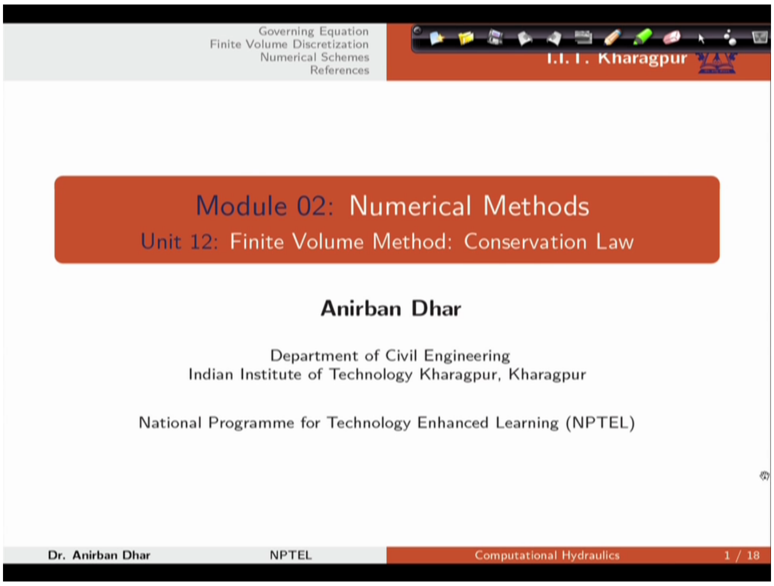
Computational Hydraulics
ProfessorAnirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 16
Finite Volume Method - Conservation Law

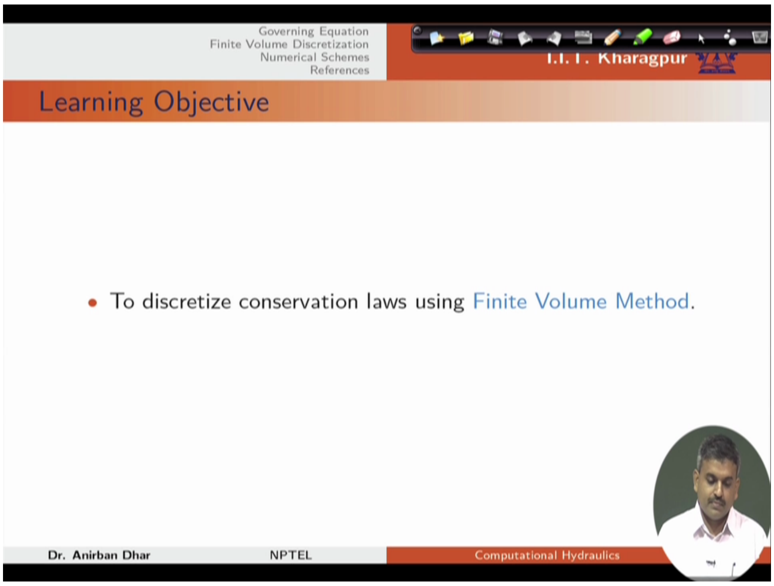
Welcome to thislecture number 16 of the course computational hydraulics.We are in module 2, numerical methods. And in this particular lecture I will be coveringunit 12, finite volume method,conservation law.

(Refer Slide Time 00:32)



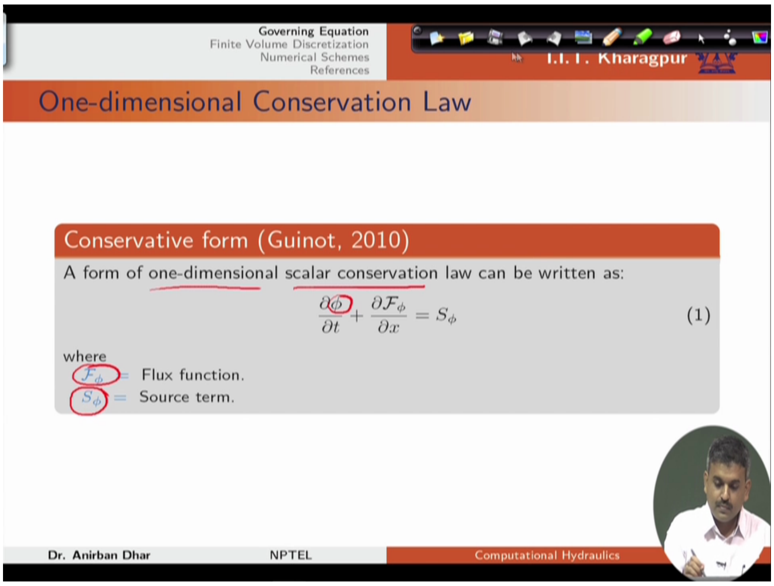
What is the learning objective for this particular unit?At the end of this unit students will be able to discretize the conservation laws using finite volume method.

(Refer Slide Time 00:58)



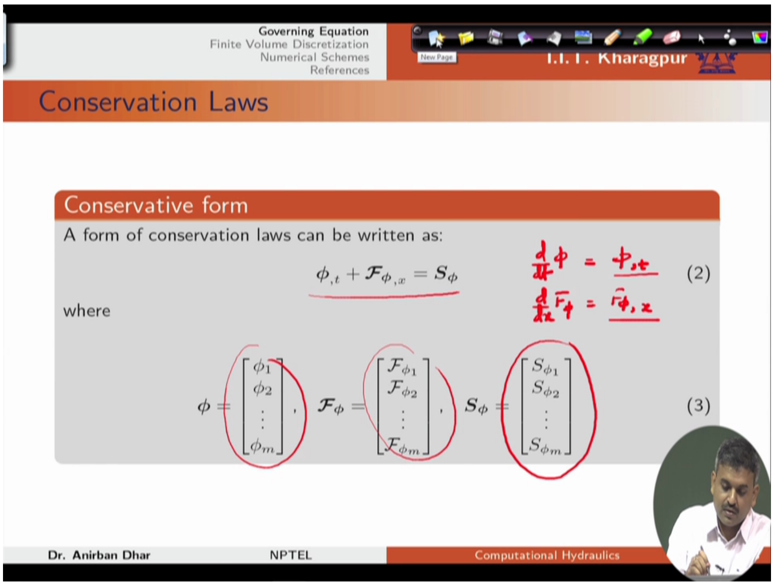
We have seen this form of equation during our finite volume discretization. This is one-dimensional scalarconservation law where phi is the general scalar variable. F phi is a flux function and Sphi is the source term for this one.

