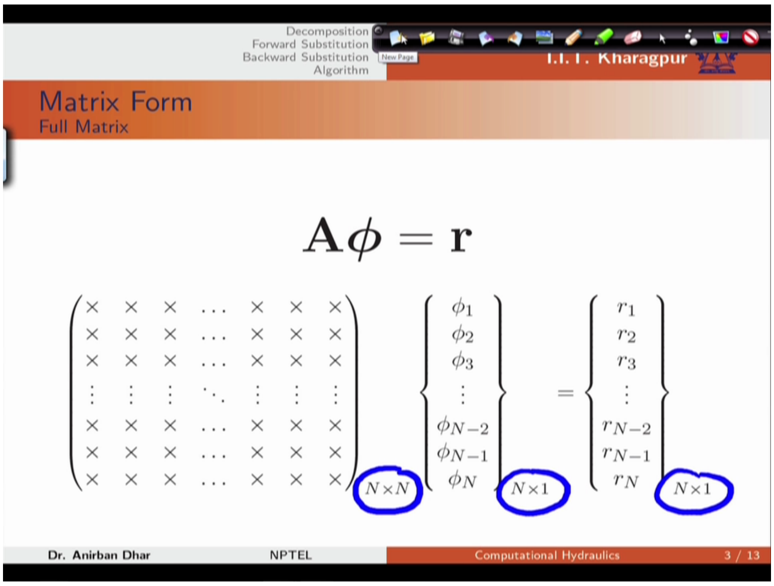

In this case our LU decomposition processit has got few steps. First step isdecomposition step. We will decompose this A matrix into one lower and one upper triangular matrix. Next step is forward substitution. So in this process we will utilize our lower triangular matrix for forward substitution step. And this psi, this is some kind ofsecondary variable that we will utilizefor forward substitution.

So in this process essentially we are solving L psi equals to r equation. And we can get this psi value here. But psi is not our ultimate variable. We need to get the phi.

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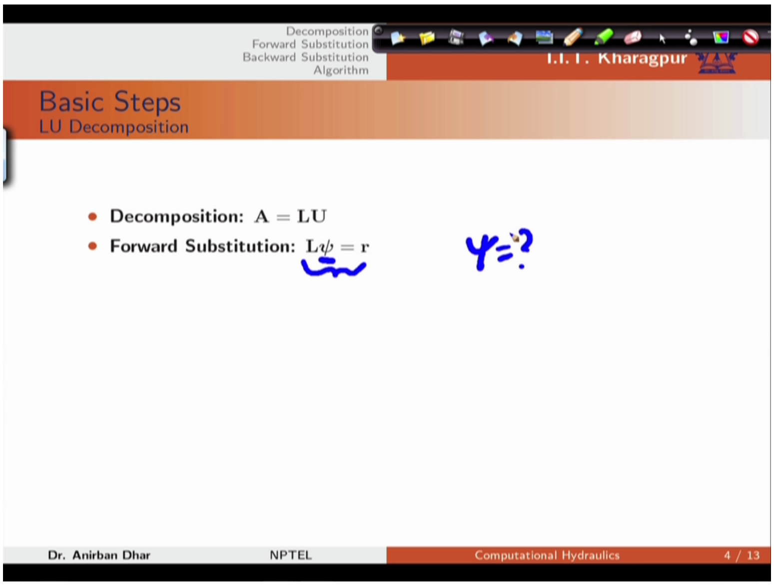

So in this last step, backward substitution step,we will utilize U phi equals to psi. So U is the upper triangular matrix that is generated from our decomposition step and phi is a variable vector. At psi whatever we have calculated from our substitution orforward substitution step or second step,we will directly utilize it in the right hand side vector and we will calculate the phi value which is our desired value.

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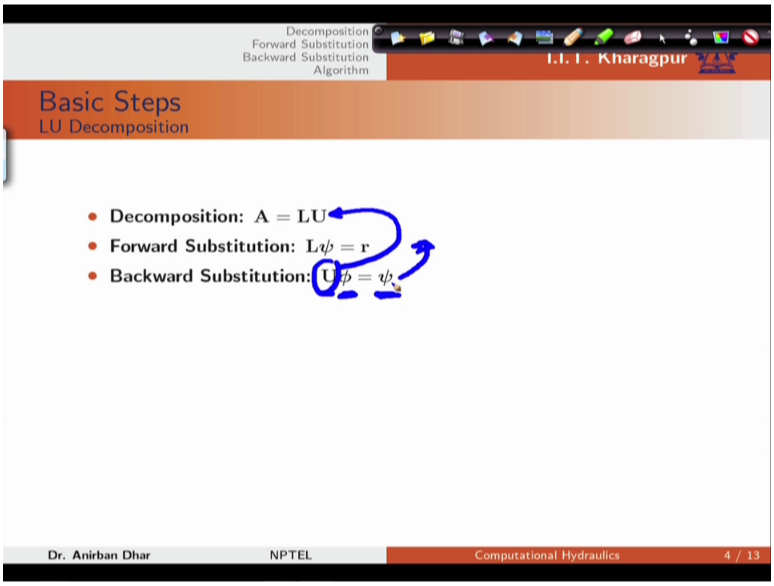

So what is the basis for this one? Basis is that if we multiply this we can write this for this step as U phi minus psi equals to zero. Now this step if you multiply L in this case,so this is essentially first multiplication it is coming as LU. LU is A. Again L psi from our forward substitution step, this is r. So essentially we are solving the same problem but with multiple steps. So LU equals to A and L psithis is equal to r.

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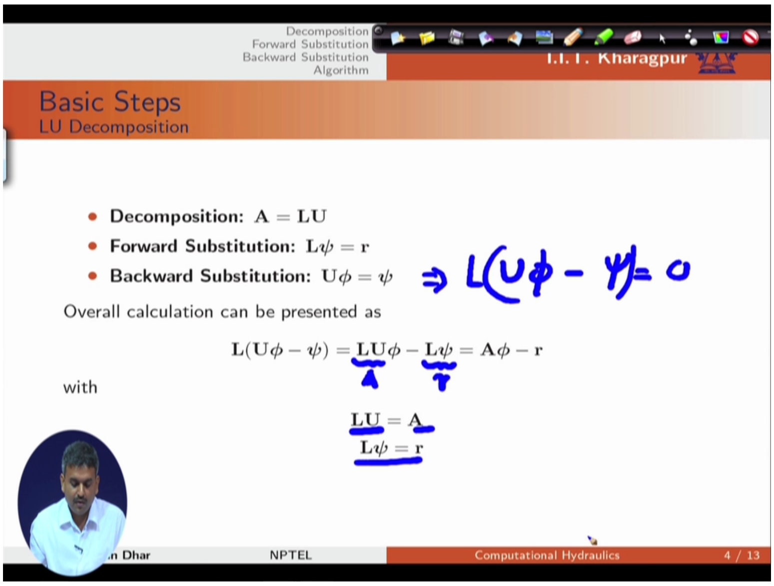

Nowwe will try to utilize this concept and we will try to implement this in algorithm format. So if we see our Gauss elimination thing, in LU decomposition we can utilize our Gauss elimination concept. So what is the basis? Basis is that matrix generated from forward elimination process is this one, wherefirst one is unchanged, second one is changed, third one changed, fourth one and fifth one, these are changed values of the coefficient matrix. And we have generated upper triangular matrix out of this Gauss elimination or forward elimination process.

