

Computational Hydraulics
Professor Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 16
Finite Volume Method - Conservation Law

Welcome to this lecture number 16 of the course computational hydraulics. We are in module 2, numerical methods. And in this particular lecture I will be covering unit 12, finite volume method, conservation law.

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The slide features a navigation menu at the top left with the following items: Governing Equation, Finite Volume Discretization, Numerical Schemes, and References. The top right corner displays the I.I.T. Kharagpur logo. The main content area is centered and contains the following text:

Module 02: Numerical Methods
Unit 12: Finite Volume Method: Conservation Law

Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur, Kharagpur
National Programme for Technology Enhanced Learning (NPTEL)

The bottom of the slide has a footer with the following information: Dr. Anirban Dhar, NPTEL, Computational Hydraulics, and 1 / 18.

What is the learning objective for this particular unit? At the end of this unit students will be able to discretize the conservation laws using finite volume method.

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Governing Equation
Finite Volume Discretization
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Learning Objective

- To discretize conservation laws using **Finite Volume Method**.

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We have seen this form of equation during our finite volume discretization. This is one-dimensional scalar conservation law where ϕ is the general scalar variable. F_ϕ is a flux function and S_ϕ is the source term for this one.

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One-dimensional Conservation Law

Conservative form (Guinot, 2010)

A form of one-dimensional scalar conservation law can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial F_\phi}{\partial x} = S_\phi \quad (1)$$

where

- F_ϕ = Flux function.
- S_ϕ = Source term.

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Now if we write the conservation laws in the form of vector then for ϕ_1 to ϕ_m , these are scalar variables. And F_{ϕ_1} to F_{ϕ_m} , these are corresponding flux functions. And S_{ϕ_1} to S_{ϕ_m} , these are actually source sink terms. So we can write it in this vector format. Here this comma it means we are differentiating this thing with respect to t . This is ϕ , this means and this is $\phi_{,t}$. Similarly for $\frac{d}{dx} F_\phi$, we can write $F_{\phi, x}$. This is short notation for derivatives. So with this form we can start our problem.

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Conservation Laws

Conservative form

A form of conservation laws can be written as:

$$\phi_t + \mathcal{F}_{\phi,x} = S_{\phi} \quad (2)$$

where

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{bmatrix}, \quad \mathcal{F}_{\phi} = \begin{bmatrix} \mathcal{F}_{\phi_1} \\ \mathcal{F}_{\phi_2} \\ \vdots \\ \mathcal{F}_{\phi_m} \end{bmatrix}, \quad S_{\phi} = \begin{bmatrix} S_{\phi_1} \\ S_{\phi_2} \\ \vdots \\ S_{\phi_m} \end{bmatrix} \quad (3)$$

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We can define Jacobian of the flux function where this A phi matrix is actually del F phi del phi. And individually these are terms. And in this case we can define our things. If we look back to our finite difference lecture then we can see that, d phi by dt plus del phi F by del x del phi and del phi by del x, this time was there for single variable. Now if we consider the same thing for a complete system, then we will have this Jacobian matrix. In place of single derivative we will have full Jacobian matrix here.

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Jacobian Matrix

Jacobian of Flux Function

$$A(\phi) = \frac{\partial \mathcal{F}_{\phi}}{\partial \phi} = \begin{bmatrix} \frac{\partial \mathcal{F}_{\phi_1}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_1}}{\partial \phi_m} \\ \frac{\partial \mathcal{F}_{\phi_2}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_2}}{\partial \phi_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial \mathcal{F}_{\phi_m}}{\partial \phi_1} & \cdots & \frac{\partial \mathcal{F}_{\phi_m}}{\partial \phi_m} \end{bmatrix}$$

$$\frac{\partial \phi}{\partial t} + \left(\frac{\partial \mathcal{F}_{\phi}}{\partial \phi} \right) \frac{\partial \phi}{\partial x} = S_{\phi}$$

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Now we have seen that derivative in one-dimensional case dictates the nature of the solution. In this case we can have the non-conservative form, similar way. This phi t is again derivative

respect to t. A phi is our Jacobian flux function and phix again vector of phi(res) derivative with respect to x. This is modified source sink function.

