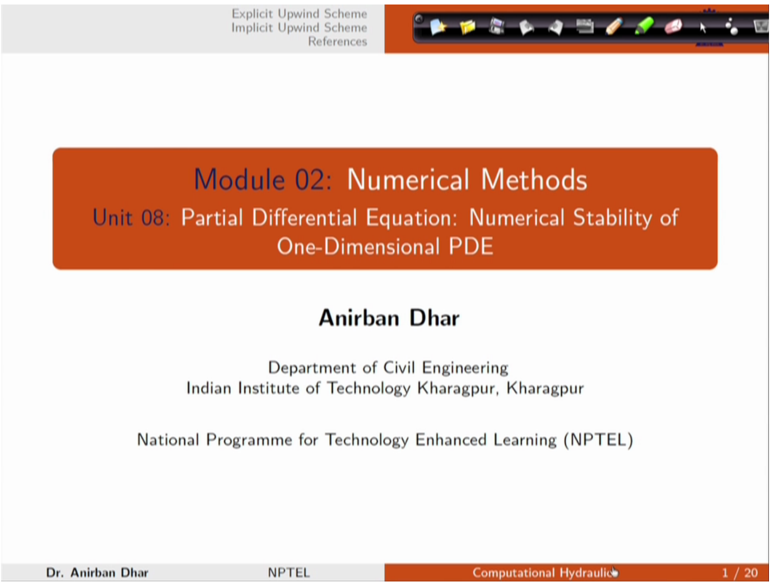
Computational Hydraulics
ProfessorAnirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 12
Partial Differential Equation - Numeric Stability of One Dimensional PDE

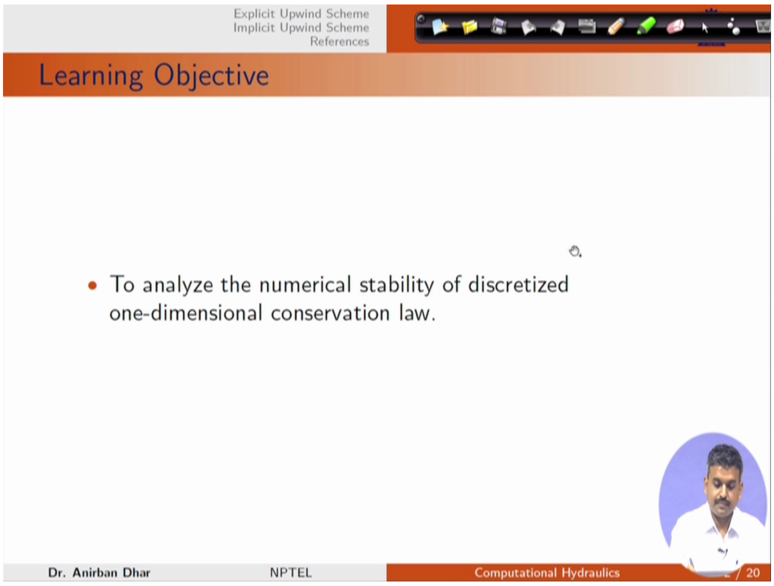
Welcome to this lecture number 12 of the course computational hydraulics. We are in module 2, numerical methods. And in today's lecture we will becovering unit 8, partial differential equation, numerical stability of one dimensional PDE.

(Refer Slide Time 00:38)



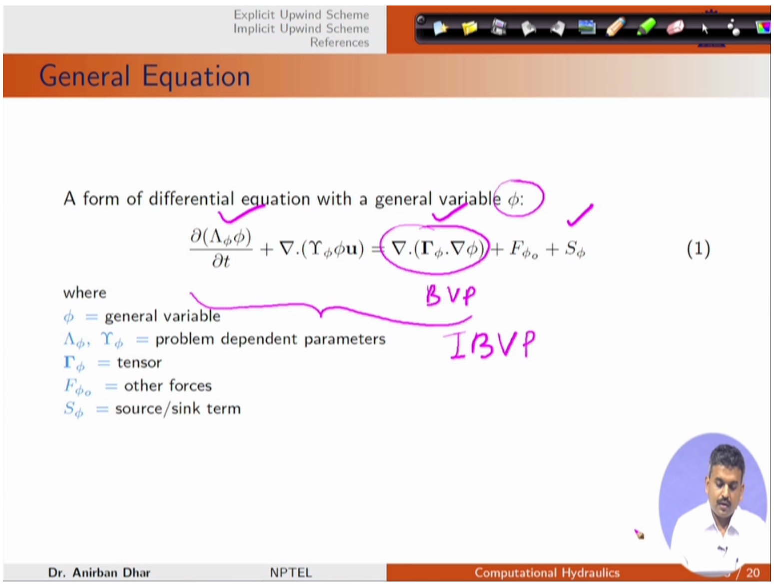
What is the learning objective for this particular unit? At the end of this unit students will be able to analyze the numerical stability of discretized one-dimensional conservation law in terms of partial differential equation.

