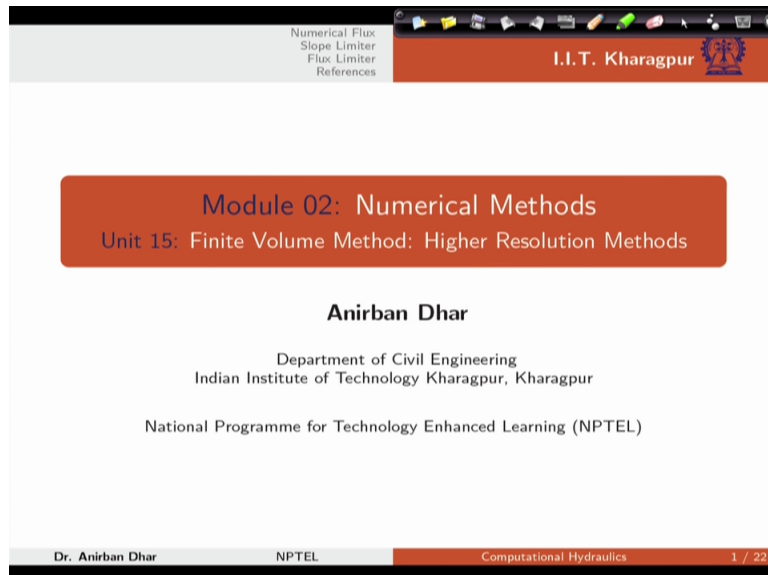


Computational Hydraulics
Professor Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 19
Finite Volume Method - High Resolution Methods

Welcome to of this lecture number 19 of the course computational hydraulics. We are in module 2, numerical methods. And in this particular lecture class I will be covering unit 15 and this is the last lecture class for finite volume method. And we will be discussing high resolution methods.

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The image shows a presentation slide with a white background and a dark blue header. The header contains the text 'Numerical Flux', 'Slope Limiter', 'Flux Limiter', and 'References' on the left, and 'I.I.T. Kharagpur' with the IIT Kharagpur logo on the right. The main content area features a dark blue rounded rectangle with the text 'Module 02: Numerical Methods' and 'Unit 15: Finite Volume Method: Higher Resolution Methods'. Below this, the name 'Anirban Dhar' is centered, followed by 'Department of Civil Engineering' and 'Indian Institute of Technology Kharagpur, Kharagpur'. At the bottom of the slide, it says 'National Programme for Technology Enhanced Learning (NPTEL)'. The footer of the slide includes 'Dr. Anirban Dhar', 'NPTEL', 'Computational Hydraulics', and '1 / 22'.

Learning objective for this particular unit. At the end of this unit students will be able to discretize conservation laws using high resolution methods.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Learning Objective

- To discretize conservation laws using Higher Resolution Methods.

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This is our well known governing equation that we have utilized for our earlier lecture classes. Phi is again general variable here and in one dimension we have only xt representation. F is our flux function. We can have different representations for this flux function. And S phi is our source sink term. So this is our one-dimensional conservation law.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \quad (1)$$

where
 \mathcal{F}_ϕ = Flux function.
 S_ϕ = Source term.

$\phi(x,t)$

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Now let us see that with high resolution methods, we will be utilizing the concept of piecewise reconstruction. Let us say that we have our general cell where in one dimension we have central cell as P. E cell is I plus 1 and west cell or let us say this is W cell. Now with this if we start, now we can have this east and west face for this one. Let us say that cell averaged

value ϕ_P can be used to construct this $\tilde{\phi}^n$ and this one is x^n which is dependent on x and t^n that is present time level value. So here x is variable.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_P^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_P^n + \sigma_P^n(x - x_P) \quad \forall x \in [x_w, x_e]$$

Diagram showing a cell with nodes W, P, and E. A horizontal line represents the cell, with nodes W, P, and E marked. A vertical line is drawn from node P down to the x-axis, indicating the cell center x_P .

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We have ϕ_P^n value and this σ_P^n which is calculated at the present time step for cell P, we will be utilizing this thing. And we have this difference $x - x_P$. x is arbitrary any distance and x_P is our co-ordinate for this P cell.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_P^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_P^n + \sigma_P^n(x - x_P) \quad \forall x \in [x_w, x_e]$$

Diagram showing a cell with nodes W, P, and E. A horizontal line represents the cell, with nodes W, P, and E marked. A vertical line is drawn from node P down to the x-axis, indicating the cell center x_P .