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The slide is titled "Jacobian Matrix" and is part of a presentation from I.I.T. Kharagpur. It contains two main sections:

- Jacobian of Flux Function:** Shows the Jacobian matrix $A(\phi) = \frac{\partial \mathcal{F}_\phi}{\partial \phi}$ as a square matrix with elements $\frac{\partial \mathcal{F}_{\phi_i}}{\partial \phi_j}$.
- Non-Conservative Form:** Shows the equation $\phi_t + A(\phi)\phi_x = \hat{S}_\phi$.

At the bottom, it lists "Dr. Anirban Dhar", "NPTEL", and "Computational Hydraulics". A small circular portrait of the speaker is visible in the bottom right corner.

Now in this case we can find out the eigenvalues of this Jacobian matrix from characteristic polynomial. So if A is the matrix, I identity matrix, only the diagonals terms will be there. Other side it will be zero and lambda is some characteristicvalue we can use here.

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The slide is titled "Eigenvalues" and is part of a presentation from I.I.T. Kharagpur. It contains the following content:

- Eigenvalues:** States that "Eigenvalues of the Jacobian matrix A can be obtained from the characteristic polynomial: $|A(\phi) - \lambda I| = 0$ ".
- Handwritten Matrix:** A handwritten identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is shown in red.

At the bottom, it lists "Dr. Anirban Dhar", "NPTEL", and "Computational Hydraulics". A small circular portrait of the speaker is visible in the bottom right corner.

And eigenvalues provide information regarding speed of propagation. And propagation of information from one side to another side. For hyperbolic system at a point xd if Jacobian matrix has m real eigenvalue. M right eigenvectors are linearly independent. We call it as

hyperbolic system and in this case we call it strictly hyperbolic, if all eigenvalues are distinct in nature.

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The slide is titled "Eigenvalues" and is part of a presentation on "Computational Hydraulics" by Dr. Anirban Dhar, NPTEL. The slide content is as follows:

Eigenvalues

Eigenvalues of the Jacobian matrix A can be obtained from the characteristic polynomial:

$$|A(\phi) - \lambda I| = 0$$

Eigenvalues provide information regarding speeds of propagation. (Toro, 2009)

Hyperbolic System

A system is hyperbolic at a point (x, t) if

- the Jacobian matrix A has m real eigenvalues
- m right eigenvectors are linearly independent

Strictly Hyperbolic: if all eigenvalues are distinct in nature

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Let us consider our one-dimensional conservation law. We have general P cell, E cell and W cell here for some interior nodes. Now for this one we can define our finite volume method. It is with space-time discretization. The first term is for temporal derivative, second term for the spatial derivative of the flux function and the last one is source sink term.

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The slide is titled "One-dimensional Conservation Law" and is part of a presentation on "Computational Hydraulics" by Dr. Anirban Dhar, NPTEL. The slide content is as follows:

The diagram shows a horizontal axis x with three points labeled W , P , and E . Above the axis, the regions are labeled "W cell", "P cell", and "E cell". The distance between W and P is w , and the distance between P and E is e . The total distance between W and E is Δx_P .

In Finite Volume Method, the governing equation is integrated over the element volume (in space) and time interval to form the discretized equation at node Point P.

$$\int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial \phi}{\partial t} d\Omega \right] dt + \int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial \mathcal{F}_\phi}{\partial x} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{\Omega_P} S_\phi d\Omega \right] dt \quad (4)$$

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And interestingly we can see that this Ω_P refers to the P cell. That means for that elemental volume we are considering this integration. And for t to t plus Δt , or we can say

that time starting from t_n to $t_n + 1$, we are discretizing and this thing. So we are taking the integration for that level.

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One-dimensional Conservation Law

In Finite Volume Method, the governing equation is integrated over the element volume (in space) and time interval to form the discretized equation at node Point P.

$$\int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial \phi}{\partial t} d\Omega \right] dt + \int_t^{t+\Delta t} \left[\int_{\Omega_P} \frac{\partial \mathcal{F}_\phi}{\partial x} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{\Omega_P} S_\phi d\Omega \right] dt \quad (4)$$

$t^n \rightarrow t^{n+1}$

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We have considered this case P cell. So P cell is running from x_w that is face w to e . This is P cell. Now for derivative here we can replace the limits or integral limits x_w to x_e , for all the cases. Now if we simplify this one, we can write it for discretized form and here this is $t + \Delta t$ at t level.

