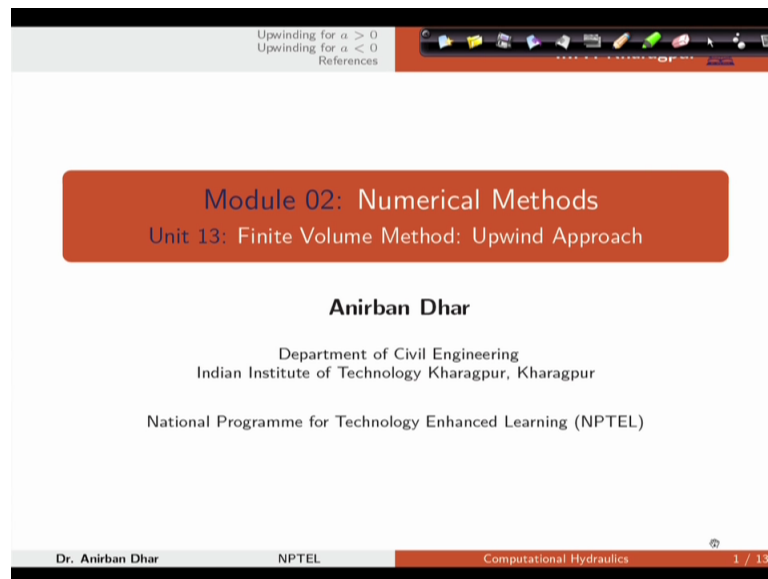


Computational Hydraulics
Professor Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 17
FVM - Upwind Approach

Welcome to lecture number 17 of the course computational hydraulics. We are in module 2, numerical methods. And this is unit 13, finite volume method, upwind approach.

(Refer Slide Time 00:25)



In our previous lecture we have seen that numerical flux function approximation is important. We cannot use arbitrary approximation of the numerical flux function. If we use averaged value of the flux up to adjacent cells for the interface, we are getting unstable schemes. At the same time adding some extra term or virtual terms in the equation gives stability to the numerical discretization. So in this particular lecture we will be concentrating on upwind approach.

Learning objective. At the end of this unit students will be able to discretize the conservation law using upwind method.

(Refer Slide Time 01:41)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

Learning Objective

- To discretize conservation laws using Upwind method.

Dr. Anirban Dhar NPTEL Computational Hydraulics 2 / 13

We already know this is our conservative form uh in terms of ϕ , F_ϕ and S_ϕ .

(Refer Slide Time 01:54)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \quad (1)$$

where
 \mathcal{F}_ϕ = Flux function.
 S_ϕ = Source term.

Dr. Anirban Dhar NPTEL Computational Hydraulics

Let us consider the simple approximation of the flux for linear case where $A\phi$ is a linear approximation.

(Refer Slide Time 02:09)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

Governing Equation

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = S_\phi \quad (1)$$


where
 \mathcal{F}_ϕ = Flux function.
 S_ϕ = Source term.

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant.

Dr. Anirban Dhar NPTEL Computational Hydraulics



Now cell average value can be approximated for this case, where the information is travelling. Let us say this is X E point, this is x axis, this is point XE. This is X E. So information is travelling at certain speed. So in this case this line is linear or this is characteristic line. So information which is travelling from level n to $n+1$, in this case $\phi_E t + \Delta t$, that is from ϕ_E minus $A \Delta t$. $A \Delta t$ is the distance travel in this case.

(Refer Slide Time 03:22)

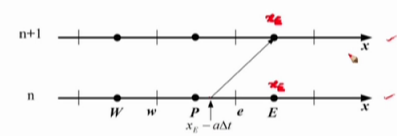
Upwinding for $a > 0$
Upwinding for $a < 0$
References

Upwind Method


$a > 0$

Cell-averaged value can be approximated as,

$$\phi_P^{n+1} \approx \phi(x_P, t + \Delta t) = \phi(x_P - a\Delta t, t)$$

$$\phi_E^{n+1} \approx \phi(x_E, t + \Delta t) = \phi(x_E - a\Delta t, t)$$


Dr. Anirban Dhar NPTEL Computational Hydraulics



So essentially we are using the previous time level value for calculation of future time level values directly, in this case. future time level values can be directly expressed in this form, where this ϕ_P or ϕ_P^{n+1} can be written in terms of convex combination of ϕ_W and

phi p. Convex combination means, in this case our consideration is A is greater than zero. That is why it is travelling on the right side.