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So in one dimension we have represented this $\tilde{\phi}^n$ in terms of cell centered value. Some difference which is with respect to our cell centered value of the co-ordinate. And σ_P^n is another coefficient. Let us see what is this coefficient?

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_p^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_p^n + \sigma_p^n(x - x_p) \quad \forall x \in [x_w, x_e]$$

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And this is valid obviously for this range, x_w and x_e . That means within P cell only this approximation is valid. This is obviously a linear approximation.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_p^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_p^n + \sigma_p^n(x - x_p) \quad \forall x \in [x_w, x_e]$$

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Now let us consider our simplified flux term where $A \phi$ is the flux function and A is constant. ϕ is our again general variable. With this if we proceed, solution for future time level that is \tilde{x}^{n+1} can be represented in terms of $\tilde{x}^n - A t$, provided that A is positive. So we can get the future time level value for a particular cell based on the previous time level value at any arbitrary location x^{n+1} .

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_p^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_p^n + \sigma_p^n(x - x_p) \quad \forall x \in [x_w, x_e]$$

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant. Solution for future time level can be written as

$$\tilde{\phi}(x, t^{n+1}) = \tilde{\phi}(x - a\Delta t, t^n) \quad a > 0$$

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Now with this we can calculate our numerical flux function. Calculation of numerical flux function is the most important thing for this exercise of high resolution methods. We know that flux functions are calculated at the interface only or that means east face x_e and x_w for the west face. So average value we can calculate considering the time interval t to t plus Δt . That is t^n to $t^n + 1$. Similarly for west face also we can calculate the same thing.

(Refer Slide Time 07:00)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Piecewise Reconstruction (LeVeque, 2002)

A piecewise linear from of cell average value ϕ_p^n can be used as

$$\tilde{\phi}^n(x, t^n) = \phi_p^n + \sigma_p^n(x - x_p) \quad \forall x \in [x_w, x_e]$$

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant. Solution for future time level can be written as

$$\tilde{\phi}(x, t^{n+1}) = \tilde{\phi}(x - a\Delta t, t^n)$$

Numerical flux function can written as

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt$$

$$\bar{\mathcal{F}}_\phi(x_w, t^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt$$

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Now approximation is important in this case. So let us consider the situation where we have A greater than zero. That means A is positive and ϕ , the average value of the flux at the face or east face, we can represent like this as per our definition.

(Refer Slide Time 07:37)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for east face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_e, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_P^n + \sigma_P^n (x_e - a(t - t^n) - x_P)] dt \\ &= a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t) \end{aligned}$$

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Now we already know that our F_{ϕ} equals to $A \phi$. Now let us use our tilde values which is the constructed value. Now if we replace this ϕ tilde with our approximation that means for this one we can get our ϕ tilde for $\phi(x_e - A t \text{ minus } t^n)$. That means we have any arbitrary time level t which is in between t^n plus 1 and t^n , we are considering this. So obviously if we are considering the future time level value, we should consider this variation within t^n to t^n plus 1.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for east face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_e, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_P^n + \sigma_P^n (x_e - a(t - t^n) - x_P)] dt \\ &= a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t) \end{aligned}$$

$\bar{f}_{\phi} = a\tilde{\phi}$

Diagram showing time levels t^n and t^{n+1} on a vertical axis, with a horizontal line at time t between them.

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Now at x_e , we have some value. So obviously we should consider Δt . That means if this is x_e , this co-ordinate should be x_e minus $A \Delta t$. In this case Δt is t minus t^n . So we can replace this and we can calculate this derivative.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for east face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_e, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \tilde{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \tilde{\phi}(x_e - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_p^n + \sigma_p^n (x_e - a(t - t^n) - x_p)] dt \\ &= a \phi_p^n + \frac{a}{2} \sigma_p^n (\Delta x - a \Delta t) \end{aligned}$$

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So in this particular case we can replace our linear approximation for this one. So obviously this is x_e minus $A \Delta t$ or in this case Δt is t minus t^n minus x_p . So this is general x . In this case, x minus x_p and ϕ_p^n . Now if we simplify this from the first term we are getting, $A \phi_p^n$. And from the second term we are getting this second part which is in this case obviously Δt means we are taking variation between t^n plus 1 and t^n . So this ϕ_p or σ_p^n , this term is important here.