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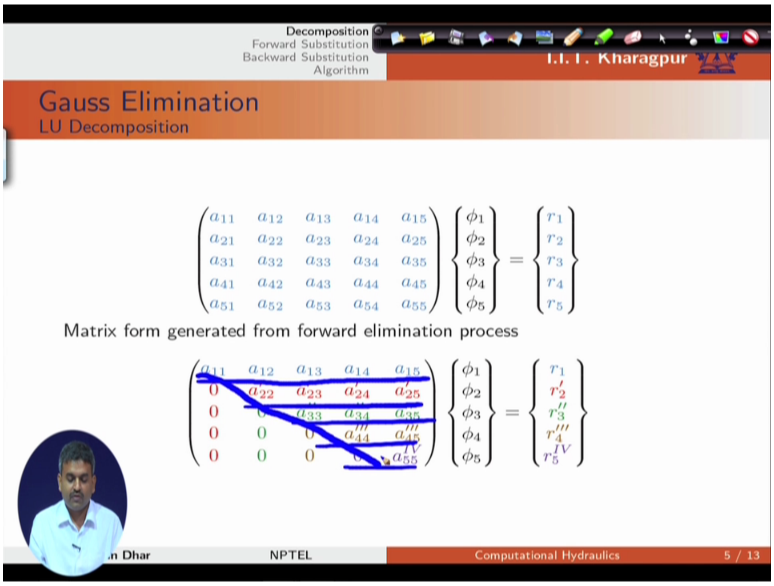

Now in this casewe can see that we have zero values stored in this lower triangular portion.

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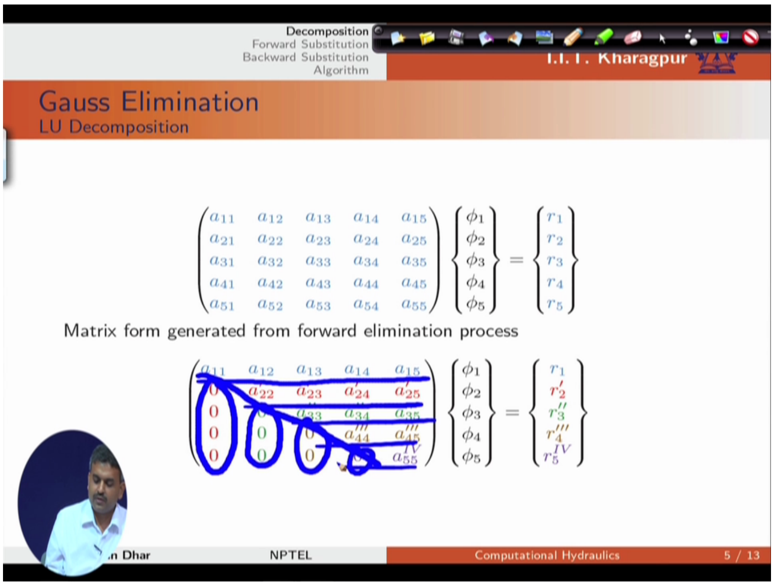

Now in this process, in the first step gamma 12. That means gamma multiplied with first row for second row, gamma multiplied with first row for third row, gamma multiplied with first row for fourth and fifth row were multiplied with row 1. So in this case multiplication factors can be stored. So multiplication factors in this case stored like this. This is gamma 1 2, gamma 13, gamma 14, gamma 15.

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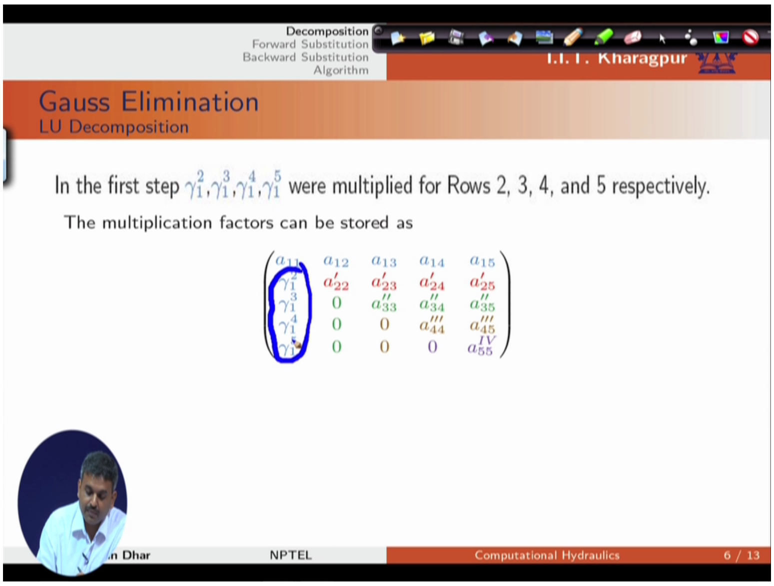

Similarly for this casewe can again store other values. Other values are generated from different steps. These are actually factors that we have utilized to multiply our reference row.

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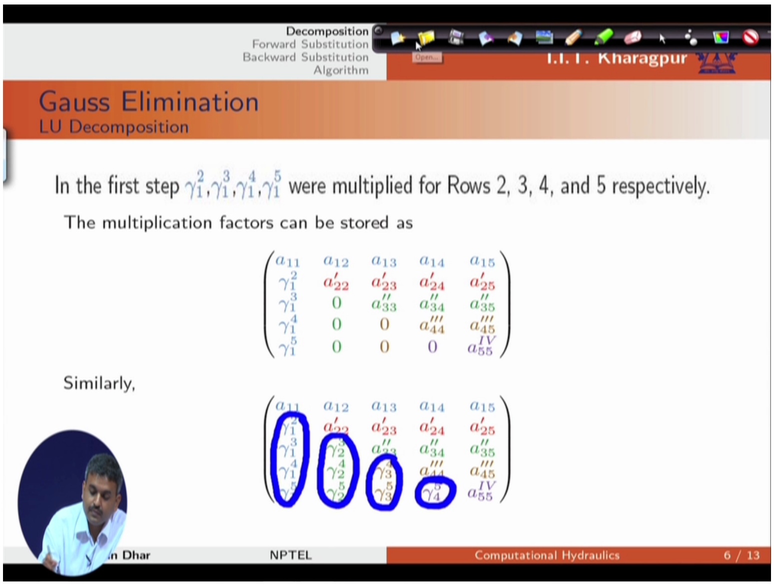

Now for this one, LU decomposition step we can write it as, this is our upper triangular matrix or U. Andthis is our lower triangular matrix with diagonal term 1, 1, 1, 1 and the factors are stored in the lower portion. These are all factors that we have utilized. And this is the thing that we have got from our Gauss elimination process.

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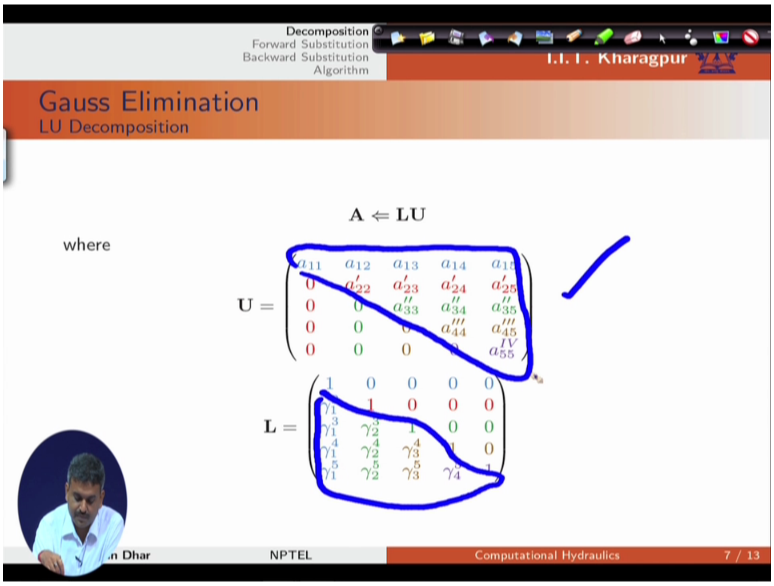

Now in this case forward substitution, we can say that in place of gamma we are writing it in a more structured way. That l 21 this is essentially gamma 1 2. So why this is forward substitution? So we are directly substituting the value in this case psi 1 equals to your r1. And in this case next is psi 2 which is r2 minus psi 1 and this l 21.