(Refer Slide Time 01:31)



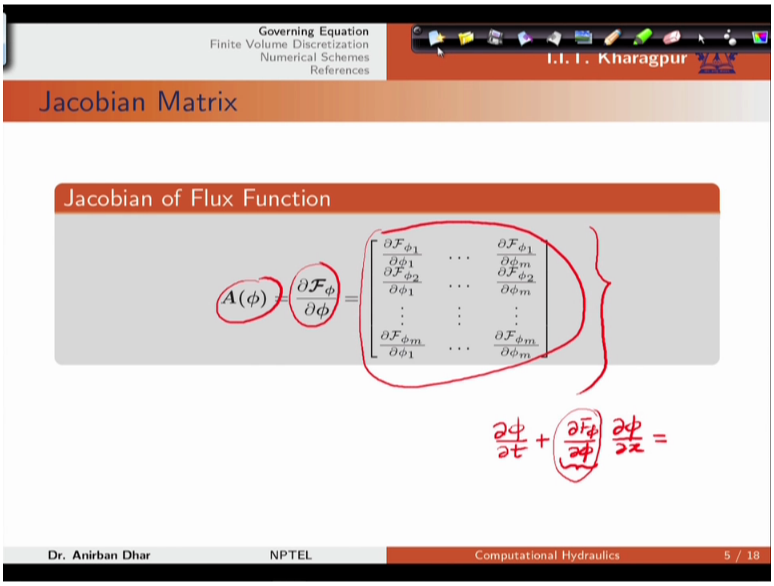
Now if we writethe conservation laws in the form of vector then for phi 1 to phi m, these are scalar variables. And F phi 1 to F phi m, these are corresponding flux functions. And Sphi 1 to S phi m, these are actually source sink terms. So we can write it in this vector format. Here this comma it means we are differentiating this thing with respect to t. This is phi, this means and this is phi comma t. Similarly for d by dx F phi, we can write Fphi comma x. This is short notation for derivatives. Sowith this form we can start our problem.

(Refer Slide Time 03:06)



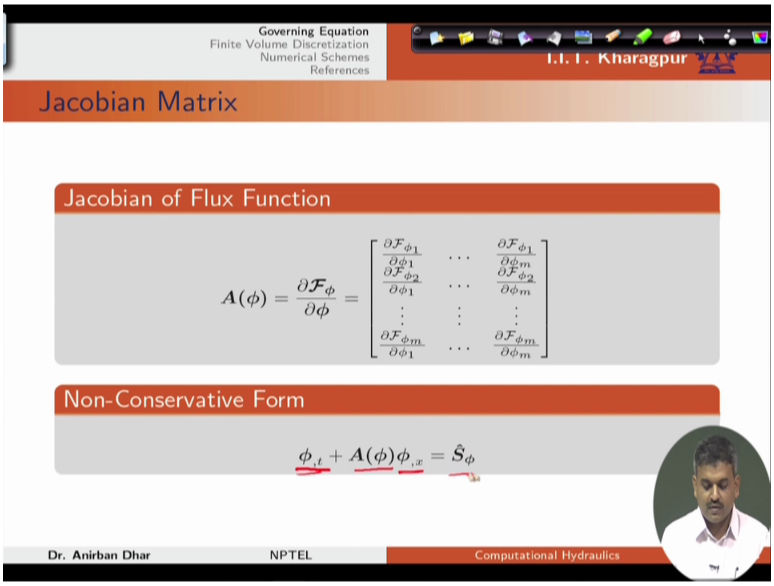
We can define Jacobian of the flux function where this A phi matrix is actually del Fphi delphi. And individually these are terms. And in this case we can define our things. If welook back to our finite difference lecture then we can see that, d phi by dt plus delphi F by del x delphi and delphi by del x, this time was there for single variable.Now if we consider the same thing for a complete system, then we will have thisJacobian matrix. In place of single derivative we will have full Jacobian matrix here.

(Refer Slide Time 04:45)



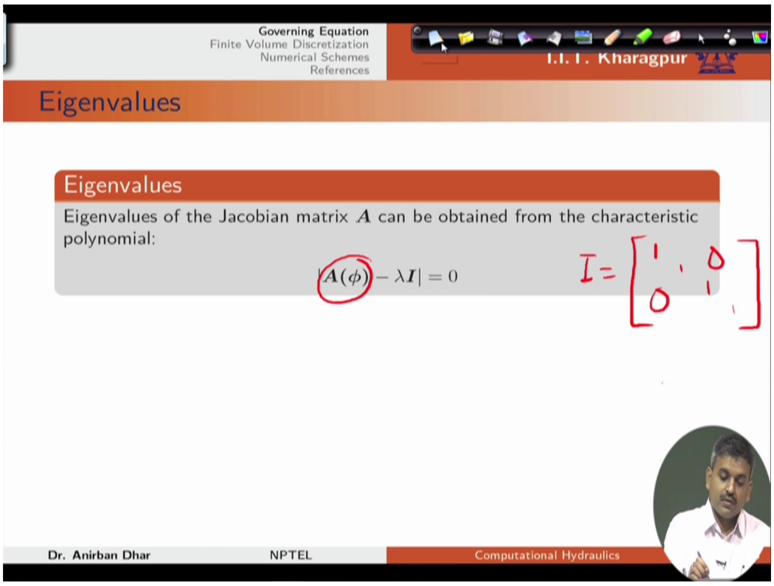
Now we have seen that derivative in one-dimensional case dictates the nature of the solution. In this casewe can have the non-conservative form, similar way. This phi t is again derivative respect to t. A phi is our Jacobian flux function and phix again vector of phi(res) derivative with respect to x. This is modified source sink function.

(Refer Slide Time 05:36)



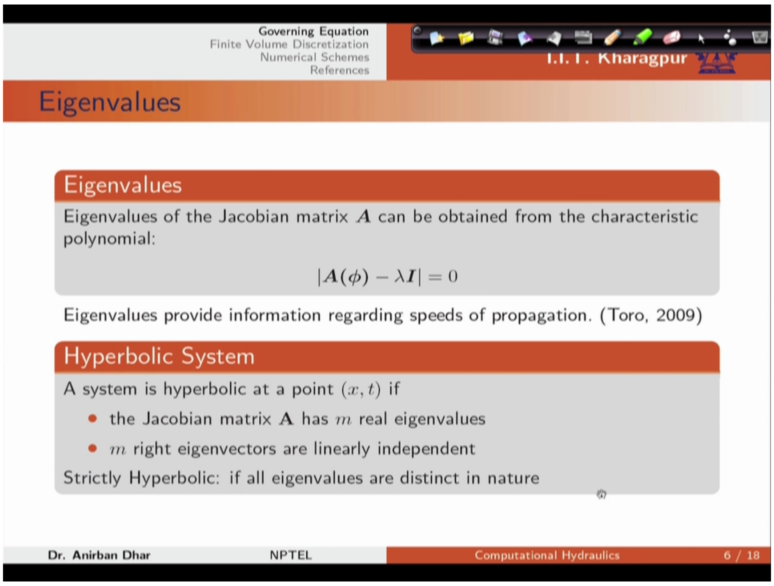
Now in this case we can find out the eigenvalues of this Jacobian matrix from characteristic polynomial. So if A is the matrix, I isidentity matrix, only the diagonals terms will be there. Other side it will be zero and lambda is some characteristicvalue we can use here.