(Refer Slide Time 10:59)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for east face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_e, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \tilde{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \tilde{\phi}(x_e - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_p^n + \sigma_p^n (x_e - a(t - t^n) - x_p)] dt \\ &= a \phi_p^n + \frac{a}{2} \sigma_p^n (\Delta x - a \Delta t) \end{aligned}$$

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Similarly if we calculate the numerical flux function for the west face. Again for A greater than zero or positive case we will have again t level and this is t^n level. So obviously in this

case if this is x_w , we can calculate this one based on x_w minus A into t minus t_n . And this term have replaced here. And in this case approximation is in terms of west cell value.

(Refer Slide Time 12:02)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for west face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_w, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_W^n + \sigma_W^n (x_w - a(t - t^n) - x_W)] dt \\ &= a \phi_W^n + \frac{a}{2} \sigma_W^n (\Delta x - a \Delta t) \end{aligned}$$

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In previous case we have seen that our calculations are based on P cell only. Because variation we have considered within cell and flow of information from that cell to this face. But in case of west face, the flow of information is from west cell. So we are considering ϕ_w , σ_w , these two values. So obviously piecewise approximation should be in terms of x_w which is the cell centered value of the W cell.

(Refer Slide Time 12:50)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for west face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_w, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_W^n + \sigma_W^n (x_w - a(t - t^n) - x_W)] dt \\ &= a \phi_W^n + \frac{a}{2} \sigma_W^n (\Delta x - a \Delta t) \end{aligned}$$

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Now from the first term we can get this approximation directly. And from the second term we can get this A by $2 \sigma_w$ and Δx minus $A \Delta t$. Obviously Δt means the change in time level from t_n to $t_n + 1$.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Numerical flux function for west face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_{\phi}(x_w, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_w^n + \sigma_w^n (x_w - a(t - t^n) - x_w)] dt \\ &= a \phi_w^n + \frac{a}{2} \sigma_w^n (\Delta x - a \Delta t) \end{aligned}$$

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Now let us consider our original finite volume equation. If you want to calculate ϕ_P^{n+1} , we need information about ϕ_P^n , Δt by Δx . And these two are numerical flux functions. So numerical flux functions, we have calculated in terms of ϕ_P and ϕ_w . Now we can replace this in this particular expression. Now you can see that for east face we have ϕ_P , σ_P and for west face we have ϕ_w and σ_w .

(Refer Slide Time 14:22)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{\mathcal{F}}_{\phi}(x_e, t^n) - \bar{\mathcal{F}}_{\phi}(x_w, t^n)]$$

or,

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} \left[\left(a \phi_P^n + \frac{a}{2} \sigma_P^n (\Delta x - a \Delta t) \right) - \left(a \phi_w^n + \frac{a}{2} \sigma_w^n (\Delta x - a \Delta t) \right) \right]$$

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Now if we (re) rearrange this one, we can write it like this. So this is our original expression where we have directly replaced this $A \phi_P$ value minus $A \phi_W$ value. And this term is extra. This term is extra in this case. Now we need to interpret this particular term physically.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a > 0$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{\mathcal{F}}_\phi(x_e, t^n) - \bar{\mathcal{F}}_\phi(x_w, t^n)]$$

or,

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} \left[\left(a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t) \right) - \left(a\phi_W^n + \frac{a}{2}\sigma_W^n(\Delta x - a\Delta t) \right) \right]$$

or,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} (\phi_P^n - \phi_W^n) - \frac{1}{2} \frac{a\Delta t}{\Delta x} (\Delta x - a\Delta t) (\sigma_P^n - \sigma_W^n)$$

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Next thing is if we consider A less than zero. In case of A less than zero that means again this is east face, this is west face and this is t level, this is t^n . So obviously the flow of information for this particular level is from east cell. So we are considering ϕ_E σ_E in this case. This is E cell, this is P cell, this is W cell. So flow of information is from east cell. So we have calculated all the values in terms of ϕ_E and σ_E . Now finally we can get the value of the integration.

(Refer Slide Time 16:11)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a < 0$

Numerical flux function for east face can be calculated as

$$\begin{aligned} \bar{\mathcal{F}}_\phi(x_e, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a\tilde{\phi}(x_e - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \left[\phi_E^n + \sigma_E^n (x_e - a(t - t^n) - x_E) \right] dt \\ &= a\phi_E^n - \frac{a}{2} \sigma_E^n (\Delta x + a\Delta t) \end{aligned}$$

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Similarly for west face. Now for west face again, this is t, this is t_n. So information will be travelling from our P cell to x_w. So we can replace this phi P sigma P and x_p, in this case. Now this is our final form of the numerical flux.

