

Computational Hydraulics
Professor Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 18
FVM - Godunov Approach

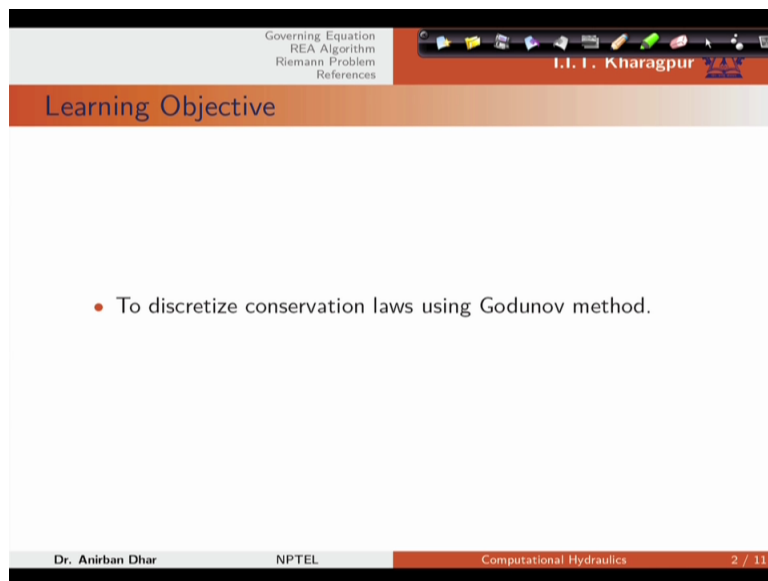
Welcome to this lecture number 18 of the course computational hydraulics. We are in module 2, numerical methods. In this particular lecture I will be covering unit 14, finite volume method and Godunov approach.

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The image shows a presentation slide with a white background and a dark red header and footer. The header contains navigation links: 'Governing Equation', 'REA Algorithm', 'Riemann Problem', and 'References'. The main content area features a dark red box with white text: 'Module 02: Numerical Methods' and 'Unit 14: Finite Volume Method: Godunov Approach'. Below this, the presenter's name 'Anirban Dhar' is listed, followed by his affiliation: 'Department of Civil Engineering, Indian Institute of Technology Kharagpur, Kharagpur'. The slide also mentions 'National Programme for Technology Enhanced Learning (NPTEL)'. The footer contains 'Dr. Anirban Dhar', 'NPTEL', 'Computational Hydraulics', and '1 / 11'.

Learning objectives for this particular unit. At the end of this unit students will be able to discretize conservation laws using Godunov method.

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Governing Equation
REA Algorithm
Riemann Problem
References

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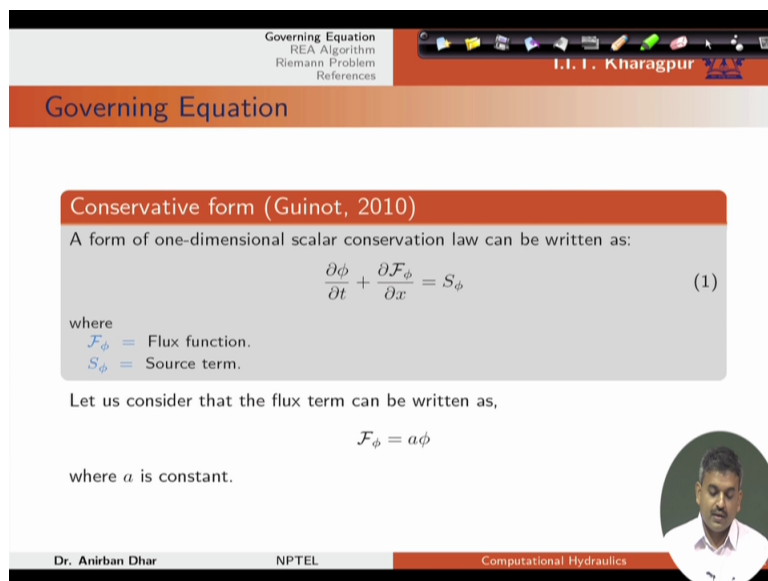
Learning Objective

- To discretize conservation laws using Godunov method.