(Refer Slide Time 06:19)



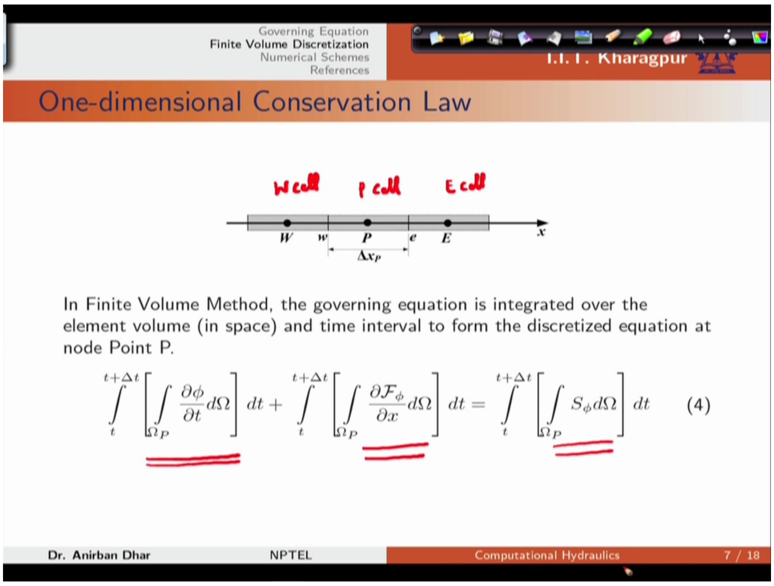
And eigenvalues provide information regarding speed of propagation. And propagation of information from one side to another side. For hyperbolic system at a point xd if Jacobian matrix has m real eigenvalue. M right eigenvectors are linearly independent. We call it as hyperbolic system and in this casewe call as strictly hyperbolic, if all eigenvalues are distinct in nature.

(Refer Slide Time 07:13)



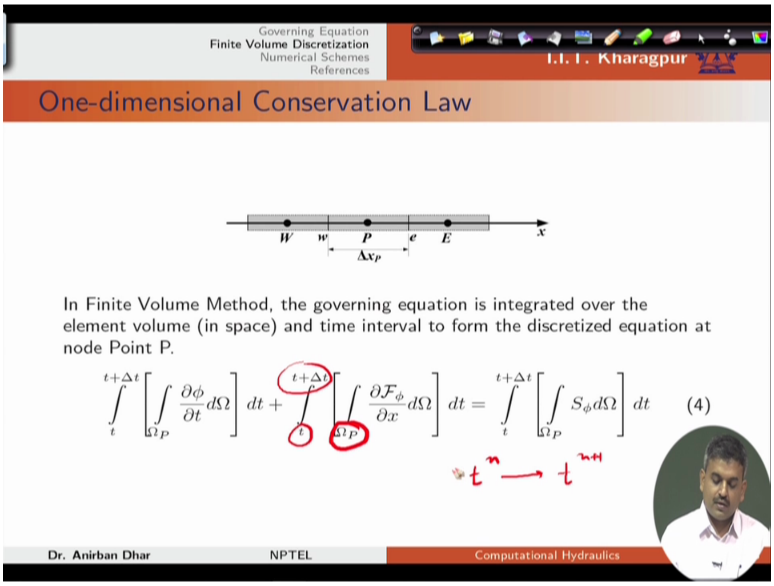
Let us consider our one-dimensional conservation law. We have general P cell, E cell and W cell here for some interior nodes. Now for this one we can define our finite volume method. It is with space-time discretization. The first term is for temporal derivative, second term for the spatial derivative of the flux function and the last one is source sink term.

(Refer Slide Time 08:08)



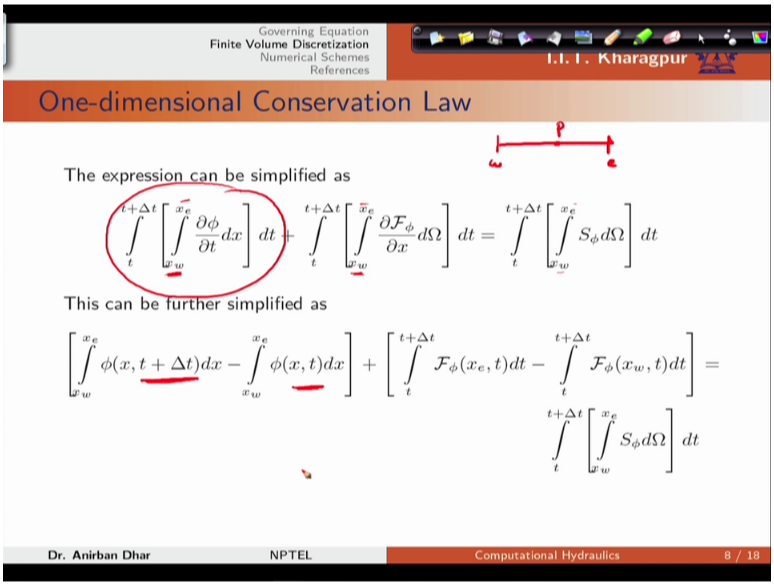
And interestingly we can see that this omega P refers to the P cell. That means for that elemental volume we are considering this integration. And for t to t plus delta t, or we can say that time starting from tn to tn plus 1, we are discretizing and this thing.So we are taking the integration for that level.

(Refer Slide Time 08:44)



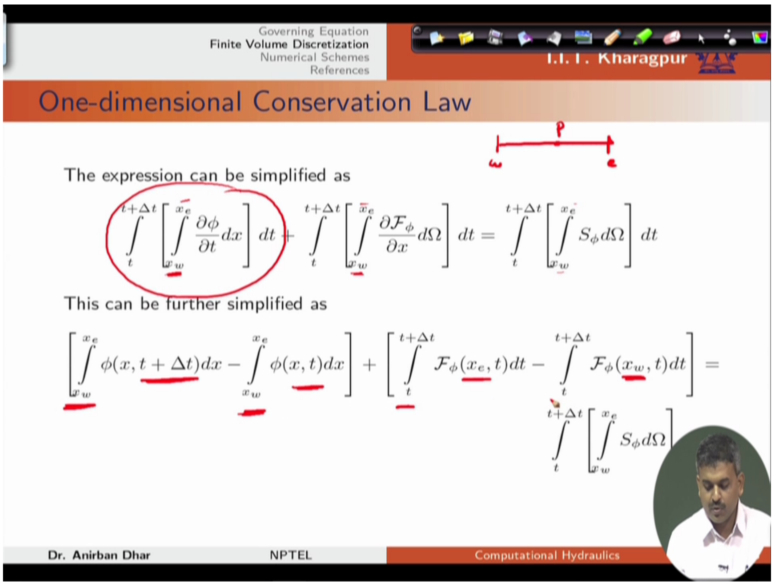
We have considered this case P cell. So P cell is running from xw that is face w to e. This is P cell. Nowfor derivative here we can replacethe limits or integral limits xw to xe, for all the cases. Now if we simplify this one, we can write it for discretized form and here this is t plus delta t at t level.