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One-dimensional Conservation Law

The expression can be simplified as

$$\int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \phi}{\partial t} dx \right] dt + \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \mathcal{F}_\phi}{\partial x} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

This can be further simplified as

$$\left[\int_{x_w}^{x_e} \phi(x, t + \Delta t) dx - \int_{x_w}^{x_e} \phi(x, t) dx \right] + \left[\int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt - \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt \right] = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

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In this case we have taken discretization in terms of space values. So e and w . In this case we have outside only spatial integral. This is temporal one.

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One-dimensional Conservation Law

The expression can be simplified as

$$\int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \phi}{\partial t} dx \right] dt + \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \mathcal{F}_\phi}{\partial x} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

This can be further simplified as

$$\left[\int_{x_w}^{x_e} \phi(x, t + \Delta t) dx - \int_{x_w}^{x_e} \phi(x, t) dx \right] + \left[\int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt - \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt \right] = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

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This is space time. We have not divided this thing.

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One-dimensional Conservation Law

The expression can be simplified as

$$\int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \phi}{\partial t} dx \right] dt + \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} \frac{\partial \mathcal{F}_\phi}{\partial x} d\Omega \right] dt = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

This can be further simplified as

$$\left[\int_{x_w}^{x_e} \phi(x, t + \Delta t) dx - \int_{x_w}^{x_e} \phi(x, t) dx \right] + \left[\int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt - \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt \right] = \int_t^{t+\Delta t} \left[\int_{x_w}^{x_e} S_\phi d\Omega \right] dt$$

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Let us define the average value of phi pm that means at any time level m cell average value we can define like this. And numerical flux function can be written because we have discretized in terms of space but still the temporal integration is there from t to t plus delta t level. So for east face we need to consider P cell and E cell values. For w face we need to consider W cell and P cell value. Depending on that we can calculate the flux function.

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One-dimensional Conservation Law

Let us define

$$\phi_P^n \approx \frac{1}{\Delta x} \int_{x_w}^{x_e} \phi(x, t) dx$$

and Numerical flux function can be written as

$$\bar{\mathcal{F}}_\phi(x_e, t) = \bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt$$

$$\bar{\mathcal{F}}_\phi(x_w, t) = \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt$$

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Final form of the discretization. If we simply replace these terms and of course in this case we are not considering S_ϕ because discretization of S_ϕ depends on its functional form and more or less that is constant. So we will not consider that discretization at this stage.

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One-dimensional Conservation Law

Let us define

$$\phi_P^n \approx \frac{1}{\Delta x} \int_{x_w}^{x_e} \phi(x, t) dx$$

and Numerical flux function can be written as

$$\bar{\mathcal{F}}_\phi(x_e, t) = \bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt$$

$$\bar{\mathcal{F}}_\phi(x_w, t) = \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt$$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) - \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n)]$$

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So let's concentrate on the first temporal derivative and then spatial derivatives of the flux function. So this is the final form but still we need to calculate, what will be the value of this flux functions?

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One-dimensional Conservation Law

Let us define

$$\phi_P^n \approx \frac{1}{\Delta x} \int_{x_w}^{x_e} \phi(x, t) dx$$

and Numerical flux function can be written as

$$\bar{\mathcal{F}}_\phi(x_e, t) = \bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_e, t) dt$$

$$\bar{\mathcal{F}}_\phi(x_w, t) = \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{F}_\phi(x_w, t) dt$$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{\Delta x} [\bar{\mathcal{F}}_\phi(\phi_P^n, \phi_E^n) - \bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n)]$$

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Riemann problem, this is conservative form. We already know that we have P cell, W cell, E cell. Now at each cell there will be different cell averaged values. So Riemann problem or Riemann problem essentially comes at the interface where, let us say we are considering this east face. If x is on the right side. So this is phi En and the left side we have phi Pn. If it is on the left side it is phi Pn. So essentially there is this continuity at this level. So this problem is called as Riemann problem.