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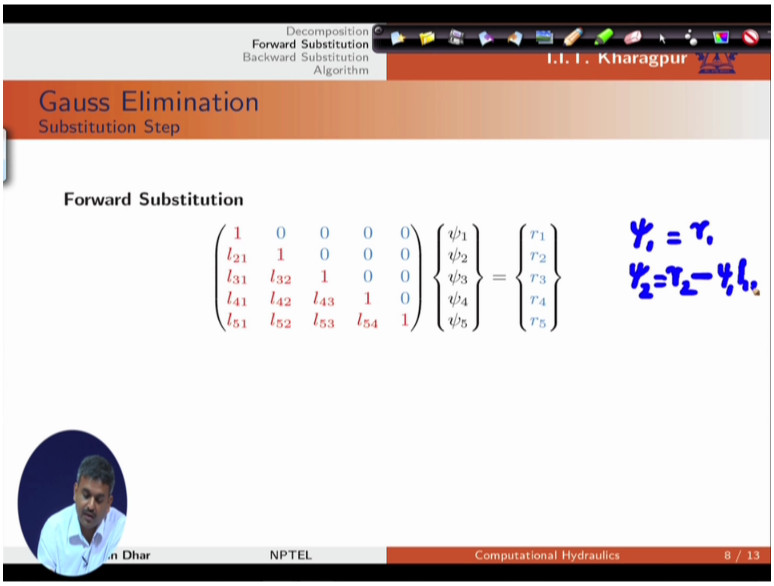

Soobviously we can utilize our valueswhich we have calculated in our first decomposition step using Gauss elimination andwe can get this forward substitution. Now in this casegeneral algorithm is like this. Psi ican be generated ri minus j equals to 1 to this I minus 1. I minus 1 because for iminus 1 number of variables only we will have updated values available and our matrixthat is a lower triangular matrix. So we will consider I values starting from 2to N.

Now this means that for all i within this set 2 to N, we can utilize this. So this is i minus 1. For 2 this will be up to 1. That means only one term. Ifour iis equal to 3, so obviously that meansthese two values should be available. That's why thisj equals to 1 to 2.

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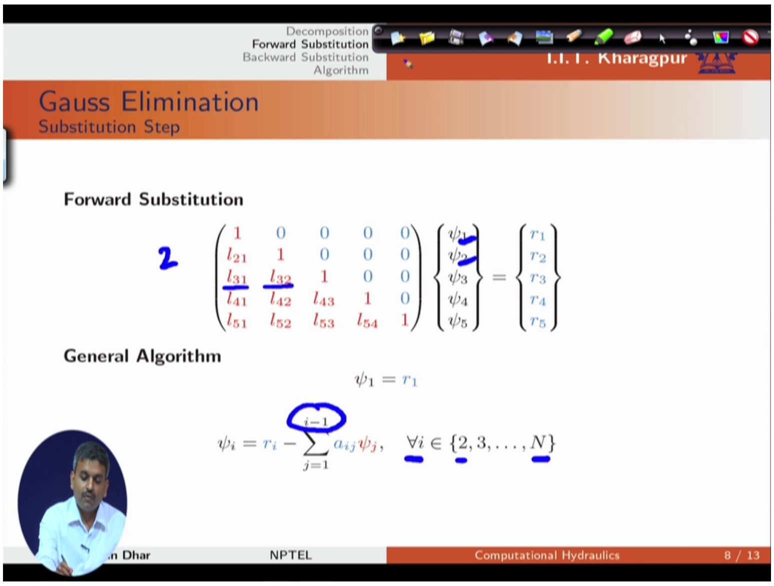

Now in this case the next step is backward substitution. Backward substitution again we can directly write in terms of phi N. Phi N is psi N divided by a NN.A NN because in this case we are storing all values in our Amatrix and in general because psi values are calculated values. That's why with the red color. And phi j values, these are also calculated values. Because for i if you are considering i equals to 4, so obviously this should be phi to 5. That means only one term.

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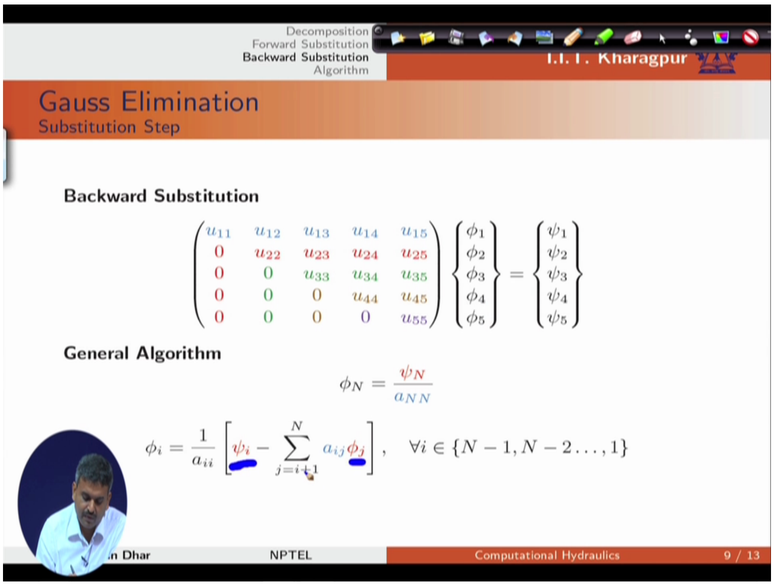

So our calculation is valid from N minus 1 to 1. And with this general steps we candirectly calculate our values. Obviously I have not utilized this U 55 here because we are storing all the values in a 11 to a nn format where a n1 to a 1n. That means N into N one matrix is our A matrix. So we are utilizing all the values here for our calculations.

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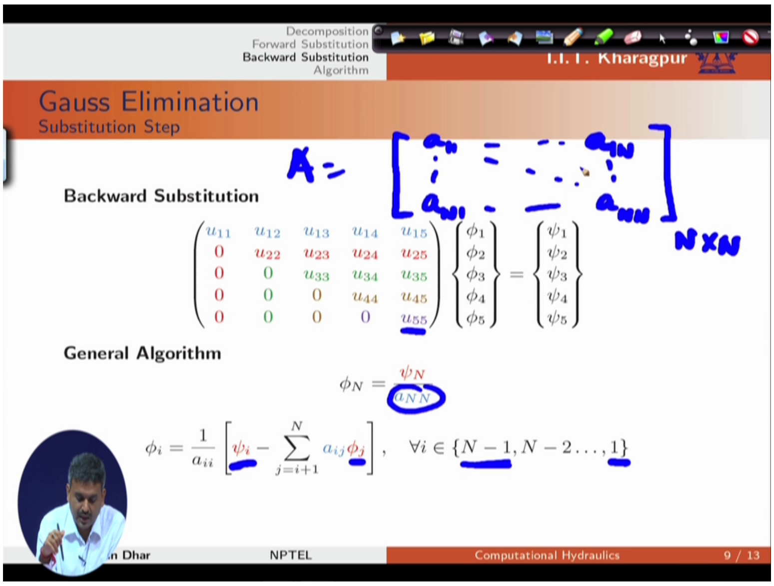

Now let us see what is there in the algorithm of our LU decomposition. In LU decompositionthis step is similar to our Gauss elimination decomposition step. So same we are starting from reference row and we are going up to n minus 1 and i, k plus 1, that means leaving that reference row we are considering other rows for calculation. And we are generating this gamma.