(Refer Slide Time 04:23)

Upwind Method
a > 0

Cell-averaged value can be approximated as,

$$\phi_P^{n+1} \approx \phi(x_P, t + \Delta t) = \phi(x_P - a\Delta t, t)$$

$$\phi_E^{n+1} \approx \phi(x_E, t + \Delta t) = \phi(x_E - a\Delta t, t)$$

Future time level value can be expressed as convex combination of nodal values

$$\phi_P^{n+1} = \frac{a\Delta t}{\Delta x} \phi_W^n + \left(1 - \frac{a\Delta t}{\Delta x}\right) \phi_P^n$$

$$\phi_E^{n+1} = \frac{a\Delta t}{\Delta x} \phi_P^n + \left(1 - \frac{a\Delta t}{\Delta x}\right) \phi_E^n$$

Dr. Anirban Dhar NPTEL Computational Hydraulics

So this weight functions if we add this to weight functions, the submission will be equal to 1. So phi p, we have written in terms of phi w and phi p. And phi e n plus 1, we can write it in terms of phi e. And phi p, phi e and phi p again convex combination for this one.

(Refer Slide Time 04:59)

Upwind Method
a > 0

Cell-averaged value can be approximated as,

$$\phi_P^{n+1} \approx \phi(x_P, t + \Delta t) = \phi(x_P - a\Delta t, t)$$

$$\phi_E^{n+1} \approx \phi(x_E, t + \Delta t) = \phi(x_E - a\Delta t, t)$$

Future time level value can be expressed as convex combination of nodal values

$$\phi_P^{n+1} = \frac{a\Delta t}{\Delta x} \phi_W^n + \left(1 - \frac{a\Delta t}{\Delta x}\right) \phi_P^n$$

$$\phi_E^{n+1} = \frac{a\Delta t}{\Delta x} \phi_P^n + \left(1 - \frac{a\Delta t}{\Delta x}\right) \phi_E^n$$

Dr. Anirban Dhar NPTEL Computational Hydraulics 4 / 13

If we simplify this with our approximation thing, we can get these forms. In this case CFL condition for A greater than zero is, A delta t delta x. We have seen this derivation of the CFL condition from stability criteria for our finite difference approximations. A delta t equals to 1,

A $\Delta t \Delta x$ equals to 1 or Courant number equals to 1. That means the information which is travelling from the n th level and reaching to the $n + 1$ level, this is exactly for point e . Whatever is there at the point p , that is directly getting transferred to point e .

(Refer Slide Time 06:22)

Upwind Method

Rearranging,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} (\phi_P^n - \phi_W^n)$$

$$\phi_E^{n+1} = \phi_E^n - \frac{a\Delta t}{\Delta x} (\phi_E^n - \phi_P^n)$$

CFL Condition for $a > 0$

$$0 \leq \frac{a\Delta t}{\Delta x} \leq 1$$

If $\frac{a\Delta t}{\Delta x} = 1$,

Diagram showing grid points W, w, p, e, E on level n and $n+1$. A red circle highlights E on level $n+1$ and p on level n , with an arrow pointing from p to E . The equation $x_E - a\Delta t = x_p - \Delta x = x_p$ is shown below the diagram.

Dr. Anirban Dhar NPTEL Computational Hydraulics

Or in this case if you have e equals to 1, so obviously this is ϕ_w^n . Or in this case if this equals to 1, again this is ϕ_p^n . That means if A greater than zero. The information of the previous cell is directly getting transferred to the right ward cell, in the future time level.

(Refer Slide Time 06:59)

Upwind Method

Rearranging,

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} (\phi_P^n - \phi_W^n) = \phi_w^n$$

$$\phi_E^{n+1} = \phi_E^n - \frac{a\Delta t}{\Delta x} (\phi_E^n - \phi_P^n) = \phi_p^n$$