(Refer Slide Time 16:50)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a < 0$

Numerical flux function for west face can be calculated as

$$\begin{aligned} \bar{F}_\phi(x_w, t^n) &= \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w, t) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a \bar{\phi}(x_w - a(t - t^n), t^n) dt \\ &= \frac{1}{\Delta t} \int_t^{t+\Delta t} a [\phi_P^n + \sigma_P^n (x_w - a(t - t^n) - x_P)] dt \\ &= a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t) \end{aligned}$$

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Now we can write similar expression for A less than zero by putting this integral values. Now in this case we can see that phi P_n, phi n plus 1 equals to phi P_n. And we have phi_E and phi_P combination. In case of A greater than zero, we have phi_P and phi_W combination.

(Refer Slide Time 17:38)

Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a < 0$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{F}_\phi(x_e, t^n) - \bar{F}_\phi(x_w, t^n)]$$

or,

$$\phi_P^{n+1} - \phi_P^n - \frac{\Delta t}{\Delta x} \left[\left(a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t) \right) - \left(a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t) \right) \right]$$

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In this case we have in natural terms, this is our original expression and this is extra one.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

$a < 0$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{\mathcal{F}}_\phi(x_e, t^n) - \bar{\mathcal{F}}_\phi(x_w, t^n)]$$

or,

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} \left[\left(a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t) \right) - \left(a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t) \right) \right]$$

or,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} (\phi_E^n - \phi_P^n) - \frac{1}{2} \frac{a\Delta t}{\Delta x} (\Delta x + a\Delta t) (\sigma_E^n - \sigma_P^n)$$

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So in this case if we summarize both the things, so for east face we have A greater than zero. It should be represented in terms of phi P. If A less than zero, it should be represented in terms of phi E and sigma E.

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Numerical Flux
Slope Limiter
Flux Limiter
References

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Higher Resolution Methods

Numerical Flux

Numerical flux values can be summarized as

$$\mathcal{F}_\phi(x_e, t^n) = \begin{cases} a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

or,

$$\bar{\mathcal{F}}_\phi(x_e, t^n) = a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t)$$

$$= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t)$$

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Now if we combine these two, we can easily represent this one and in this case we can use this expression where we have A plus and A minus that we have utilized in our previous lecture class. A plus, A minus obviously for A greater than zero this is our expression. And we have just written it in terms of phi P, phi E, sigma P and sigma E, in general form.

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Higher Resolution Methods

Numerical Flux

Numerical flux values can be summarized as

$$\bar{F}_\phi(x_c, t^n) = \begin{cases} a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

or,

$$\begin{aligned} \bar{F}_\phi(x_c, t^n) &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \\ &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \end{aligned}$$

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Similarly we can also write the numerical flux function in terms of this compact representation.

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Higher Resolution Methods

Numerical Flux

Numerical flux values can be summarized as

$$\bar{F}_\phi(x_c, t^n) = \begin{cases} a\phi_P^n + \frac{a}{2}\sigma_P^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_E^n - \frac{a}{2}\sigma_E^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

or,

$$\begin{aligned} \bar{F}_\phi(x_c, t^n) &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \\ &= a^+\phi_P^n + a^-\phi_E^n + \frac{a^+}{2}\sigma_P^n(\Delta x - a\Delta t) - \frac{a^-}{2}\sigma_E^n(\Delta x + a\Delta t) \end{aligned}$$

$$\bar{F}_\phi(x_w, t^n) = \begin{cases} a\phi_W^n + \frac{a}{2}\sigma_W^n(\Delta x - a\Delta t), & a > 0 \\ a\phi_P^n - \frac{a}{2}\sigma_P^n(\Delta x + a\Delta t), & a < 0 \end{cases}$$

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So what is this sigma P? Sigma P is nothing but this is slopes per Godunov, zero slope. That means if we use sigma Pn equals to zero, then we will get our Godunov scheme that is the basic one. Now for higher order or higher resolution method we need to use different slope values. If we use this centered slope, that means in this case this sigma P is dependent on phi E and phi W value (divi) and divided by 2 delta x that means the distance between them. So if we use this, this is called as Fromm slope.