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We already know this is our conservative form in terms of phi. F_ϕ is the flux function. S_ϕ is the source sink term. Let us consider from our general case one simple approximation. That is, flux can be written in terms of $A\phi$, where A is constant, ϕ is the scalar variable.

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Governing Equation

Governing Equation
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Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \quad (1)$$


where

- \mathcal{F}_ϕ = Flux function.
- S_ϕ = Source term.

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant.



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Basic structure of Godunov method is with REA algorithm. REA means, reconstruct, evolve and average. In this case we reconstruct a piecewise polynomial from cell average value ϕ_P^n . As ϕ_{tilde} this ϕ_{tilde}^n, x, t_n , that is present time level. This is with the present time level value and where x is local.

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Governing Equation
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Godunov Method

REA Algorithm (LeVeque, 2002)

Reconstruct-Evolve-Average

- Reconstruct a piecewise polynomial from cell average value ϕ_p^n as

$$\phi_p^n(x, t^n) = \phi_p^n \quad \forall x \in [x_w, x_e]$$

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So in the next step we evolve the hyperbolic equation with base condition. Base condition or initial condition to obtain the future time level value ϕ_p^{n+1} . And at this level we use our existing knowledge or previous methods to evolve this hyperbolic equation with base condition. Then in the last stage we take average of the polynomial function at cell level to obtain cell average value at future time level. So it has got 3 steps. One is reconstruct, second is evolve and third one is average or taking average of the value.

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Governing Equation
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References

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Godunov Method

REA Algorithm (LeVeque, 2002)

Reconstruct-Evolve-Average

- Reconstruct a piecewise polynomial from cell average value ϕ_p^n as

$$\phi_p^n(x, t^n) = \phi_p^n \quad \forall x \in [x_w, x_e]$$

- Evolve the hyperbolic equation with base condition to obtain $\phi_p^{n+1}(x, t + \Delta t)$ at future time $t + \Delta t$.
- Average the polynomial function at cell level to obtain cell average value at future time $t + \Delta t$ as

$$\phi_p^{n+1} = \frac{1}{\Delta x} \int_{x_w}^{x_e} \phi(x, t^{n+1}) dx$$

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And in this case, this ϕ_p^n or ϕ_p^{n+1} , this is subscript or superscript, is constant over time interval t^n . Where t is within n to $n + 1$. And within this time interval, this ϕ function is constant.

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Godunov Method

REA Algorithm (LeVeque, 2002)

Reconstruct-Evolve-Average

- Reconstruct a piecewise polynomial from cell average value ϕ_p^n as

$$\tilde{\phi}^n(x, t^n) = \phi_p^n \quad \forall x \in [x_w, x_e]$$
- Evolve the hyperbolic equation with base condition to obtain $\tilde{\phi}^n(x, t + \Delta t)$ at future time $t + \Delta t$.
- Average the polynomial function at cell level to obtain cell average value at future time $t + \Delta t$ as

$$\phi_p^{n+1} = \frac{1}{\Delta x} \int_{x_w}^{x_e} \tilde{\phi}(x, t^{n+1}) dx$$

Steps are repeated at every time level.

$\tilde{\phi}(x, t^n)$ is constant over time interval $t^n < t < t^{n+1}$

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Now we already know from our Riemann problem that at interface there is discontinuity in the information on the each side. On the left side we have p cell and this side we have e cell. So there is discontinuity in the value.

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Governing Equation
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Riemann Problem

Conservative Form

Riemann Problem

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

$$\phi(x, t) = \begin{cases} \phi_p^n & \text{if } x < x_e \\ \phi_e^n & \text{if } x > x_e \end{cases}$$

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Now at this e face we can construct Riemann solution. For construction of Riemann solution we can define single variable. So the flux at cellface depends on the exact solution of this \tilde{x} of the Riemann problem along with the axis. Considering along the t axis. Considering local coordinates, we have this \tilde{x} t . Where in the east face we have x minus x_e , t minus t_n . That means we are taking this point as a starting point.