(Refer Slide Time 10:00)



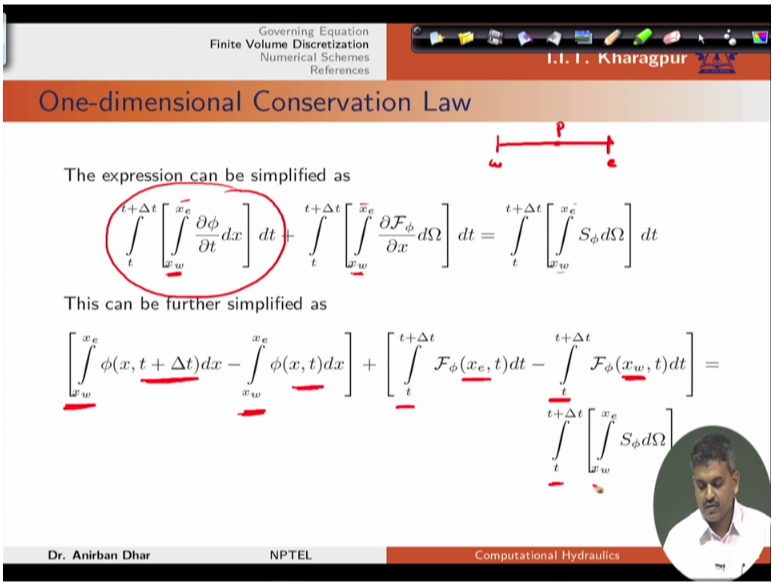
In this case we have taken discretization in terms of space values. So e and w. In this case we have outside only spatial integral. This is temporal one.

(Refer Slide Time 10:24)



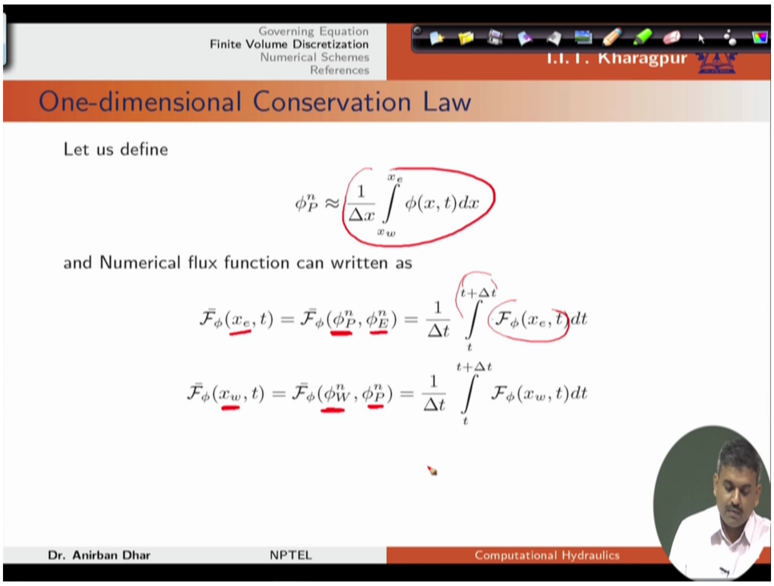
This is space time. We have not divided this thing.

(Refer Slide Time 10:26)



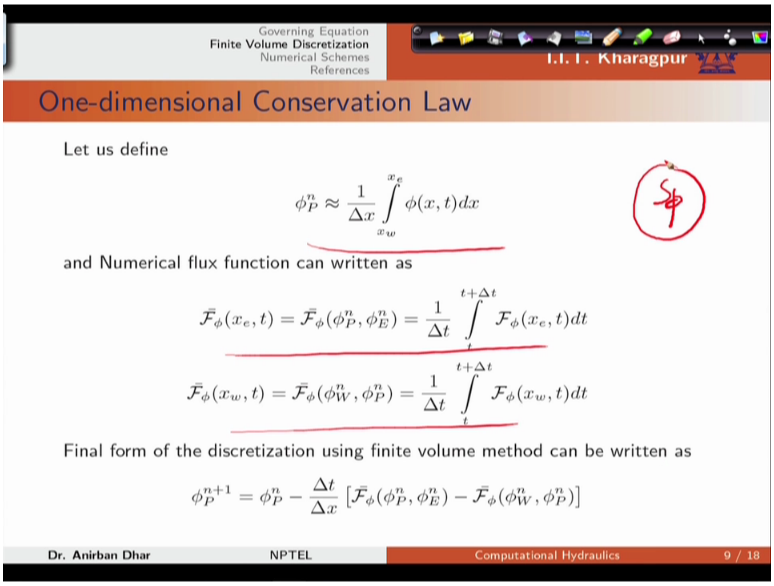
Let us define the average value of phi pm that means at any time level m cell average value we can define like this. And numerical flux function can be written because we havediscretizedin terms of space but stillthe temporal integration is there from t to t plus delta t level. So for east face we need to consider P cell and E cell values. For w face we need to consider W cell and P cell value. Depending on that we can calculate the flux function.

(Refer Slide Time 11:45)



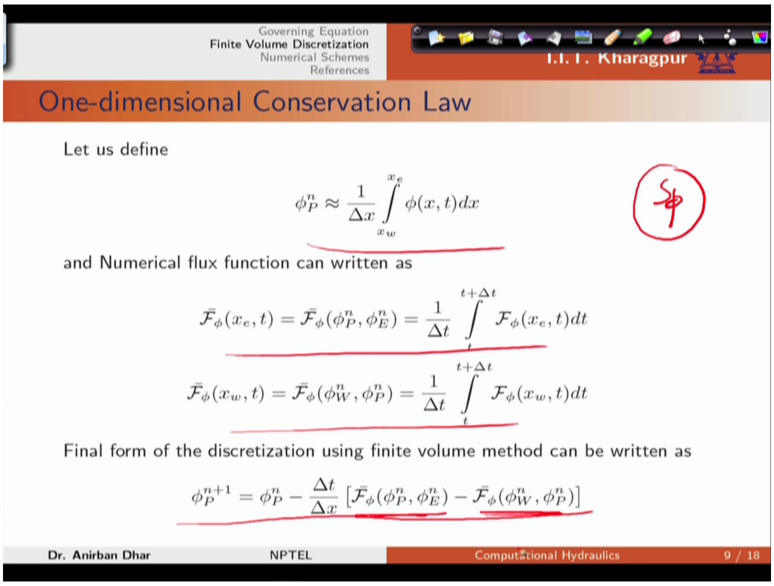
Final form of the discretization. If we simply replace these terms and of course in this case we are not considering S phi because discretization of Sphi depends on its functional form and more or less that is constant. So we will not consider that discretization at this stage.