CFL Condition for $a > 0$

$$0 \leq \frac{a\Delta t}{\Delta x} \leq 1$$

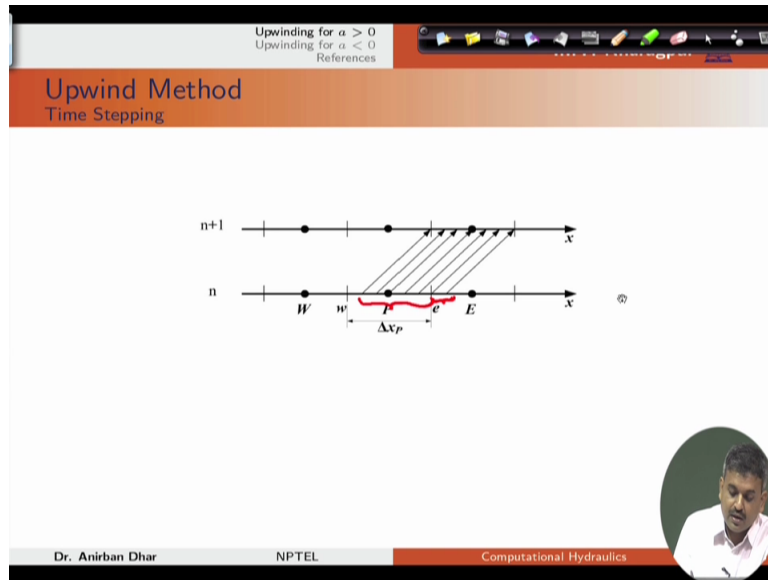
If $\frac{a\Delta t}{\Delta x} = 1$,

Diagram showing grid points W, w, p, e, E on level n and $n+1$. A red circle highlights E on level $n+1$ and p on level n , with an arrow pointing from p to E . The equation $x_E - a\Delta t = x_p - \Delta x = x_p$ is shown below the diagram.

Dr. Anirban Dhar NPTEL Computational Hydraulics

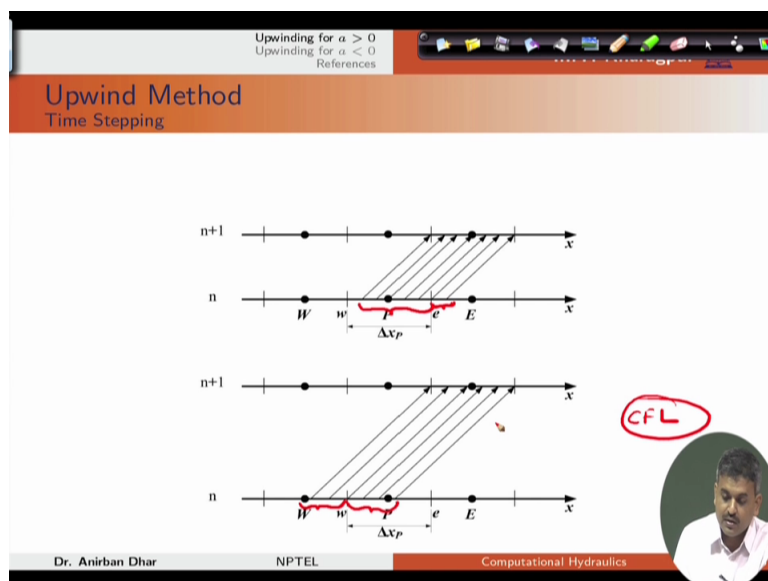
In upwind, time stepping is important. So (fo) in this case information is travelling from p cell to e cell or e cell to. This part is information is travelling from p cell to e cell. In this case information is travelling from e cell to E cell.

(Refer Slide Time 07:34)



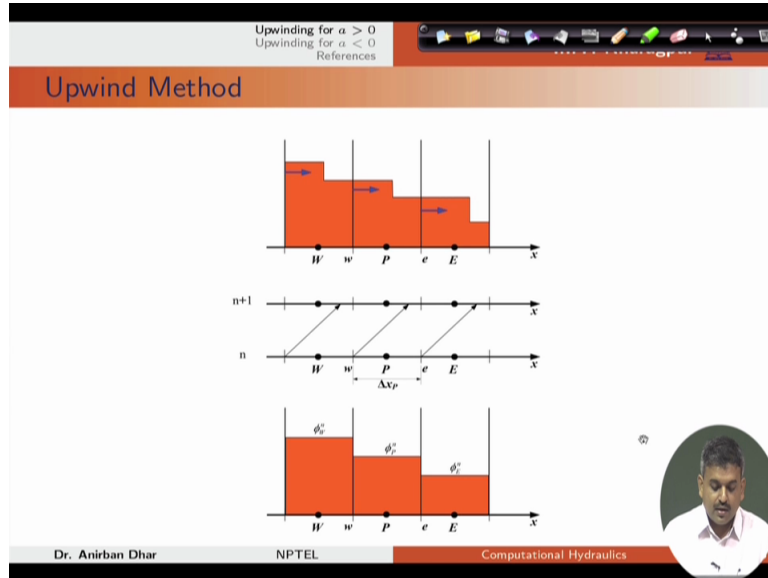
But if we increase this time step, then we can see that information transfer will be there from w cell and p cell. So time stepping is important. This is controlled by CFL condition or Courant Friedrich Lewy condition.