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Riemann Problem Conservative Form

Riemann Problem

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

$$\phi(x, t) = \begin{cases} \phi_P^n & \text{if } x < x_e \\ \phi_E^n & \text{if } x > x_e \end{cases}$$

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Now in this case first hand approximation of the flux function can be averaged flux values from both the cells. So at east face we can consider that phi P for flux and phi E for flux and we can take average. And westside also we are taking average of flux values of W and P.

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Unstable Flux

Numerical flux can be calculated by taking arithmetic average of cell centred values

$$\bar{F}_\phi(x_e, t) = \bar{F}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{2} (\mathcal{F}_{\phi_P}^n + \mathcal{F}_{\phi_E}^n)$$

$$\bar{F}_\phi(x_w, t) = \bar{F}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{2} [\mathcal{F}_{\phi_W}^n + \mathcal{F}_{\phi_P}^n]$$

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Now this is first hand approximation. And if we consider our numerical discretization then only flux value for the east face and west face will be required for this calculation, and 2 is coming here.

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Unstable Flux

Numerical flux can be calculated by taking arithmetic average of cell centred values

$$\bar{F}_\phi(x_e, t) = \bar{F}_\phi(\phi_P^n, \phi_E^n) = \frac{1}{2} [\mathcal{F}_{\phi_P}^n + \mathcal{F}_{\phi_E}^n]$$

$$\bar{F}_\phi(x_w, t) = \bar{F}_\phi(\phi_W^n, \phi_P^n) = \frac{1}{2} [\mathcal{F}_{\phi_W}^n + \mathcal{F}_{\phi_P}^n]$$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \phi_P^n - \frac{\Delta t}{2\Delta x} (\mathcal{F}_{\phi_E}^n - \mathcal{F}_{\phi_W}^n)$$

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Now let us consider the simplest form of the flux where A is constant, phi is our general scalar variable. Then we can write our final form of the discretization using finite volume technique as a delta t, 2 delta x, phi_E minus W. We can also represent it in terms of ij or I. I plus 1, I minus 1. If we have P cell then at the center we can consider it as I. On the left side it is I minus 1, on the right side we have I plus 1. So for east cell this is I plus 1 and west cell we have I minus 1. And A delta t by 2 delta x, this is coming as constant.

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Unstable Flux

Let us consider that the flux term can be written as,

$$\mathcal{F}_\phi = a\phi$$

where a is constant.

$$\phi_P^{n+1} = \phi_P^n - a \frac{\Delta t}{2\Delta x} (\phi_E^n - \phi_W^n)$$

or,

$$\phi_i^{n+1} = \phi_i^n - a \frac{\Delta t}{2\Delta x} (\phi_{i+1}^n - \phi_{i-1}^n)$$

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Now we need to see whether this discretization is stable or not. Like our finite difference technique, we can use our error equation. Error equation, we are writing it in terms of epsilon in plus 1, var epsilon. And this is var epsilon i n and this term is constant.

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Unstable Flux

The error equation can be written as

$$\epsilon_i^{n+1} = \epsilon_i^n - a \frac{\Delta t}{2\Delta x} (\epsilon_{i+1}^n - \epsilon_{i-1}^n)$$

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Now for this one we can define our terms as $A \Delta t$ by Δx . So this is C_r by 2. And as per our convention, we can directly write as, e to the power square root of minus 1 varphi x . Actually we have seen that this epsilon i n, this is A^n and e to the power minus 1 i varphi x , in this case. So for var phi this var epsilon i plus 1, this will be n , square root minus 1 phi plus 1 phi minus n . So we can just simplify this one and get 1 minus square root of minus 1 C_r var phi x .

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Unstable Flux

The error equation can be written as


$$\varepsilon_i^{n+1} = \varepsilon_i^n - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

With $Cr = \frac{a\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = 1 - \frac{Cr}{2} (e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= 1 - \sqrt{-1}Cr \sin \varphi_x$$

$\xi_i^n = A^n e^{\sqrt{-1}i\varphi_x}$
 $\xi_{i+1}^n = A^n e^{\sqrt{-1}(i+1)\varphi_x}$
 $\xi_{i-1}^n = A^n e^{\sqrt{-1}(i-1)\varphi_x}$



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And in this case we can write our G or growth factor which is G into G star which is conjugate complex number. So this is minus, this will be plus. So always 1 plus Cr square sin square var phix. In this case it is clear that 1 plus some quantity. So Cr is always positive because square term is there, sin square is again positive. So 1 plus some positive term will be always greater than 1. So this scheme is unstable scheme.