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Godunov Method

$\mathcal{F}_\phi(\tilde{\phi}(x,t))$ at cell face depends on the exact solution $\tilde{\phi}(x,t)$ of the Riemann problem along the taxis. Considering local coordinates

$$\tilde{\phi}(x,t) = \phi_e\left(\frac{x-x_e}{t-t^n}\right), \quad x_P \leq x \leq x_E, \quad t^n \leq t \leq t^{n+1}$$

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We have p cell, this is e face, this is east cell, west cell. At interface we are considering the Riemann problem at e face at this point only. So x in this case varies from xp. This is xp to xe. That means center of the cell e and p. And the time, this is again, if this is level n, this is n plus 1. So time interval will be considered in this case. So everything varies between this n and n plus 1 time level. And we have defined one single variable. Single variable, we can also write like this xi, x minus xe divided by t minus tn. So this is only definition for e face.

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Governing Equation
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Godunov Method

$\mathcal{F}_\phi(\tilde{\phi}(x,t))$ at cell face depends on the exact solution $\tilde{\phi}(x,t)$ of the Riemann problem along the taxis. Considering local coordinates

$$\tilde{\phi}(x,t) = \phi_e\left(\frac{x-x_e}{t-t^n}\right), \quad x_P \leq x \leq x_E, \quad t^n \leq t \leq t^{n+1}$$

$\xi = \frac{x-x_e}{t-t^n}$

W P E
| | |
x_P x_E

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We can again define another Riemann problem for this face considering w and p. That will be bounded between xp again. We need to consider that. So in terms of local co-ordinate we can

define number of Riemann problem and we can individually solve them to get the values at future time level. So tilde xt. This is w again, x minus xw divided by t minus tn.

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Godunov Method

$\mathcal{F}_\phi(\tilde{\phi}(x, t))$ at cell face depends on the exact solution $\tilde{\phi}(x, t)$ of the Riemann problem along the axis. Considering local coordinates

$$\tilde{\phi}(x, t) = \phi_e \left(\frac{x - x_e}{t - t^n} \right), \quad x_p \leq x \leq x_e, \quad t^n < t \leq t^{n+1}$$

$\xi = \frac{x - x_e}{t - t^n}$

$\phi(x, t) = \phi_w \left(\frac{x - x_w}{t - t^n} \right), \quad x_w \leq x \leq x_p, \quad t^n < t < t^n$

Diagram showing the x-t plane with points x_w, x_p, x_e and t^n, t^{n+1} . The region between x_w and x_p is labeled 'W' and the region between x_p and x_e is labeled 'E'. A red circle highlights the origin (x_p, t^n) in the (ξ, τ) coordinate system.

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So with this basic knowledge we can start our finite volume discretization. From Riemann problems this is at w face. If we have w face obviously x will be xw, x equals to xw. So x equals to xw, this is essentially phi w0. Because we have define our new variable like that. Xi so this is equal to zero.

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Godunov Method

From Riemann problems:

$$\tilde{\phi}(x_w, t) = \phi_w \left(\frac{x_w - x_w}{t - t^n} \right) = \phi_w(0) \quad \text{with} \quad t^n \leq t \leq t^{n+1}$$

$\xi = 0$

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Similarly for east face again this will be 0.

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Godunov Method

From Riemann problems:

$$\tilde{\phi}(x_w, t) = \phi_w \left(\frac{x_w - x_w}{t - t^n} \right) = \phi_w(0) \text{ with } t^n \leq t \leq t^{n+1}$$

$$\tilde{\phi}(x_e, t) = \phi_e \left(\frac{x_e - x_e}{t - t^n} \right) = \phi_e(0) \text{ with } t^n \leq t \leq t^{n+1}$$

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So with that information we can say that the information is travelling from w face and e face either on the right hand side or left hand side through this constant line or any chord line or characteristics line. But the main thing is that, within this time interval there should not be intersection of these characteristic lines. So this should not intersect each other.