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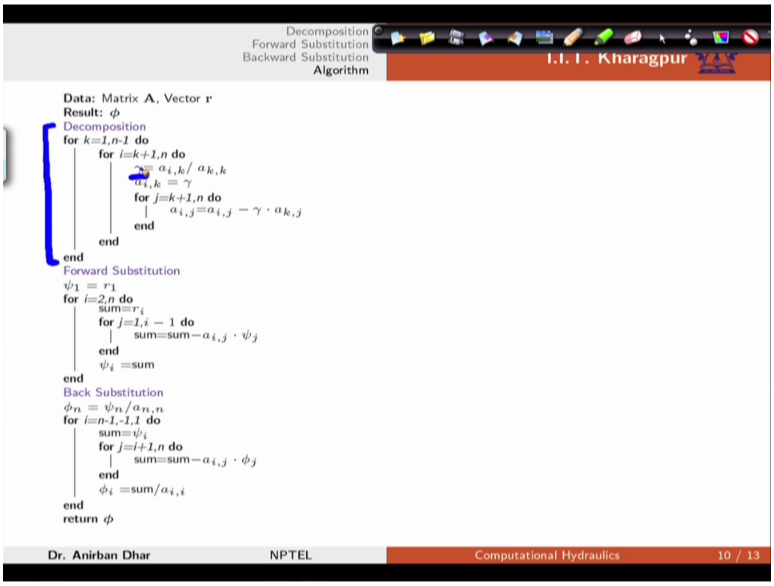

So gamma is your actor ik and divided by kk. So ik again we are storing this gamma because this is for storage of lower triangular matrix here. And this process is as usual aij, aij minus gamma akj. Now in this process please remember one thing that we are not considering changes in r. So leaving r, other calculations are same. Only thing is that this step where we are storing aik. In place of aik we are storing this gamma factor values.

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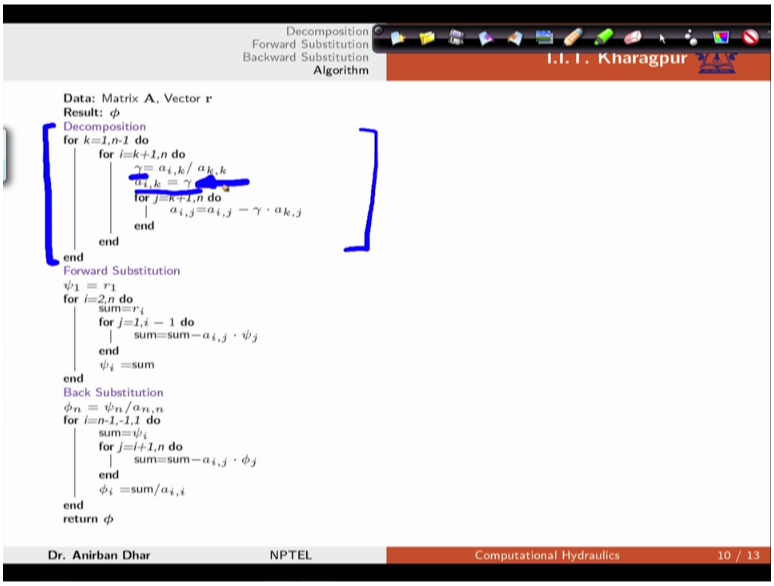

So next step is forward substitution, psi 1 equals to r1. And we are running this loop from 2 to N. Sum equals to ri, j equals to 1 to i minus 1. That means to considerthe diagonal term only. Corresponding to diagonal term we are calculating the psi values. So psi should be sum directly,because we have considered initial value as i and from that we are subtracting other j related values. So directly we are getting psi i equals to sum. Because in diagonal term for lower triangular matrixonly unity is there. So no division is required.

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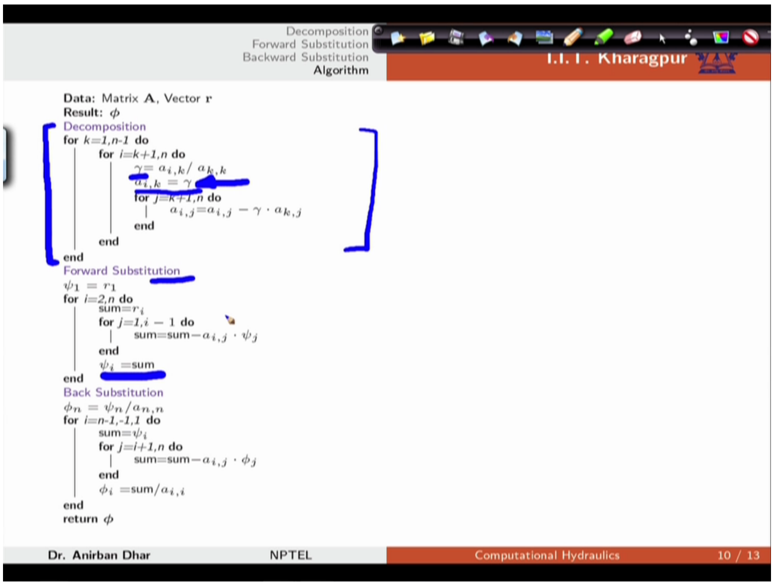

Next is backward or back substitution. Back substitution phin equals to psi n divided by a nn. And this ismore or less similar to our Gauss elimination process. So except the psi, in place of psithis r was there. So in this case psi i we are placing here, so sum equals to sum minus aij phi j.Phi is already calculated value from our calculation process. So in this case coefficient terms are there. Because diagonal term we have upper triangular matrix including the diagonal term.

So we need to divide it with a ii which is part of upper triangular matrix. So phi i equals to sum divided by a ii. Now in this process we have generated this vector phi. And this is our solution. So result is phi.

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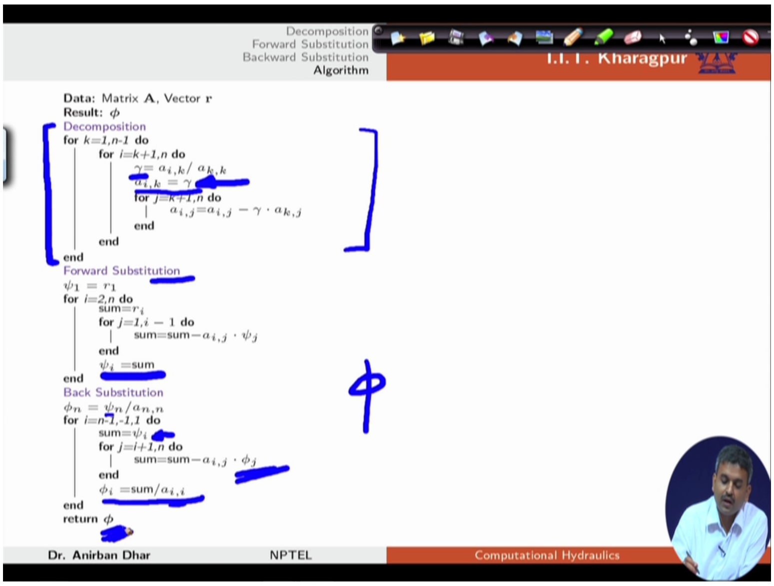

Now we need to implement this again with the scilab code. Now I will discuss the implementation of the code so that you will understand what is therewithin the process. Now again we will considertwo example problems. One with a simple our banded matrix, one diagonal and 2 of diagonal terms. And second one is bit complicated with multiple number of variables.