(Refer Slide Time 01:02)



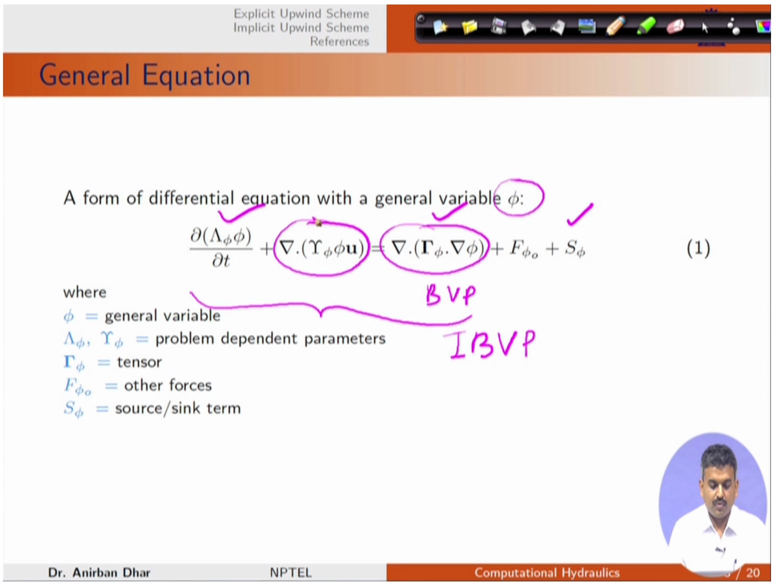
Let us consider our general equationwith general variable phi. We have seen thatinitially with thisfirst term in the right hand side we have solved boundary value problem. With our temporal term and boundary value problem or right hand term, this one and this one we have solved initial boundary value problem. And in these boundary value problem or initial boundary value problems, we have considered Sphi or source sink term.

(Refer Slide Time 02:02)



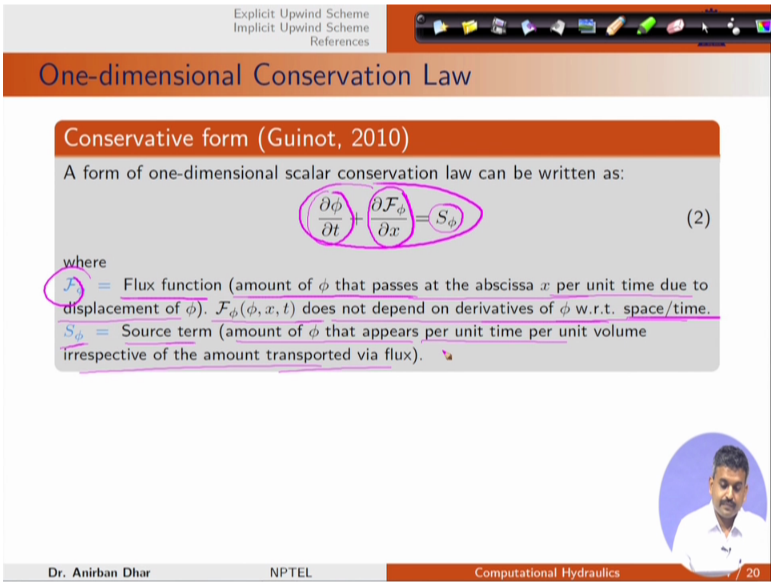
Now we need to consider the second term in the left hand side. This is somewhat related to velocity term. And we should analyze this term before starting our finite volume approach.