(Refer Slide Time 12:20)



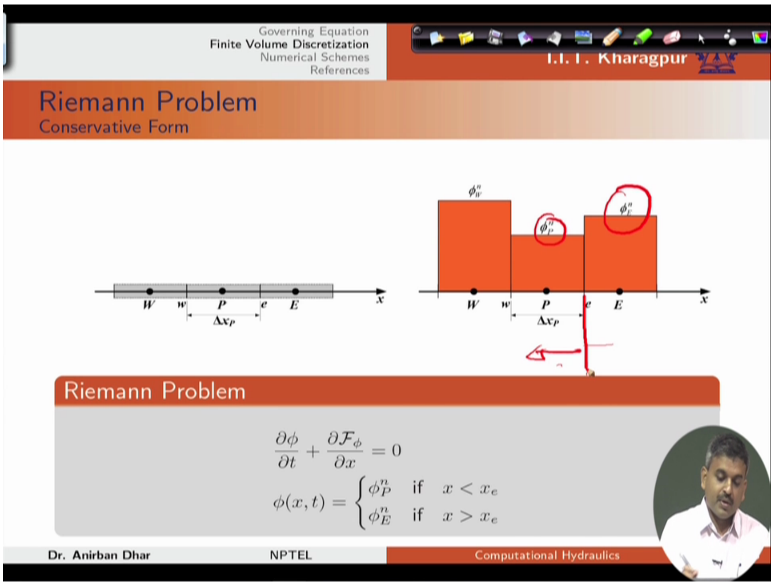
So let's concentrate on the first temporal derivative and then spatial derivatives of the flux function. So this is the final form but still we need to calculate, what will be the value of this flux functions?

(Refer Slide Time 12:43)



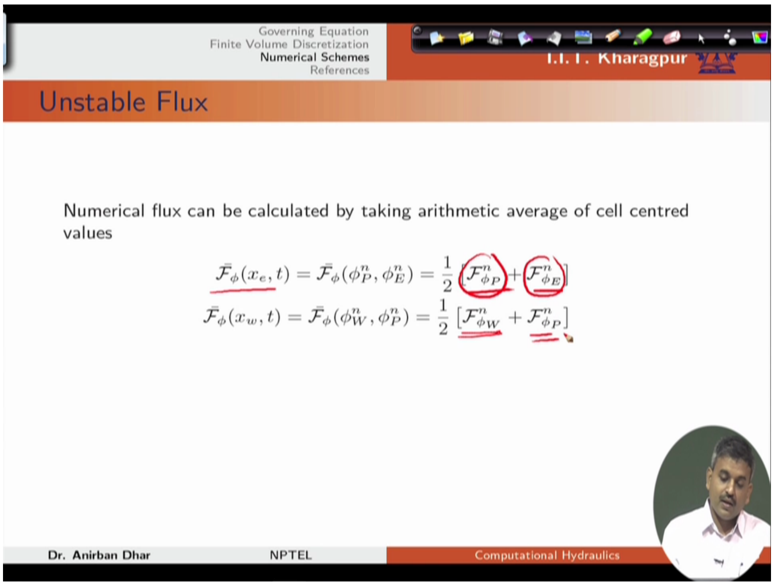
Riemann problem, this is conservative form.We already know that we have P cell, W cell, E cell. Now at each cell there will be different cell averaged values. So Riemann in problem or Riemann problem essentiallycomes at the interface where, let us say we are considering this east face. If x is on the right side. So this is phi En and the left side we have phi Pn. If it is on the left side it is phi Pn. So essentially there is this continuity at this level. So this problem is called as Riemann problem.

(Refer Slide Time 13:57)



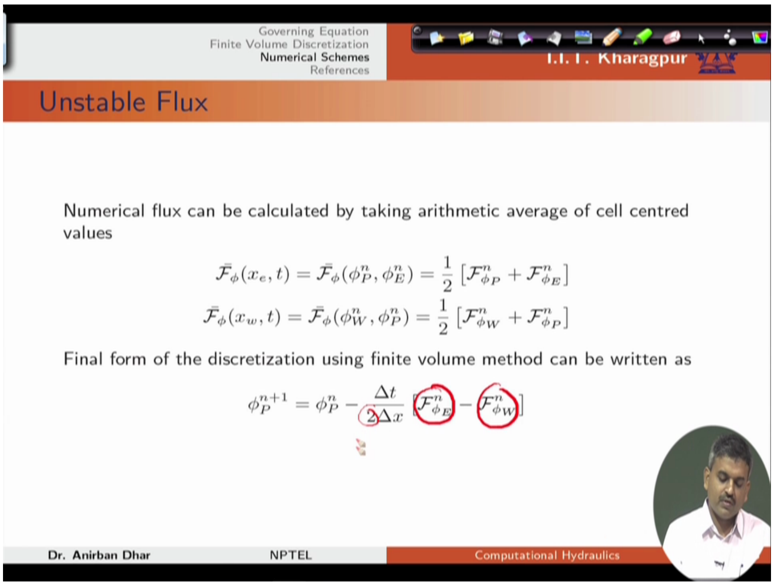
Now in this case first hand approximation of the flux function can be averaged flux values from both the cells. So at east face we can consider that phi P for flux and phi E for flux and we can take average. And westside also we are taking average of flux values of W and P.

(Refer Slide Time 14:42)



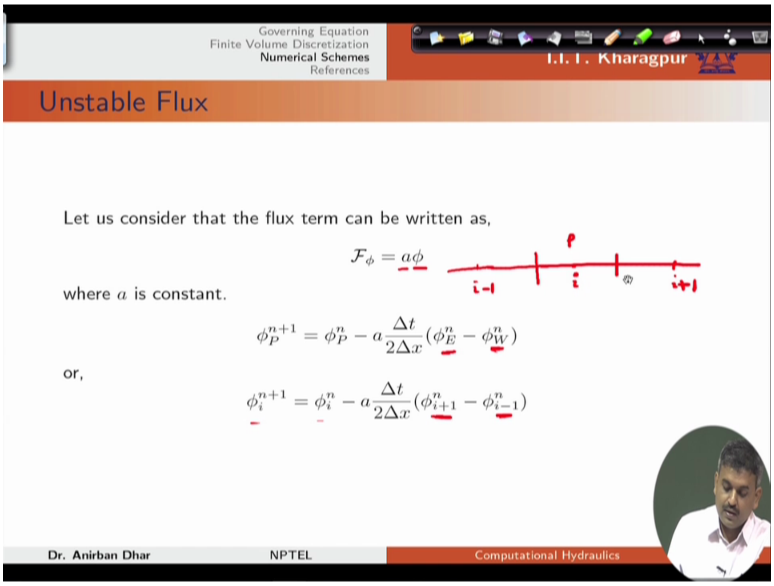
Now this is first hand approximation. And if we consider our numerical discretization then only flux value for the east face and west face will be required for this calculation, and 2 is coming here.