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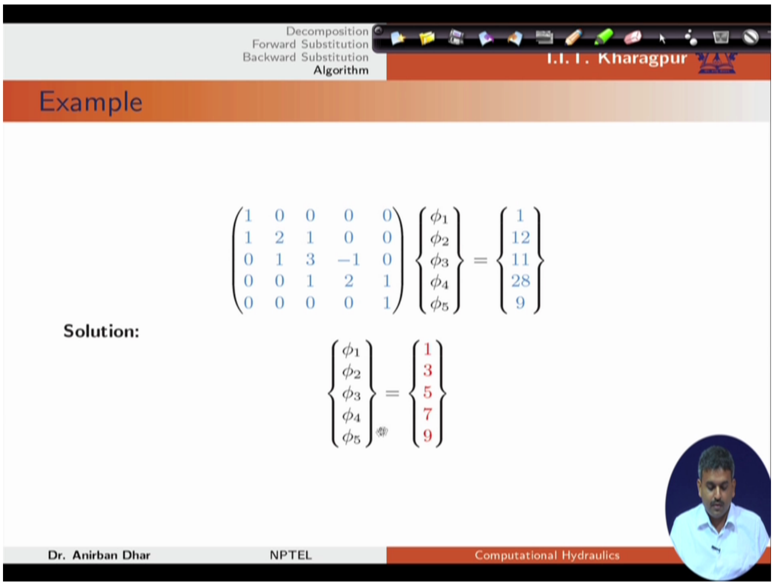

Now if we consider our problem with scilab thenwe can say that this is our LU decomposition. So it starts with again the clcr clear command or clear screen command. Then clear, clear means clear of variables. Again ludcomp is our function name and function output is phi and input required is a, r. And this is linear equation because we are considering A phi equals to r problem in our case.

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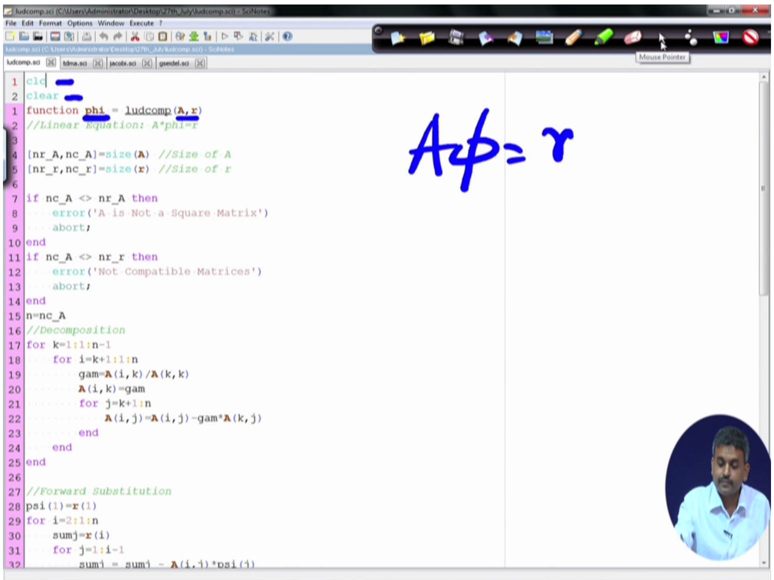

So this part is common. We can check whether it is a square matrix or not. Or compatibility issues, because we need to operate on a and r. If there is problem in the inputit can create further calculation related problem and your LU decomposition thing is not valid in that case.

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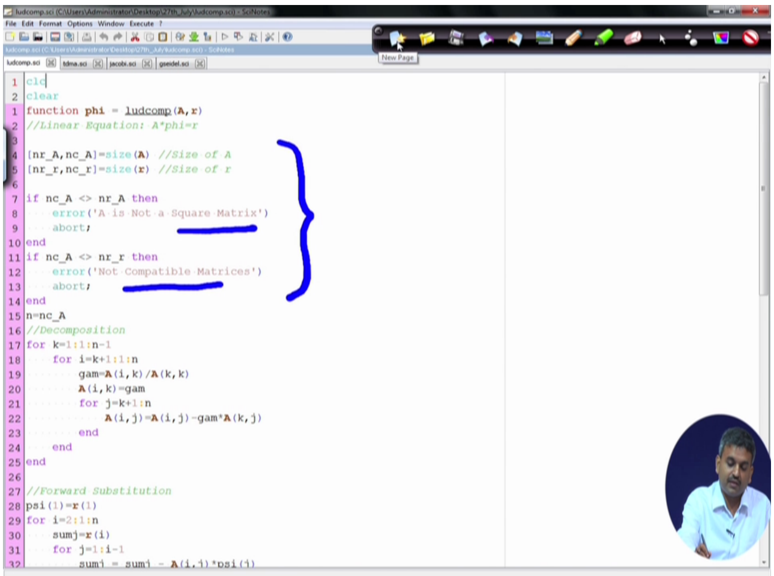

Now this is ourdecomposition step. Now in this decomposition step we are following the same algorithm that we have used in the Gauss elimination. We are starting from 1 to N minus 1. And i is running from k plus 1 to N. And this is also k plus 1 to N. This gamma is aik divided by k, k. K is your reference row and aik we are storing this gamma. Now this part is different with respect to our Gauss elimination.

And for j k plus 1 again we are changing our aij value. So obviously in this case these are updated aij values. So this is all aboutthe decomposition step. Now in this step we have dividedthe matrix into lower and upper.

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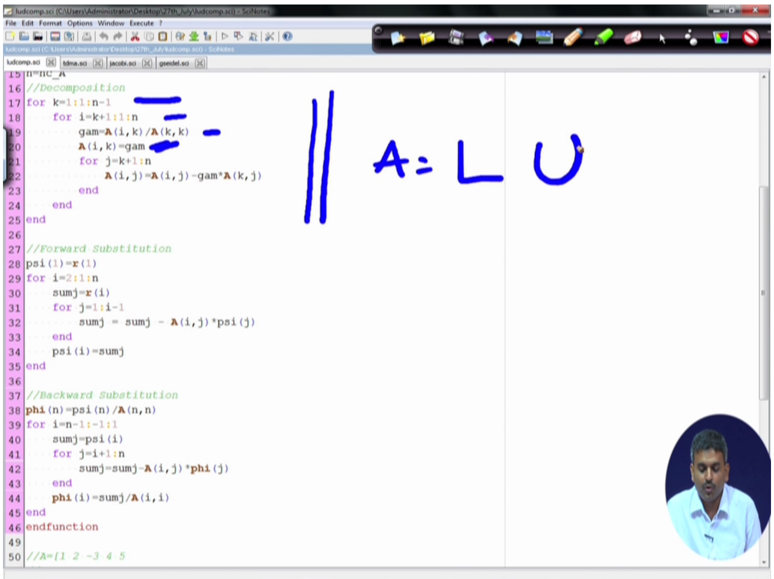

Now one thing maybe question is there that in lower triangular matrix we have considered that all values are in diagonal as one. But we have not written that thing here, because there is no scope for incorporation of this unit value in diagonal. So this diagonal thing we are directly utilizing in this forward substitution process.