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Higher Resolution Methods

Choice of Slopes

- Zero Slope: $\sigma_P^n = 0$ Godunov
- Centred Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_W^n}{2\Delta x}$ Fromm

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Next one is obtained for upwind. Depending on the direction we change the slope ϕ_P minus ϕ_W . And this is divided by delta x. So we are reconsidering the adjacent cell values in this case.

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Higher Resolution Methods

Choice of Slopes

- Zero Slope: $\sigma_P^n = 0$ Godunov
- Centred Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_W^n}{2\Delta x}$ Fromm
- Upwind Slope: $\sigma_P^n = \frac{\phi_P^n - \phi_W^n}{\Delta x}$ BeamWarming
- Downwind Slope: $\sigma_P^n = \frac{\phi_E^n - \phi_P^n}{\Delta x}$ LaxWendroff

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And downwind, this is ϕ_E minus ϕ_P divided by delta x. This is called as Lax Wendroff scheme. Now the choice of slope may create some problem or oscillation within the method. So in this case we can see that the σ_P , physically it approximates the derivative $\phi_{,x}$. That means $\frac{d\phi}{dx}$ for over the pth cell, if you are considering σ_P^n .

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Higher Resolution Methods

Choice of Slopes

- Zero Slope: $\sigma_P^n = 0$ Godunov
- Centred Slope: $\sigma_P^n = \frac{\phi_P^n - \phi_W^n}{2\Delta x}$ Fromm
- Upwind Slope: $\sigma_P^n = \frac{\phi_P^n - \phi_W^n}{\Delta x}$ BeamWarming
- Downwind Slope: $\sigma_P^n = \frac{\phi_P^n - \phi_E^n}{\Delta x}$ LaxWendroff

σ_P^n approximates the derivative $\phi_{,x}$ over the P^{th} cell.

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Now we need to see what can be done for this particular slope. Is there any limit for the slope? We can check the total variation. Total variation of function can be defined as subtraction over all I and this is phi I minus phi I minus 1. That is previous cell value.

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Total Variation

TVD

Is there any limit for the Slope ?

Total variation of a function can be defined as

$$TV(\phi) = \sum |\phi_i - \phi_{i-1}|$$

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Now total variation diminishing scheme such that if we have total variation for nth n plus 1 level should be less than equal to total variation in phi or nth level. That means TV, total variation for n plus 1 level, this is less than equal to total variation at our nth level.

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Total Variation TVD

Is there any limit for the Slope ?
Total variation of a function can be defined as

$$TV(\phi) = \sum_{\forall i} |\phi_i - \phi_{i-1}|$$

Total Variation Diminishing (TVD)

$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

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Now monotonicity preserving method. If ϕ_i^n is greater than ϕ_{i+1}^n , then ϕ_{i+1}^{n+1} this should be satisfied. This is called as monotonicity preserving method.

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Total Variation TVD

Is there any limit for the Slope ?
Total variation of a function can be defined as

$$TV(\phi) = \sum_{\forall i} |\phi_i - \phi_{i-1}|$$

Total Variation Diminishing (TVD)

$$TV(\phi^{n+1}) \leq TV(\phi^n)$$

Monotonicity Preserving Method

If $\phi_i^n \geq \phi_{i+1}^n, \forall i$
then $\phi_i^{n+1} \geq \phi_{i+1}^{n+1}, \forall i$

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Now TVD is monotonicity preserving method. Now slope limiters are required for high resolution method to check the oscillation. So first order upwind, sigma P equals to zero. That is our Godunov method. Minmod slope, in Minmod slope we generally use again the first order derivative of the function from both the sides and we use this Minmod function.

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Higher Resolution Methods

Slope-Limiter

First Order Upwind

$$\sigma_P^n = 0$$

Minmod

$$\sigma_P^n = \minmod\left(\frac{\phi_P^n - \phi_W^n}{\Delta x}, \frac{\phi_E^n - \phi_P^n}{\Delta x}\right)$$

where

$$\minmod(\alpha, \beta) = \begin{cases} \alpha & \text{if } |\alpha| < |\beta| \text{ and } \alpha\beta > 0 \\ \beta & \text{if } |\beta| < |\alpha| \text{ and } \alpha\beta > 0 \\ 0 & \text{if } \alpha\beta \leq 0 \end{cases}$$

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Interestingly if we have alpha and beta in Minmod function, this is equal to alpha if mod alpha is less than mod beta. And alpha beta greater than zero equals to beta if mod beta is less than mod alpha and alpha beta greater than zero. And equals to zero if alpha beta less than equals to zero.