(Refer Slide Time 02:24)



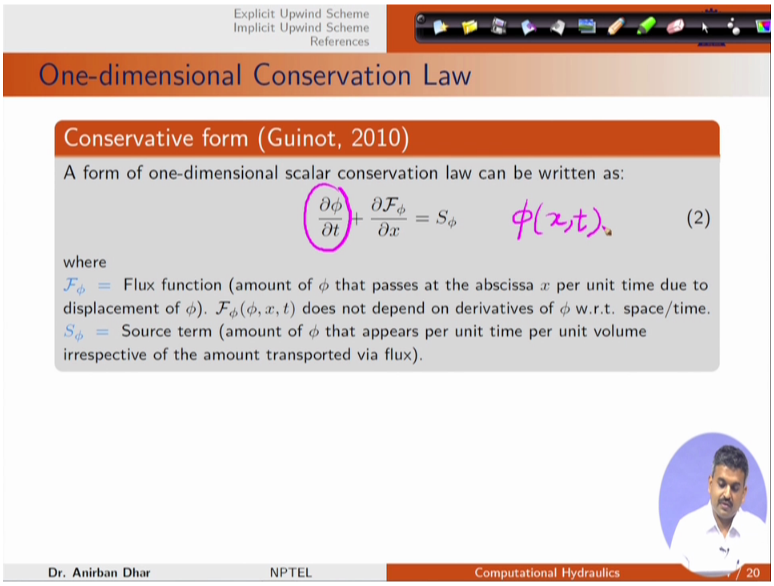
So in this case, one-dimensional conservation law in terms of phi can be written as del phi by del t and del F phidel x Sphi. S phi is source sink term. This is temporal term and this is spatial term only in one dimension that is x. So what is this F phi? Fphi is the flux function. Amount of phi that passes at the abscissa x per unit time due to displacement of phi. And Fphi does not depend on derivative of phi with respect to space or time. And S phi is source sinkterm. This is amount of phi that appears per unit volume irrespective of the amount transported viaflux.

(Refer Slide Time 03:36)



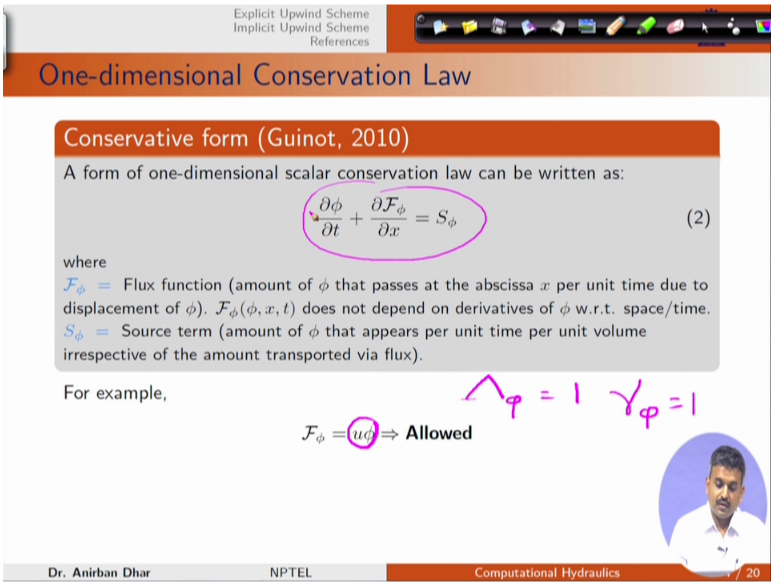
So with thisinformation we can proceed. We can see that we have phi which is function of x and t only. So one-dimensional in space and we have one time dimension.

(Refer Slide Time 04:00)