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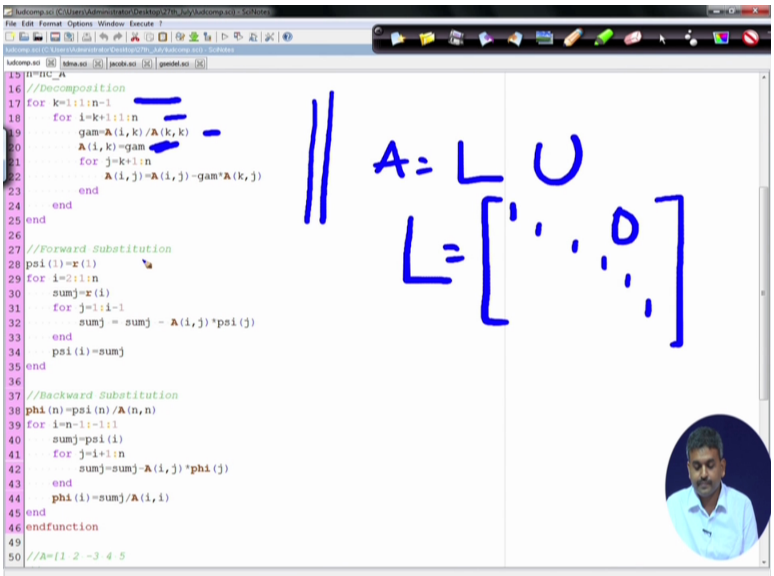

We are considering that the multiplication term with psi, so L multiplied psi equals to r is your forward substitution process. So directly we are writing it psi equals to r1.

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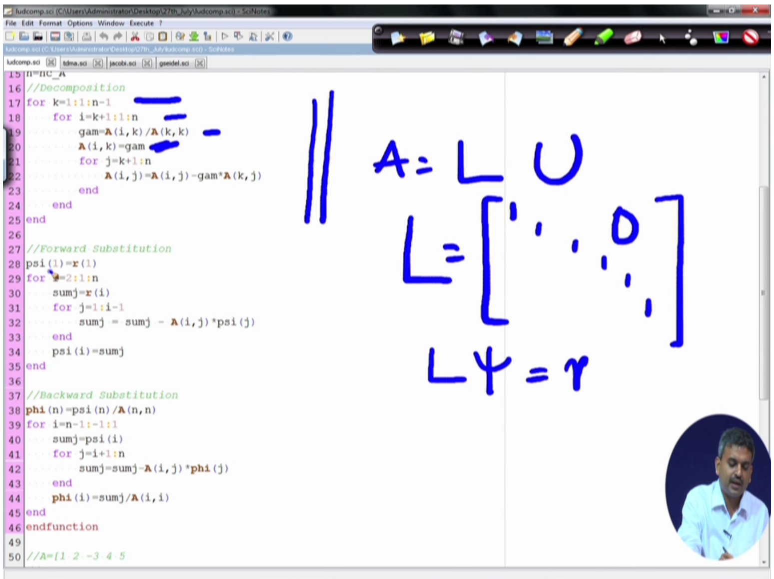

So psi 1 equals to r1. That means we are implicitly considering our coefficient as one.So we are not multiplying any quantityhere or dividing any quantity here and we are not storing this value in our A matrix. We are implicitly implementing this thing.

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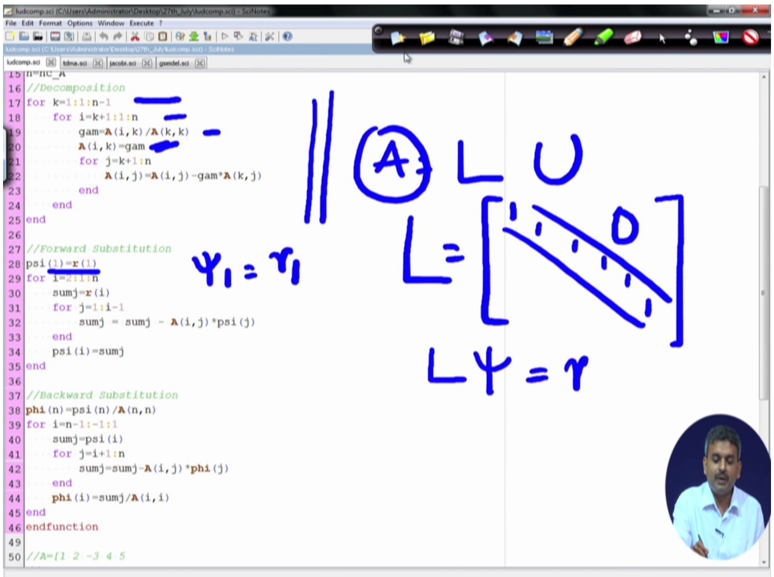

Now forward substitution, it is varying from 2 to N. So sum equals to ri. And sum j, aij, psi j, whatever value we have andpsi i equals to sum j. As I have told that implicitly we are considering that our diagonal term already coefficient one is there. So we are not multiplying or dividing any quantity here.

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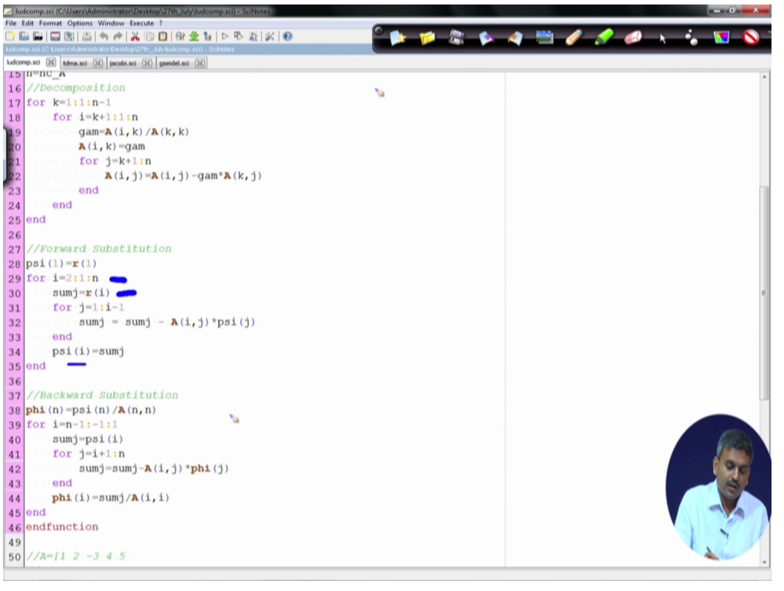

Then last step is our backward substitution. Again we are following the Gauss eliminationsteps. So phi n equals to psi n divided by a nn. So this is the firstcalculation step. But we are starting from last row and we are moving upward direction. In the forward direction moment is there in the forward elimination. Decomposition step is also forward. In forward step we are constructing our L and U matrixes.