(Refer Slide Time 15:06)



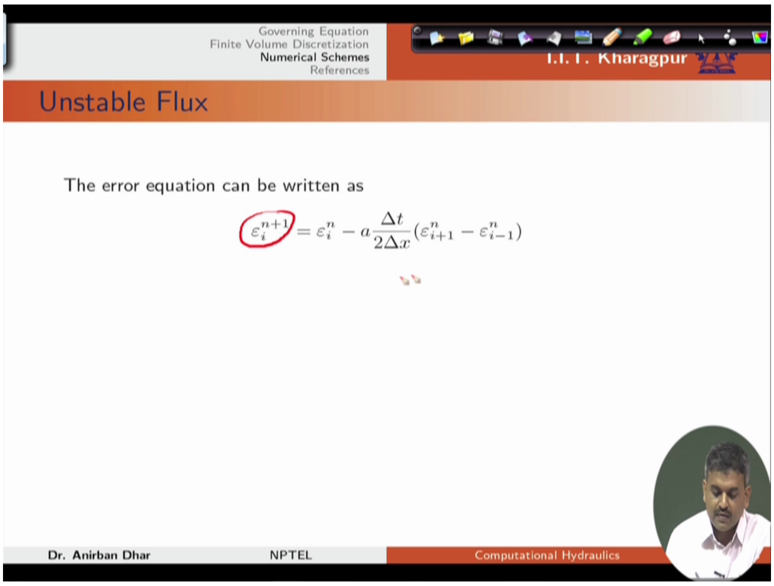
Now let us consider the simplest form of the flux where A is constant,phi is our generalscalar variable. Then we can write our final form of the discretization using finite volume technique as a delta t, 2 delta x,phiE minus W. We can also represent it in terms of ij or I. I plus 1, I minus 1. If we have P cell then at the center we can consider it as I. On the left side it is I minus 1, on the right side we have I plus 1. So for east cell this is I plus 1 and westcell we have I minus 1. And A delta t by 2 delta x, this is coming as constant.

(Refer Slide Time 16:33)



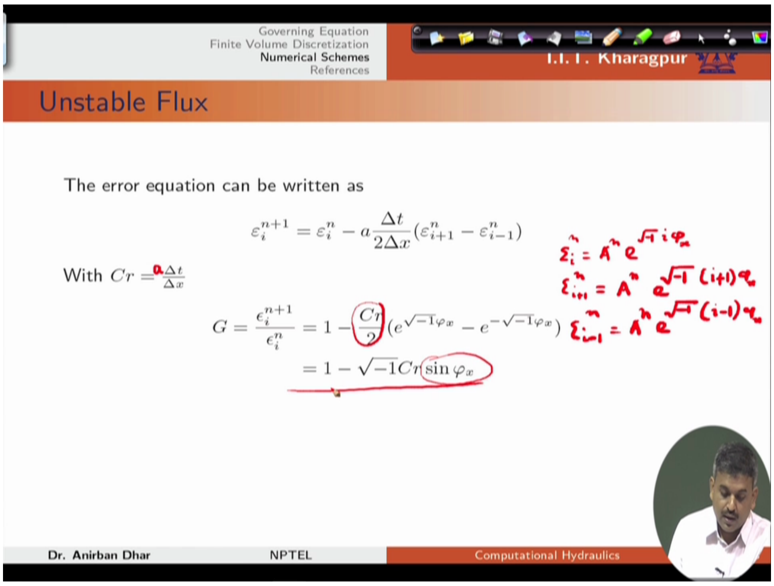
Now we need to see whether this discretization is stable or not. Like our finite difference technique, we can use our error equation. Error equation, we are writing it in terms of epsilon i n plus 1,var epsilon. And this is var epsilon i n and this term is constant.

(Refer Slide Time 17:05)



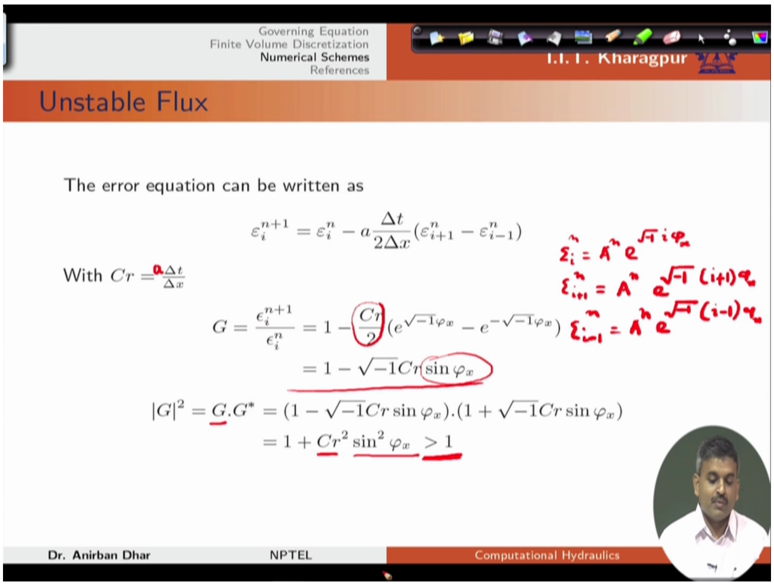
Now for this onewe can define our terms as A delta t by delta x. So this is Cr by 2. And as per our convention, we can directly write as, e to the power square root of minus 1 varphi x. Actually we have seen that this epsilon i n, this is A n and e to the power minus 1 i varphi x, in this case. So for var phi this var epsilon i plus 1, this will be n, square root minus 1 phi plus 1phi minus n. So we can just simplify this one and get 1 minus square root of minus 1 Crour sin var phi x.