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Governing Equation
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Godunov Method

From Riemann problems:

$$\tilde{\phi}(x_w, t) = \phi_w \left(\frac{x_w - x_w}{t - t^n} \right) = \phi_w(0) \text{ with } t^n \leq t \leq t^{n+1}$$

$$\tilde{\phi}(x_e, t) = \phi_e \left(\frac{x_e - x_e}{t - t^n} \right) = \phi_e(0) \text{ with } t^n \leq t \leq t^{n+1}$$

n+1

n

W w P e E

ϕ_r^{n+1}

ϕ_r^n

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Now, in this case the basic thing for the Godunov method is that with that approximate tilde value we can approximate our numerical flux. So numerical flux in this case can be written like this, where this flux function. We know that our original thing was $\phi_n + 1$ equals to $\phi_n - \Delta t / \Delta x$. This was $f(\phi_n) - f(\phi_w)$, without considering the source sink terms.

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Godunov Method


$$\phi_r^{n+1} = \phi_r^n - \frac{\Delta t}{\Delta x} \left[\bar{F}_\phi(x_e, t) - \bar{F}_\phi(x_w, t) \right]$$

Numerical flux values can be written as

$$\bar{F}_\phi(x_e, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_e, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_e(0)) dt = \mathcal{F}_\phi(\phi_e(0))$$

$$\bar{F}_\phi(x_w, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_w, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_w(0)) dt = \mathcal{F}_\phi(\phi_w(0))$$

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So in this case we need to approximate these two derivatives or numerical fluxes. So numerical fluxes essentially, these are average values within the time intervals. Now if we consider our case here, we already know that phi tilde x e t, this is phi e 0. And at west face we have w 0. We can use these two values directly within our flux function calculation.

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Godunov Method

$$\phi_r^{n+1} = \phi_r^n - \frac{\Delta t}{\Delta x} \left[\bar{F}_\phi(x_e, t) - \bar{F}_\phi(x_w, t) \right]$$

Numerical flux values can be written as


$$\bar{F}_\phi(x_e, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_e, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_e(0)) dt = \mathcal{F}_\phi(\phi_e(0))$$

$$\bar{F}_\phi(x_w, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_w, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_w(0)) dt = \mathcal{F}_\phi(\phi_w(0))$$

$$\tilde{\phi}(x_e, t) = \phi_e(0)$$

$$\tilde{\phi}(x_w, t) = \phi_w(0)$$

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Obviously this f phi, this is again written as function of phi tilde e. And f phi, phi tilde x w t. This can be written with this approximation.

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Godunov Method

$$\phi_r^{n+1} = \phi_r^n - \frac{\Delta t}{\Delta x} \left[\bar{F}_\phi(x_e, t) - \bar{F}_\phi(x_w, t) \right]$$

Numerical flux values can be written as


$$\bar{F}_\phi(x_e, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_e, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_e(0)) dt = \mathcal{F}_\phi(\phi_e(0))$$

$$\bar{F}_\phi(x_w, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_w, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_w(0)) dt = \mathcal{F}_\phi(\phi_w(0))$$

$$\tilde{\phi}(x_e, t) = \phi_e(0)$$

$$\tilde{\phi}(x_w, t) = \phi_w(0)$$

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And for w again we can write it like this with this approximation that this is in terms of secondary variable, this is zero.