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Higher Resolution Methods

Slope-Limiter

First Order Upwind

$$\sigma_P^n = 0$$

Minmod

$$\sigma_P^n = \minmod\left(\frac{\phi_P^n - \phi_W^n}{\Delta x}, \frac{\phi_E^n - \phi_P^n}{\Delta x}\right)$$

where

$$\minmod(\alpha, \beta) = \begin{cases} \underline{\alpha} & \text{if } |\alpha| < |\beta| \text{ and } \underline{\alpha\beta} > 0 \\ \underline{\beta} & \text{if } |\beta| < |\alpha| \text{ and } \underline{\alpha\beta} > 0 \\ 0 & \text{if } \alpha\beta \leq 0 \end{cases}$$

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Another slope limit is Superbee limiter, where sigma P1 is Minmod. You can see that in this case only the derivative value and twice from the westside. And second one is twice the east side and this is single. So this is Maxmod function.

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We have flux limiter and what is this flux limiter? In our previous case we have seen that positive values and negative values can be represented like this. And slope related terms are these two terms.

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Now in this particular case if we write this with first two terms directly and we approximate the second term. That means we are using some kind of limit for the slope. But in this case we are considering sigma E this is phiE minus phi P and this psi function is used. Now we need to define different forms of this psi for flux limiters.

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In this case, this theta En, this is defined as phi P minus phi W divided by phi E minus phi P for A greater than zero. And this is phiE minusphi EE. EE means east-east. If we have central cell as P, this is E, this is W, and extreme left we will have WW and extreme right we will have our east-east cell. This is actually east-east cell value. And this is for A less than zero.

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Now if we consider the west face, again we can approximate the flux for the west face with this theta. And in this case we are utilizing this WW. That means west-west cell value or extreme west cell value.

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Flux Limiter

$$\bar{F}_\phi(x_w, t^n) = a^+ \phi_W^n + a^- \phi_P^n + \frac{a^+}{2} \sigma_W^n (\Delta x - a \Delta t) - \frac{a^-}{2} \sigma_P^n (\Delta x + a \Delta t)$$

$$= a^+ \phi_W^n + a^- \phi_P^n + \frac{1}{2} |a| \left(1 - \left| \frac{a \Delta t}{\Delta x} \right| \right) \Psi(\theta_w^n) (\phi_P^n - \phi_W^n)$$

where

$$\theta_w^n = \begin{cases} \frac{\phi_W^n - \phi_{WW}^n}{\phi_P^n - \phi_W^n} & \text{for } a \geq 0 \\ \frac{\phi_E^n - \phi_P^n}{\phi_P^n - \phi_W^n} & \text{for } a \leq 0 \end{cases}$$

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Now what can be this flux limiter? For upwind scheme we have psi theta equals to zero. These are linear models. Lax Wendroff, we have psi theta equals to 1. Beam-Warming, this is psi theta equals to theta. And Frommslope, psi theta equals to half 1 plus theta.

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Flux Limiter

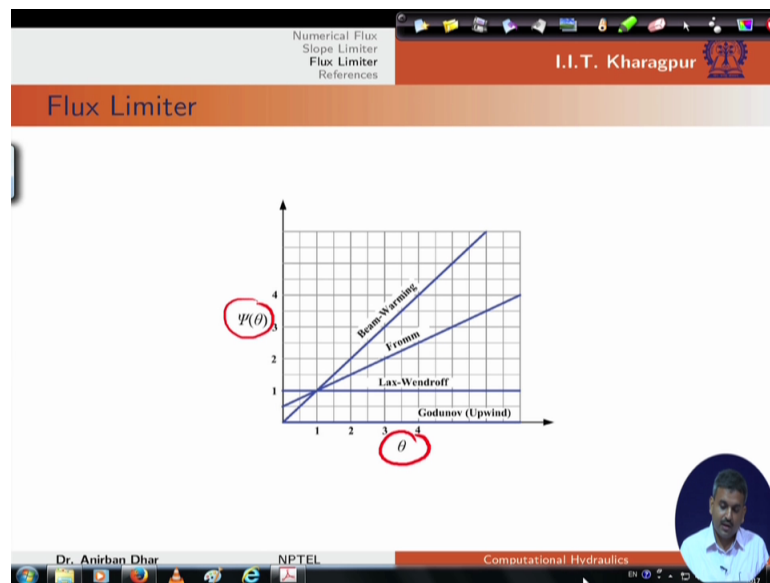
Linear Models

- Upwind: $\Psi(\theta) = 0$
- Lax-Wendroff: $\Psi(\theta) = 1$
- Beam-Warming: $\Psi(\theta) = \theta$
- Fromm: $\Psi(\theta) = \frac{1}{2}(1 + \theta)$