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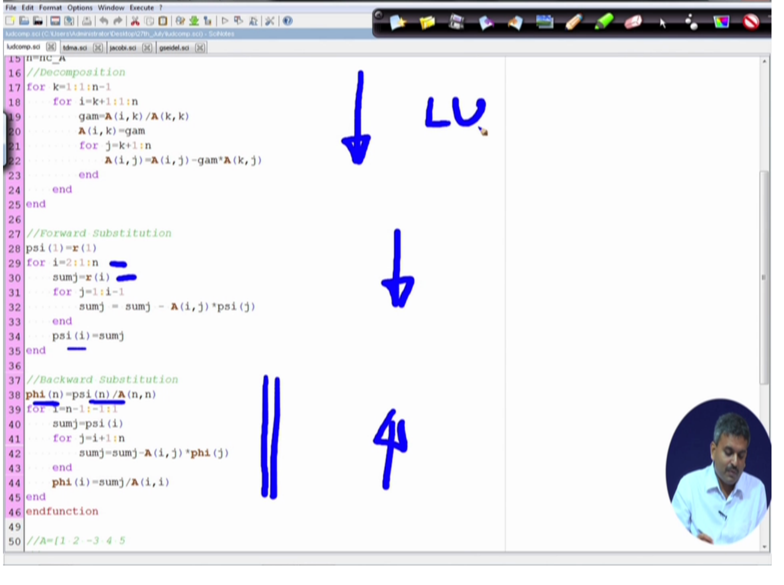

So in backward step we have already got the value of phi n. And from this phi nwe are calculating the values for n minus 1 to 1with the increment of minus 1. So sum j is equals topsi i, because you havethis thing. In the next step or backward substitution step this is upper triangular matrix.

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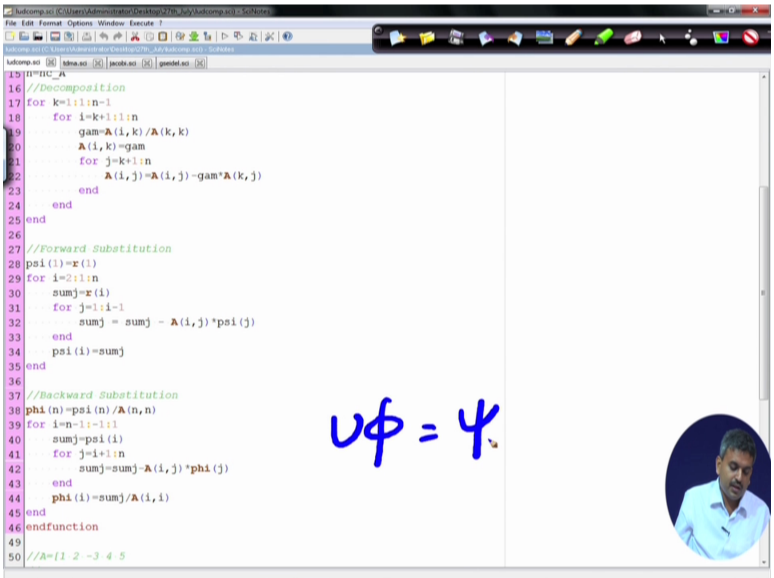

Now in this case psi values are already known valuesand these are acting as right hand side vector. So aijinto phi j, we are subtracting this thing and calculating sum j. But in this case, in upper triangular matrix we have diagonals available. And those diagonals terms are either 1 or any value. So we cannot generalize that. So we are utilizing those diagonal terms for calculation of phi i. So phi i equals to i divided by sum j equals to sum jdivided by a ii. So we're getting phi ivalues.

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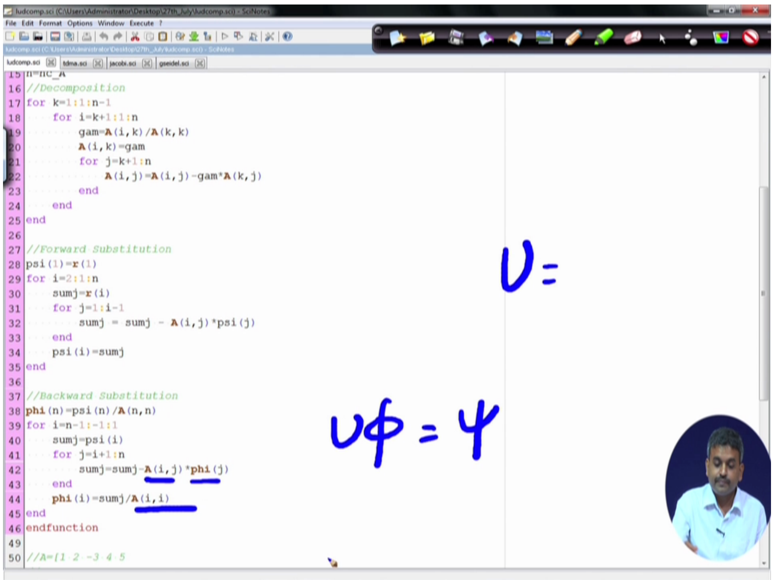

Now if we utilize our concept wise thing, solet us consider our simplified case wherewe have only phi1, phi2, phi3, phi4 and phi5. So these are odd numbers 1, 3, 5, 7 and last one is 9.

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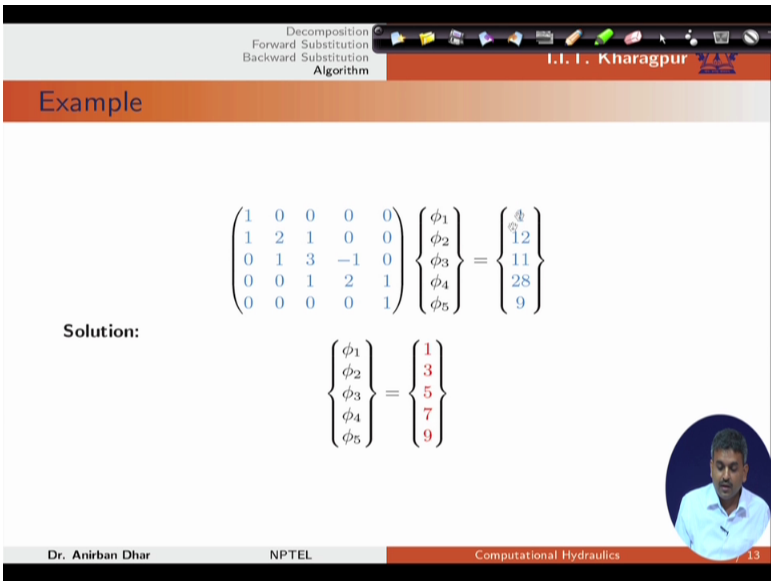

And storing this A value here and this is r. So with thiswe should get 1, 3, 5, 7, 9 as our solution. Now we can select this with the right click, evaluate command, we can check whether we are getting.

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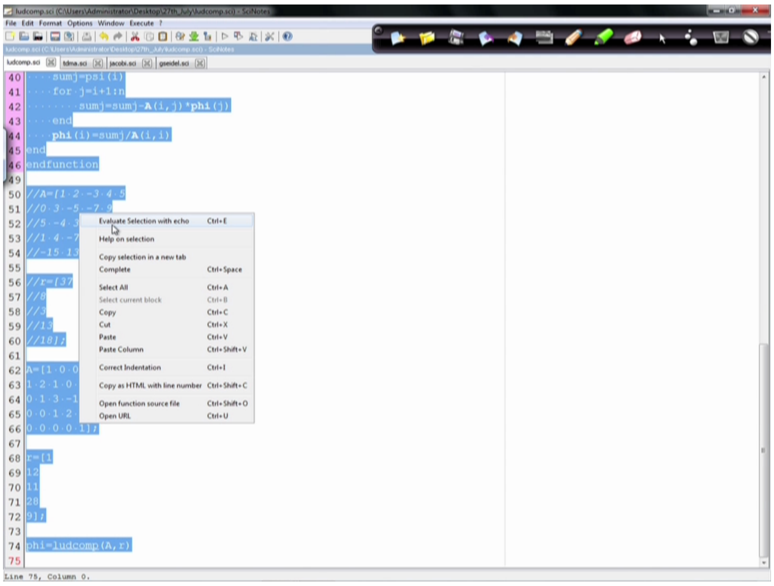

So we can see that exactly we are getting the solution from our LU decomposition step. This is 1, 3, 5, 7, 9.