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Unstable Flux

The error equation can be written as

$$\varepsilon_i^{n+1} = \varepsilon_i^n - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$


With $Cr = \frac{a\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = 1 - \frac{Cr}{2} (e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= 1 - \sqrt{-1}Cr \sin \varphi_x$$

$\xi_i^n = A^n e^{\sqrt{-1}i\varphi_x}$
 $\xi_{i+1}^n = A^n e^{\sqrt{-1}(i+1)\varphi_x}$
 $\xi_{i-1}^n = A^n e^{\sqrt{-1}(i-1)\varphi_x}$

$$|G|^2 = G.G^* = (1 - \sqrt{-1}Cr \sin \varphi_x).(1 + \sqrt{-1}Cr \sin \varphi_x)$$

$$= 1 + Cr^2 \sin^2 \varphi_x > 1$$


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So this approximation or firsthand approximation is not appropriate. This is unstable scheme. So Lax Friedrichs scheme. In that one this same type of flux, average flux is used. But one extra term is added here which is difference between east and our central P cell. And west face, this is P minus W.

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Lax-Friedrichs Scheme

Numerical flux can be calculated by taking arithmetic average of cell centred values (LeVeque, 2002):

$$\bar{\mathcal{F}}_{\phi}(x_e, t) = \bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = \frac{1}{2} [\mathcal{F}_{\phi_P}^n + \mathcal{F}_{\phi_E}^n] - \frac{\Delta x}{2\Delta t} (\phi_E^n - \phi_P^n)$$

$$\bar{\mathcal{F}}_{\phi}(x_w, t) = \bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = \frac{1}{2} [\mathcal{F}_{\phi_W}^n + \mathcal{F}_{\phi_P}^n] - \frac{\Delta x}{2\Delta t} (\phi_P^n - \phi_W^n)$$

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So final form is averaged value of W and E instead of phi phi Pn. In previous case we have seen that our discretization was with phi Pn value. But in this case we are taking average of W and E.

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Lax-Friedrichs Scheme

Numerical flux can be calculated by taking arithmetic average of cell centred values (LeVeque, 2002):

$$\bar{\mathcal{F}}_{\phi}(x_e, t) = \bar{\mathcal{F}}_{\phi}(\phi_P^n, \phi_E^n) = \frac{1}{2} [\mathcal{F}_{\phi_P}^n + \mathcal{F}_{\phi_E}^n] - \frac{\Delta x}{2\Delta t} (\phi_E^n - \phi_P^n)$$

$$\bar{\mathcal{F}}_{\phi}(x_w, t) = \bar{\mathcal{F}}_{\phi}(\phi_W^n, \phi_P^n) = \frac{1}{2} [\mathcal{F}_{\phi_W}^n + \mathcal{F}_{\phi_P}^n] - \frac{\Delta x}{2\Delta t} (\phi_P^n - \phi_W^n)$$

Final form of the discretization using finite volume method can be written as

$$\phi_P^{n+1} = \frac{1}{2} (\phi_W^n + \phi_E^n) - \frac{\Delta t}{2\Delta x} [\mathcal{F}_{\phi_E}^n - \mathcal{F}_{\phi_W}^n]$$

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Now with this again we can write it in i, i minus 1, i plus 1 format. If we see our actual equation then this is our actual equation. But virtually what we are doing, we are adding some extra second order diffusion term here, in the Lax Friedrichs scheme for stabilization.

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Lax-Friedrichs Scheme

Numerical Diffusion

Actual Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

Modified Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = \beta \frac{\partial^2 \phi}{\partial x^2}$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$

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So in this case, this diffusing flux, we can see that this is some beta into phi P minus phi W, for west face. So this extra term is added for stabilization of the scheme which is equivalent to adding one extra term on the right hand side.