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Godunov Method

$$\phi_r^{n+1} = \phi_r^n - \frac{\Delta t}{\Delta x} \left[\bar{F}_\phi(x_e, t) - \bar{F}_\phi(x_w, t) \right]$$

Numerical flux values can be written as


$$\bar{F}_\phi(x_e, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_e, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_e(0)) dt = \mathcal{F}_\phi(\phi_e(0))$$

$$\bar{F}_\phi(x_w, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_w, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_w(0)) dt = \mathcal{F}_\phi(\phi_w(0))$$

$$\tilde{\phi}(x_e, t) = \phi_e(0) \checkmark$$

$$\tilde{\phi}(x_w, t) = \phi_w(0) \checkmark$$

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So essentially this tilde this thing is not varying within the time period or solution is not varying within that time interval. So we can directly write it as $\phi_e(0)$ and $\phi_w(0)$. This is from the basic definition here that the quantities are not changing within this time interval.

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Governing Equation
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Godunov Method

$$\phi_r^{n+1} = \phi_r^n - \frac{\Delta t}{\Delta x} \left[\bar{f}_\phi(x_e, t) - \bar{f}_\phi(x_w, t) \right]$$

Numerical flux values can be written as


$$\bar{F}_\phi(x_e, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_e, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_e(0)) dt = \mathcal{F}_\phi(\phi_e(0))$$

$$\bar{F}_\phi(x_w, t) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\tilde{\phi}(x_w, t)) dt = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_\phi(\phi_w(0)) dt = \mathcal{F}_\phi(\phi_w(0))$$

$$\tilde{\phi}(x_e, t) = \phi_e(0) \checkmark$$

$$\tilde{\phi}(x_w, t) = \phi_w(0) \checkmark$$

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Now if we see our Godunov method that means we are solving this problem at this interface and westface where left hand side we have w, right hand side we have p cell. For east face we have east cell and p cell.

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Governing Equation
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References

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Godunov Method

If $\mathcal{F}_\phi = a\phi$, then numerical flux can be written as,


$$\mathcal{F}_\phi(\phi_e(0)) = a^- \phi_e^n + a^+ \phi_p^n$$

$$\mathcal{F}_\phi(\phi_w(0)) = a^- \phi_p^n + a^+ \phi_w^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

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Again in this case depending on the nature of the characteristic line which is coming from this point.

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Godunov Method

If $\mathcal{F}_\phi = a\phi$, then numerical flux can be written as,

$$\mathcal{F}_\phi(\phi_e(0)) = a^- \phi_e^n + a^+ \phi_p^n$$

$$\mathcal{F}_\phi(\phi_w(0)) = a^- \phi_p^n + a^+ \phi_w^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

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We need to consider the situation if it is in on the positive side or it is in the negative side. In this case also this is on the positive side or in the negative side. Essentially we have converted into the zero point and we are calculating the future time level values. Like our previous definition we have expressed it in terms of A minus, A plus. This is essentially our A plus line. This is A plus line. This is A minus line, this is A minus line.

That means information is travelling from this face to the positive side or the negative side. Depending on that we can write it like this. So this is again solution of the Riemann problem that the solution at this interface e depends on the cell values on both sides, e and p. And if it is A plus, so this will be phi p. If it is negative then phi e. Again at w interface if we have A positive, then this is w. And if it is negative again this phi pn, this is coming because it is a negative one.

And the definition like our upwind approach this is similar. That means for A greater than zero only A plus will be A. A minus will be zero and A (great) less than zero will have A positive equals to zero and A negative equals to A.

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Godunov Method

If $\mathcal{F}_\phi = a\phi$, then numerical flux can be written as,

$$\mathcal{F}_\phi(\phi_e(0)) = a^- \phi_e^* + a^+ \phi_p^*$$

$$\mathcal{F}_\phi(\phi_w(0)) = a^- \phi_p^* + a^+ \phi_w^*$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

$$a > 0 \quad \begin{cases} a^+ = a \\ a^- = 0 \end{cases}$$

$$a < 0 \quad \begin{cases} a^+ = 0 \\ a^- = a \end{cases}$$

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Now you can see that this is same as upwind approach. So we have used over approximations but still we are getting similar results like our upwind method. So Godunov method in basic form this is essentially first order upwind approach. Thank you.