(Refer Slide Time 19:29)



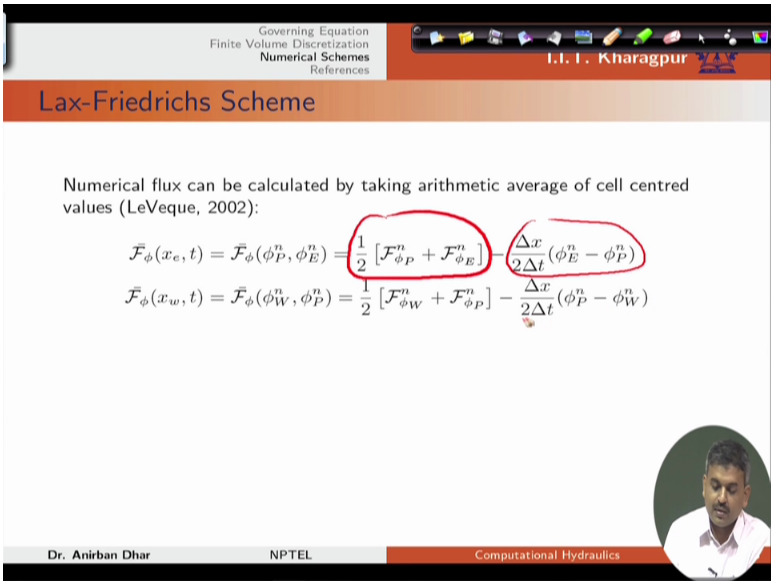
And in thiscase we can write our G or growth factor which is G into G star which is conjugatecomplex number. So this is minus, this will be plus. So always 1 plus Cr square sin square var phix. In this case it is clear that 1 plus some quantity. So Cr is always positive because square term is there, sin square is again positive. So 1 plus some positive term will be always greater than 1. So this scheme is unstable scheme.

(Refer Slide Time 20:20)



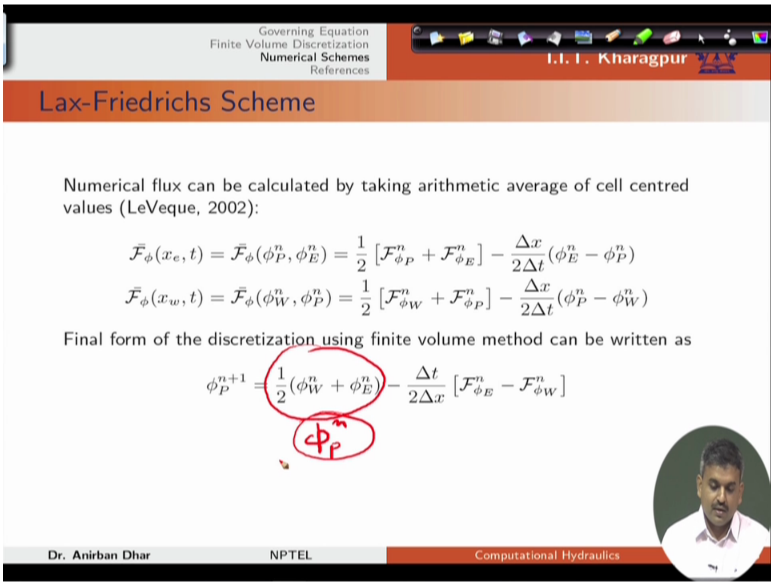
So this approximation or firsthand approximation is not appropriate. This is unstable scheme. So Lax Friedrichs scheme. In that one this same type of flux, average flux is used. But one extra term is added here which is difference between east and our central P cell. And west face, this is P minus W.

(Refer Slide Time 21:09)



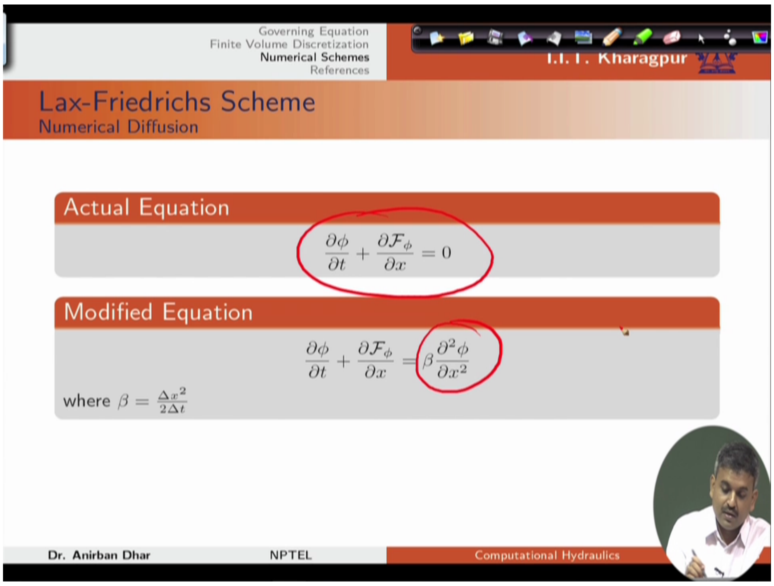
Sofinal form is averaged value of W and E instead of phi phi Pn. In previous case we have seen that our discretization was with phi Pn value. But in this case we are taking average of Wand E.

(Refer Slide Time 21:45)



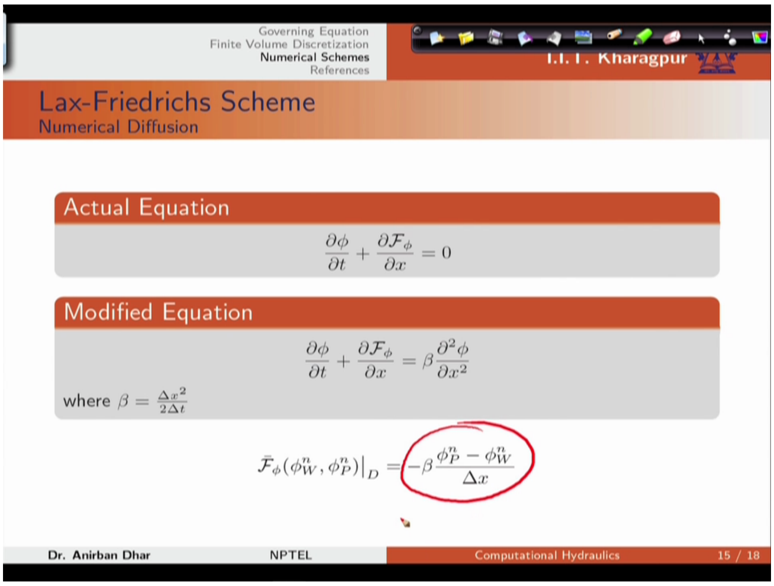
Now with this again we can write it in i,i minus 1, i plus 1 format. If we see ouractual equation thenthis is our actual equation. But virtually what we are doing, we are adding some extra second order diffusion term here, in the Lax Friedrichs scheme for stabilization.