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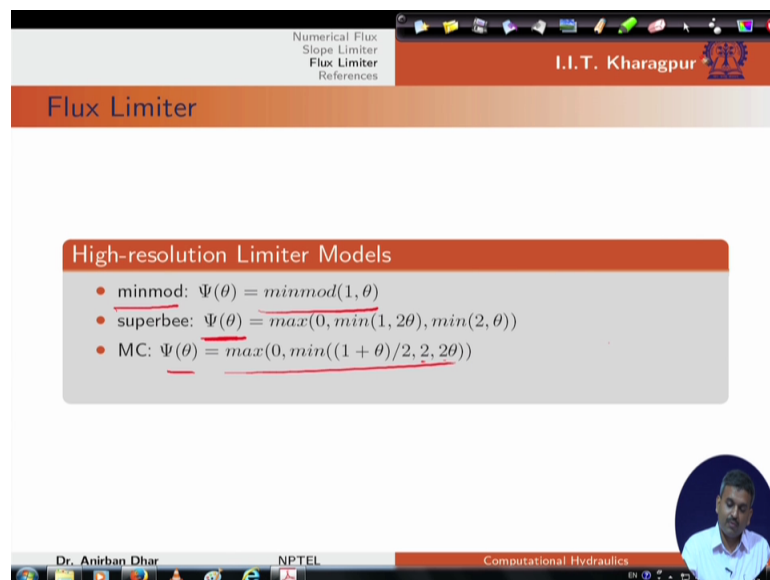
In this case these are the different slopes. With variation of theta, what is the change in psi theta? We can see from this particular diagram.

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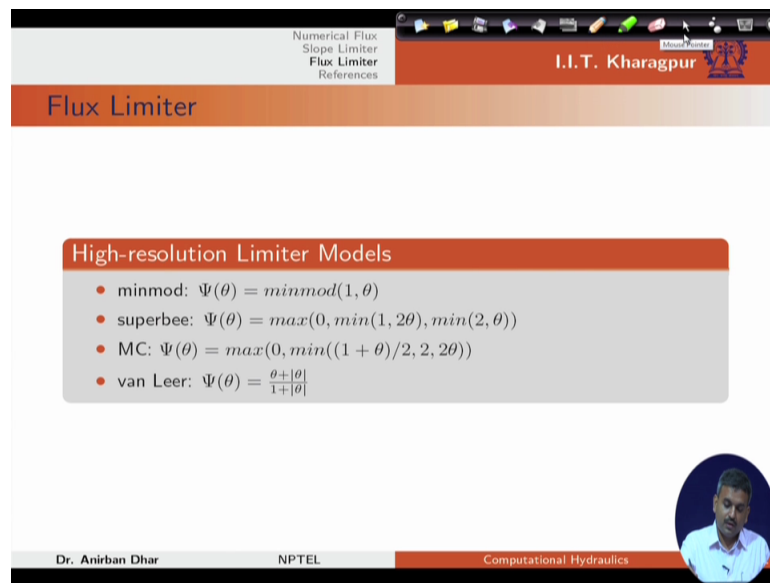
Now another kind of flux limiter can be therewhich are higher resolution limiter or nonlinear limiters. We can have other methods but we have men Minmod, Minmod of 1 theta. Then we have maximum of this is our Superbee. This is psi theta again written in terms of theta thing. And this is MC. This is psi theta again max, zero minimum 1 plus theta by 2 and 2 and theta. This is a minimum of that we can utilize.

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With this slope limiter and flux limiterwe can get different type of approximation for numerical flux and we can utilize these approximations for our cell centered value calculation for future time level. So another one is Van Leer, this one.

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Flux Limiter

High-resolution Limiter Models

- minmod: $\Psi(\theta) = \minmod(1, \theta)$
- superbee: $\Psi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$
- MC: $\Psi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$
- van Leer: $\Psi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$

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Thank you.