(Refer Slide Time 08:00)



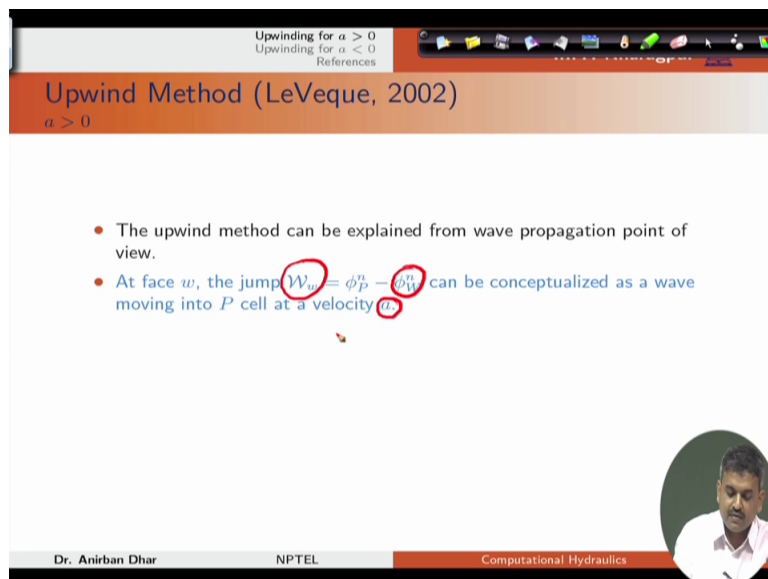
Upwind method, let us say that we are considering A greater than zero condition. Here P cell, W cell, E cell, we have different values. Now information at the cell interface that is getting transferred to the next cell. And we are getting the average values.

(Refer Slide Time 08:37)



So upwind method can be explained from the wave propagation point of view. At w face, we can see that Ww is the jump from this ϕ_w to ϕ_P can be conceptualized as wave moving into P cell at velocity a , in this case.

(Refer Slide Time 09:18)



And this wave modifies the value of ϕ at each point by minus Ww , this value.

(Refer Slide Time 09:36)

Upwind Method (LeVeque, 2002)
a > 0

- The upwind method can be explained from wave propagation point of view.
- At face w , the jump $\mathcal{W}_w = \phi_p^n - \phi_w^n$ can be conceptualized as a wave moving into P cell at a velocity a_w .
- The wave modifies the value of ϕ by $-\mathcal{W}_w$ at each point.

Dr. Anirban Dhar NPTEL Computational Hydraulics

And wave moves at distance $A \Delta t$ over time step and passes through $A \Delta t$ by Δx fraction of the cell. So we can write it like this.

(Refer Slide Time 09:59)

Upwind Method (LeVeque, 2002)
a > 0

- The upwind method can be explained from wave propagation point of view.
- At face w , the jump $\mathcal{W}_w = \phi_p^n - \phi_w^n$ can be conceptualized as a wave moving into P cell at a velocity a_w .
- The wave modifies the value of ϕ by $-\mathcal{W}_w$ at each point.
- The wave moves a distance $a\Delta t$ over a time-step and passes through $\frac{a\Delta t}{\Delta x}$ fraction of the cell.

$$\phi_p^{n+1} = \phi_p^n - \frac{a\Delta t}{\Delta x} \mathcal{W}_w$$

Dr. Anirban Dhar NPTEL Computational Hydraulics

In this case we have general equation where we have general flux function for e cell and w cell. A greater than zero, we can approximate this numerical flux as a ϕ_p . That means at the interface of the cell p and e information is travelling from p cell to e cell. So we can say that the approximate value of numerical flux will depend on this cell p which is up gradient cell. And for w face this is w because information is travelling from w to p , for w face.

(Refer Slide Time 11:31)

Upwind Method
Upwinding for $a > 0$
Upwinding for $a < 0$
References

From general equation it can be written as,

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\mathcal{F}_\phi(\phi_P^n, \phi_E^n) - \mathcal{F}_\phi(\phi_W^n, \phi_P^n)]$$

For $a > 0$ numerical flux values can be written as

$$\mathcal{F}_\phi(\phi_P^n, \phi_E^n) = a\phi_P^n$$

$$\mathcal{F}_\phi(\phi_W^n, \phi_P^n) = a\phi_W^n$$

Dr. Anirban Dhar NPTEL Computational Hydraulics 9 / 13

So if we consider A less than zero, this will be modified because at cell interface e , information will be travelling from e to p side, and at w face, from p to w side. So the corresponding values are utilized here. At e face the wave propagation again we can write it like this.