(Refer Slide Time 22:32)



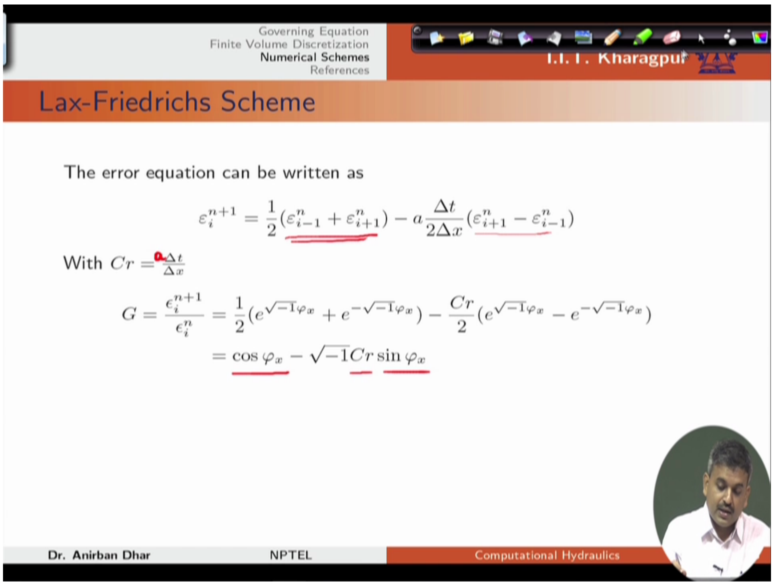
So in this case, this diffusing flux, we can see that this is some beta into phi P minusphi W, for west face. Sothis extra term is added for stabilization of the scheme which is equivalent to adding one extra term on the right hand side.

(Refer Slide Time 22:58)



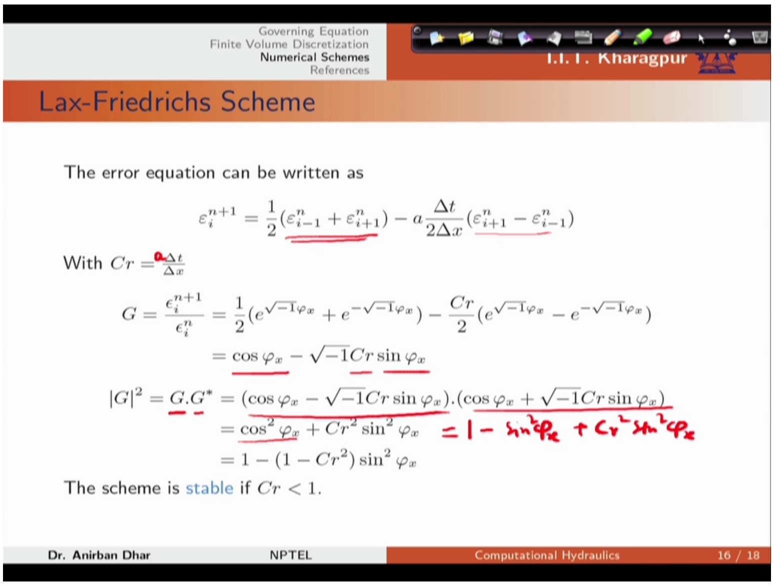
So if we check the stability of the scheme, again this is averaged value. So if we check our stabilityfor this case, the Cr is defined withA delta t by delta x. We can get this cos var phi x minus square root of minus, 1 Cr sin var phi x.

(Refer Slide Time 23:56)



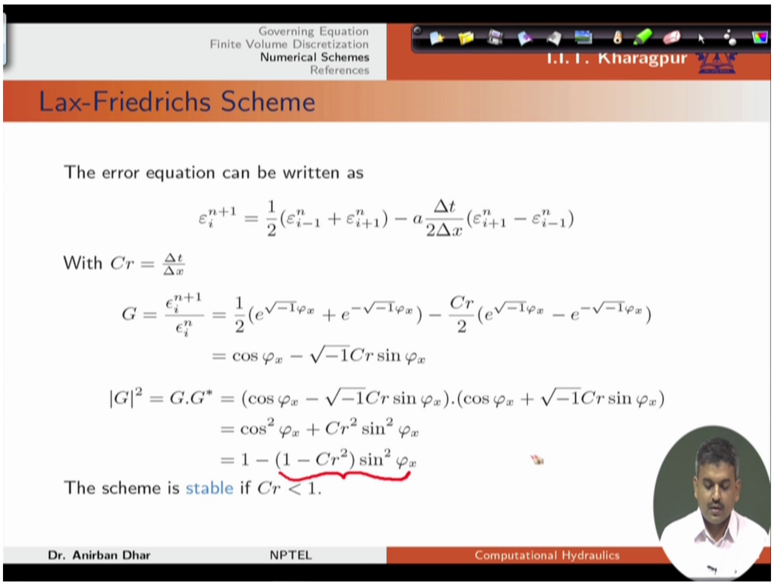
And if we write our growth factor again that is G into G star which is conjugate complex number. So in this case we are getting cos square var phi plus Cr square plus sin square var phi. So we can rewrite this 1 minus sin square var phi x plus Cr square sin square var phi x. And if we rewrite this, we can write it like this. And in this case scheme is stable if Cr is less than 1.

(Refer Slide Time 24:48)



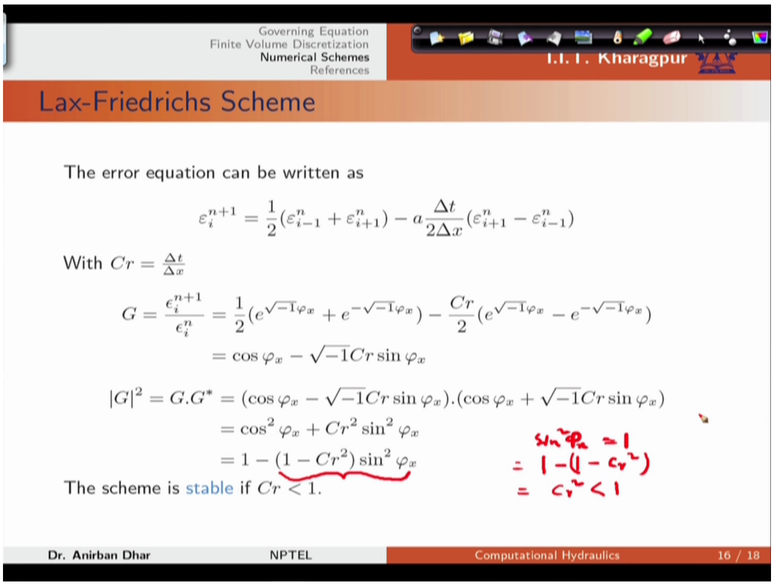
If in this particular case, if we have 1 minus some quantity. If Cr is greater than 1 then this quantity will benegative. And negative-negative this will be positive. So 1 plus some quantity. In extreme case if we consider that sin var phi square, this is equal to 1.

(Refer Slide Time 25:28)



This is extreme case that sin phi x, this equals to 1. So this comes as 1 minus Cr square. This means that this is Cr square. So obviously this should beless than 1for that stable scheme. Another extreme is sin var phi x square. This term is zero. So then obviously this is (neu) neutrally stable.

(Refer Slide Time 26:15)



So we can see that depending on theapproximate form of the flux or numerical flux function, we are getting stable or unstable scheme. In the next lecture we will see another kind of approximation for this flux function. Thank you.