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Lax-Friedrichs Scheme

Numerical Diffusion

Actual Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = 0$$

Modified Equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \mathcal{F}_\phi}{\partial x} = \beta \frac{\partial^2 \phi}{\partial x^2}$$

where $\beta = \frac{\Delta x^2}{2\Delta t}$

$$\bar{\mathcal{F}}_\phi(\phi_W^n, \phi_P^n)|_D = -\beta \frac{\phi_P^n - \phi_W^n}{\Delta x}$$

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So if we check the stability of the scheme, again this is averaged value. So if we check our stability for this case, the Cr is defined with A delta t by delta x. We can get this cos var phi x minus square root of minus, 1 Cr sin var phi x.

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Lax-Friedrichs Scheme

The error equation can be written as

$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

With $Cr = \frac{a\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{1}{2}(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}) - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= \cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x$$

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And if we write our growth factor again that is G into G star which is conjugate complex number. So in this case we are getting cos square var phi plus Cr square plus sin square var phi. So we can rewrite this 1 minus sin square var phi x plus Cr square sin square var phi x. And if we rewrite this, we can write it like this. And in this case scheme is stable if Cr is less than 1.

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Lax-Friedrichs Scheme

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$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

With $Cr = \frac{a\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{1}{2}(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}) - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= \cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x$$

$$|G|^2 = G.G^* = (\cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x) \cdot (\cos \varphi_x + \sqrt{-1}Cr \sin \varphi_x)$$

$$= \cos^2 \varphi_x + Cr^2 \sin^2 \varphi_x = 1 - \sin^2 \varphi_x + Cr^2 \sin^2 \varphi_x$$

$$= 1 - (1 - Cr^2) \sin^2 \varphi_x$$

The scheme is stable if $Cr < 1$.

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If in this particular case, if we have 1 minus some quantity. If Cr is greater than 1 then this quantity will be negative. And negative-negative this will be positive. So 1 plus some quantity. In extreme case if we consider that sin var phi square, this is equal to 1.

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Lax-Friedrichs Scheme

The error equation can be written as

$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

With $Cr = \frac{\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{1}{2}(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}) - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= \cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x$$

$$|G|^2 = G \cdot G^* = (\cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x) \cdot (\cos \varphi_x + \sqrt{-1}Cr \sin \varphi_x)$$

$$= \cos^2 \varphi_x + Cr^2 \sin^2 \varphi_x$$

$$= 1 - (1 - Cr^2) \sin^2 \varphi_x$$

The scheme is **stable** if $Cr < 1$.

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This is extreme case that $\sin \phi_x$, this equals to 1. So this comes as 1 minus Cr square. This means that this is Cr square. So obviously this should be less than 1 for that stable scheme. Another extreme is $\sin \phi_x$ square. This term is zero. So then obviously this is (neu) neutrally stable.

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Lax-Friedrichs Scheme

The error equation can be written as

$$\varepsilon_i^{n+1} = \frac{1}{2}(\varepsilon_{i-1}^n + \varepsilon_{i+1}^n) - a \frac{\Delta t}{2\Delta x} (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)$$

With $Cr = \frac{\Delta t}{\Delta x}$

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{1}{2}(e^{\sqrt{-1}\varphi_x} + e^{-\sqrt{-1}\varphi_x}) - \frac{Cr}{2}(e^{\sqrt{-1}\varphi_x} - e^{-\sqrt{-1}\varphi_x})$$

$$= \cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x$$

$$|G|^2 = G \cdot G^* = (\cos \varphi_x - \sqrt{-1}Cr \sin \varphi_x) \cdot (\cos \varphi_x + \sqrt{-1}Cr \sin \varphi_x)$$

$$= \cos^2 \varphi_x + Cr^2 \sin^2 \varphi_x$$

$$= 1 - (1 - Cr^2) \sin^2 \varphi_x$$

The scheme is **stable** if $Cr < 1$.

$\sin^2 \varphi_x = 1$
 $= 1 - (1 - Cr^2)$
 $= Cr^2 < 1$

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So we can see that depending on the approximate form of the flux or numerical flux function, we are getting stable or unstable scheme. In the next lecture we will see another kind of approximation for this flux function. Thank you.