(Refer Slide Time 12:10)

Upwind Method
 $a < 0$

Similarly, if $a < 0$ numerical flux values can be written as

$$\mathcal{F}_\phi(\phi_P^n, \phi_E^n) = a\phi_E^n$$

$$\mathcal{F}_\phi(\phi_W^n, \phi_P^n) = a\phi_P^n$$

At face e , the jump $\mathcal{W}_e = \phi_E^n - \phi_P^n$ can be conceptualized as a wave moving from P cell at a velocity a . In wave-propagation form

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} \mathcal{W}_e$$

Dr. Anirban Dhar NPTEL Computational Hydraulics

And CFL condition, if you do not consider absolute value of A , then this will be just opposite to our original condition. Original condition was $A \Delta t / \Delta x \leq 1$. Sometimes the use combine at criteria where $A \Delta t / \Delta x$ is utilized. This will be always 1 in this case.

(Refer Slide Time 13:01)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

Upwind Method

$a < 0$

Similarly, if $a < 0$ numerical flux values can be written as

$$\bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = a\phi_E^n$$

$$\bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = a\phi_P^n$$

At face e , the jump $\mathcal{W}_e = \phi_E^n - \phi_P^n$ can be conceptualized as a wave moving from P cell at a velocity a . In wave-propagation form

$$\phi_P^{n+1} = \phi_P^n - \frac{a\Delta t}{\Delta x} \mathcal{W}_e$$

CFL Condition for $a < 0$

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 0$$

Handwritten notes: $0 \leq \frac{a\Delta t}{\Delta x} < 1$ and $0 \leq \frac{|a|\Delta t}{\Delta x} < 1$

Dr. Anirban Dhar NPTEL Computational Hydraulics 10 / 13

In general form, if we write, then this is written as considering both A greater than zero and A less than zero. A minus e and A plus p . At e cell if A or information is moving in the positive direction or positive X direction, then we should consider p cell value or in negative direction e cell value. In this case if it is forward face. If it is our negative direction, we should consider e cell value. At w face if we have positive direction thing then we should consider w cell value.

(Refer Slide Time 14:16)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

General Form

First Order Upwinding

Handwritten notes: $a > 0$ and $a < 0$

In general form numerical flux can be written as,

$$\bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = a^- \phi_E^n + a^+ \phi_P^n$$

$$\bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = a^- \phi_P^n + a^+ \phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Dr. Anirban Dhar NPTEL Computational Hydraulics

So this is general form. But what is the interpretation of this? If A greater than zero, some number. So obviously A plus will be maximum of A or 0 . That means this will be A . And minimum of A 0 . That means if A is positive here, this quantity will be 0 , but this will be A . That means only p will be considered and w will be considered, if A greater than zero.

(Refer Slide Time 15:03)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

General Form

First Order Upwinding

In general form numerical flux can be written as,

$$\bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = a^- \phi_E^n + a^+ \phi_P^n$$

$$\bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = a^- \phi_P^n + a^+ \phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$

Handwritten annotations: $a > 0$ in red at the top right; a^+ underlined in red; $(a, 0)$ with an arrow pointing to 0 in red.

Dr. Anirban Dhar NPTEL Computational Hydraulics

But if we have A less than zero, less than zero means in this case uh minimum will be considered. So this will be A and maximum of negative quantity and zero. So this will give zero value. So in this case we will have e and p terms or first terms.

(Refer Slide Time 15:36)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

General Form

First Order Upwinding

In general form numerical flux can be written as,

$$\bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = a^- \phi_E^n + a^+ \phi_P^n$$

$$\bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = a^- \phi_P^n + a^+ \phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Handwritten annotations: $a < 0$ in red at the top right; a^- underlined in red; a underlined in red.

Dr. Anirban Dhar NPTEL Computational Hydraulics

Now we can write our final form like this. We have just regrouped it with the coefficients A minus and A plus. Now you can see that from our previous case we have unstable schemes and stability with some modification in the equation itself. But in this case we do not require any modification as such. But depending on the nature of the problem or characteristics of the problem, we are defining the interface values and at the same time we are defining the numerical flux.

(Refer Slide Time 16:23)

Upwinding for $a > 0$
Upwinding for $a < 0$
References

General Form

First Order Upwinding

In general form numerical flux can be written as,

$$\bar{F}_\phi(\phi_P^n, \phi_E^n) = a^- \phi_E^n + a^+ \phi_P^n$$
$$\bar{F}_\phi(\phi_W^n, \phi_P^n) = a^- \phi_P^n + a^+ \phi_W^n$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [a^- (\phi_E^n - \phi_P^n) + a^+ (\phi_P^n - \phi_W^n)]$$

Dr. Anirban Dhar NPTEL Computational Hydraulics

So this approach is morephysically consistent. And this is the basic thing for our upwind approach. This approach is called as first order upwind method. Thank you.