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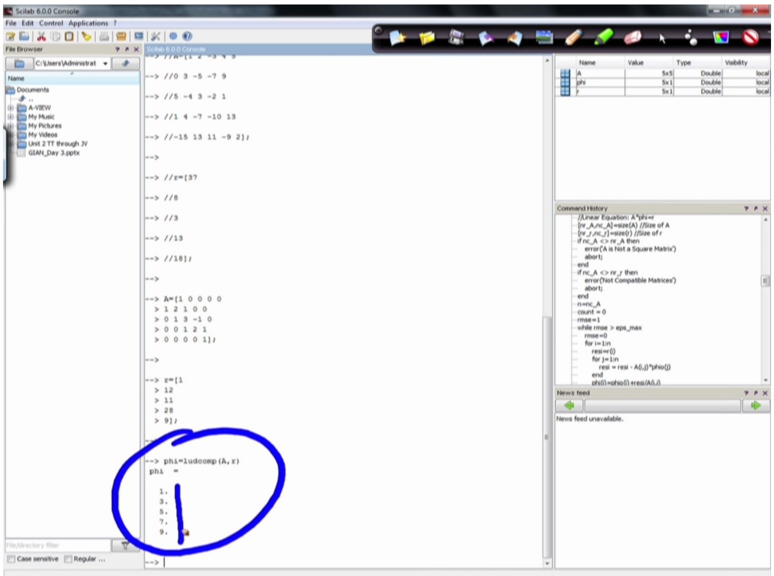

Now in this casewe are calling the function phi LU decomposition a, r with this. But if we have let us say we're changing it to a1 and we are utilizingour that general matrix structure problem, 5 by 5 although it is 5 by 5for this LU decomposition, we can check that we are getting the solution 1, 2, 3, 4, 5.Again this is our solution as discussed in our Gauss elimination slides.

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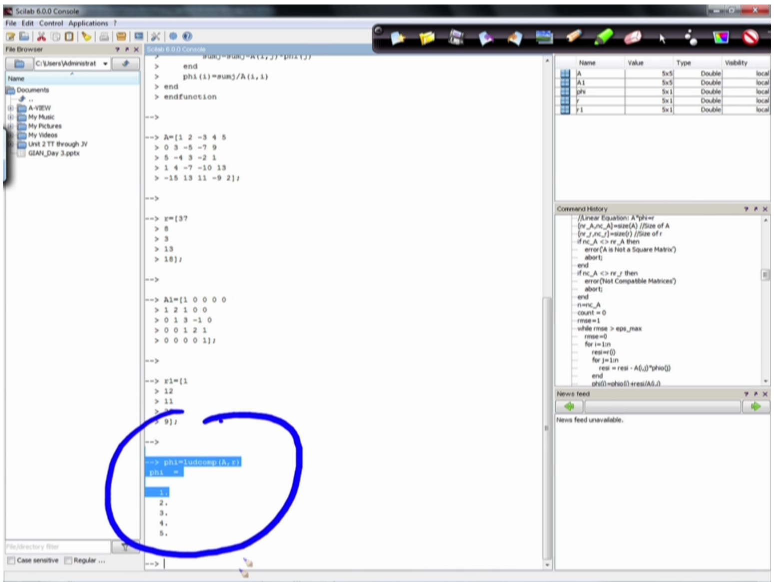

Now in LU decomposition also we are getting the same solution. For large matrixes these things areapplicable. But one problem is that we need thisdirect thing. Another wayof defining it is with indirect method. Butin our previous problem, this problem ismore general where we have full structure or coefficient matrix structure here. But most of them arehaving non zero values.

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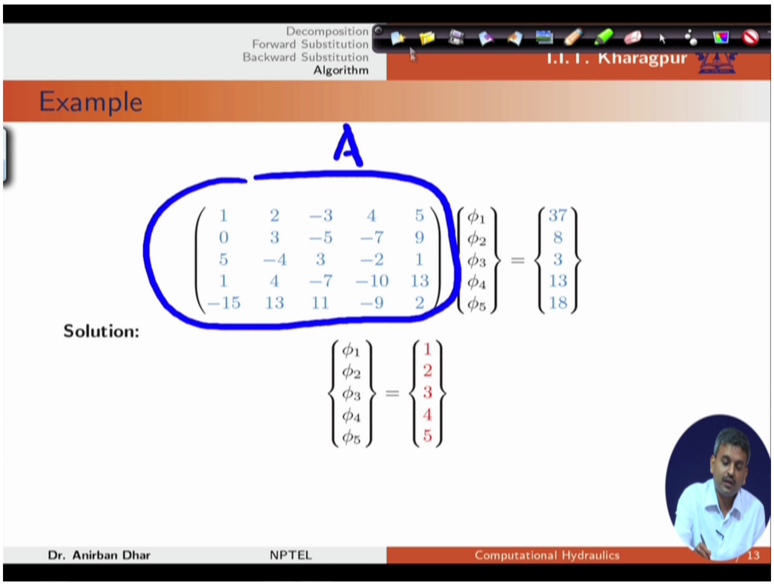

But if you have large number of zero values available and we have a general structure available, for that we can definitely use our concepts for that purpose. And for banded matrix we can write our generalized algorithm. Andbecause in this casestorage is a problem. You need to store N by N matrix for our calculation. Finally A matrix is N by N matrix.

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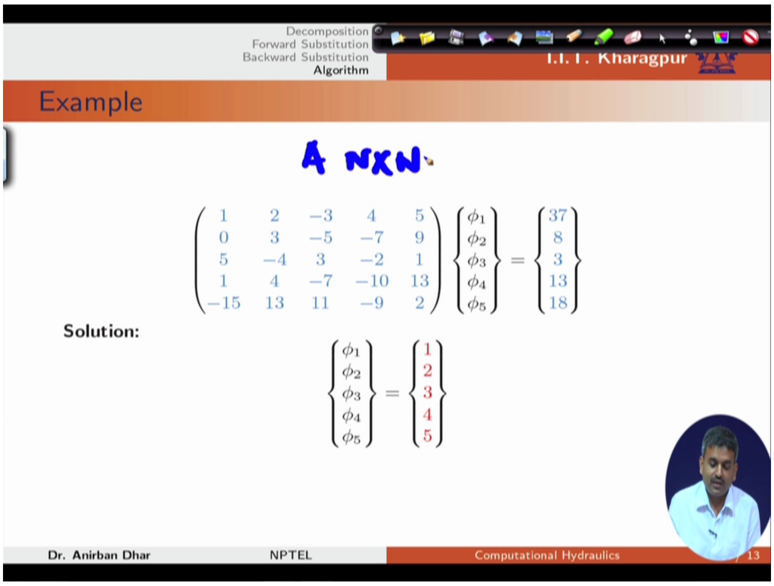

$$\begin{pmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & 3 & -5 & -7 & 9 \\ 5 & -4 & 3 & -2 & 1 \\ 1 & 4 & -7 & -10 & 13 \\ -15 & 13 & 11 & -9 & 2 \end{pmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 37 \\ 8 \\ 3 \\ 13 \\ 18 \end{Bmatrix}$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$

But if you have some diagonal structure or banded structure of the matrix, you don't need to store the full matrix. Without storing the full matrix, we can solve the problem. In the next lecture we will be discussing that special kind of algorithms. Thank you.