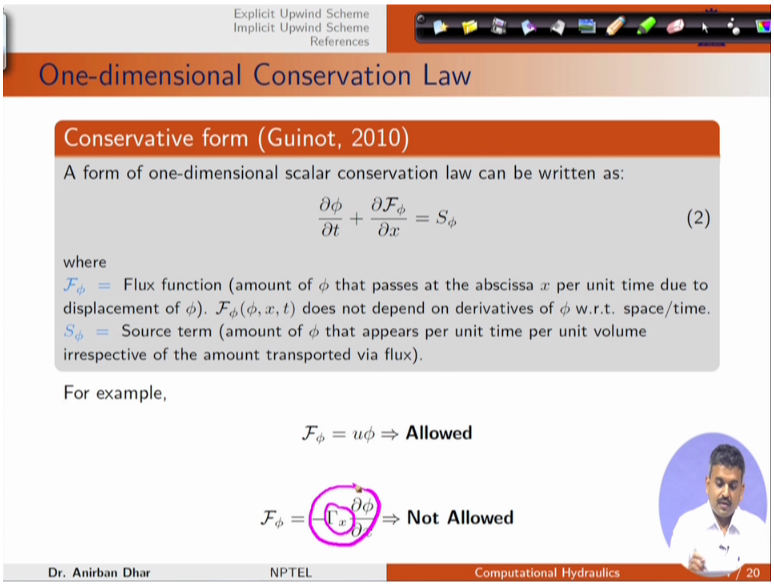
So for example, in this case phi can be u phi. If phi is equal to u phi and lambda phi equals to 1 and our upsilonphi is equal to 1, then we can get this equation without the right hand first term and other force term on the right hand side.

(Refer Slide Time 04:34)



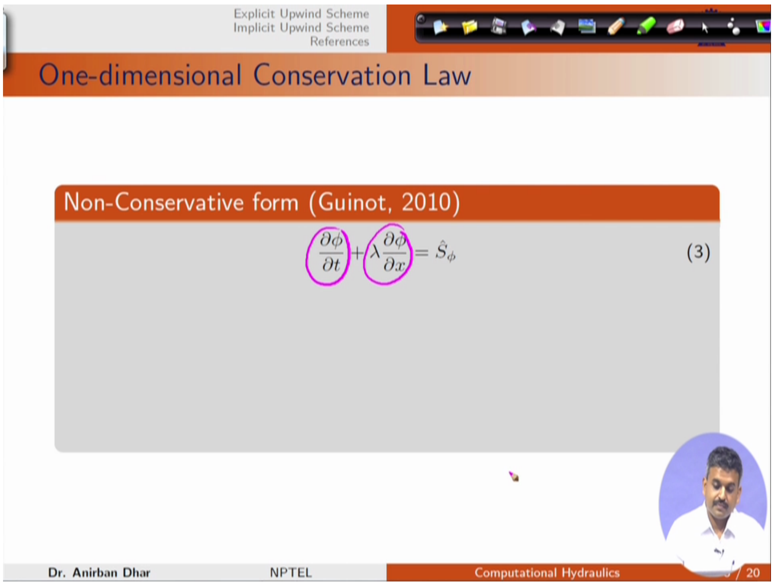
However any derivative in this Fphi is not allowed. So this is related to diffusion. So this gamma x is like diffusion coefficient. So this is not allowed for this kind ofequation because we have derivative of phi with respect to space or in one dimensional case.

(Refer Slide Time 04:57)



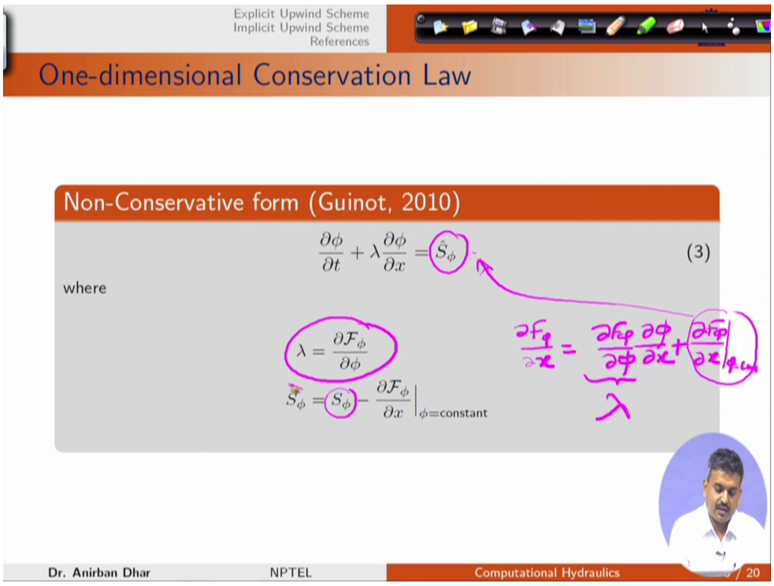
Sowe can write the non-conservative form where del phi by del t. And this is actually lambda into delphi by del x. This is written in terms of ourgeneral variable phi, not in terms of flux Fphi.

(Refer Slide Time 05:44)



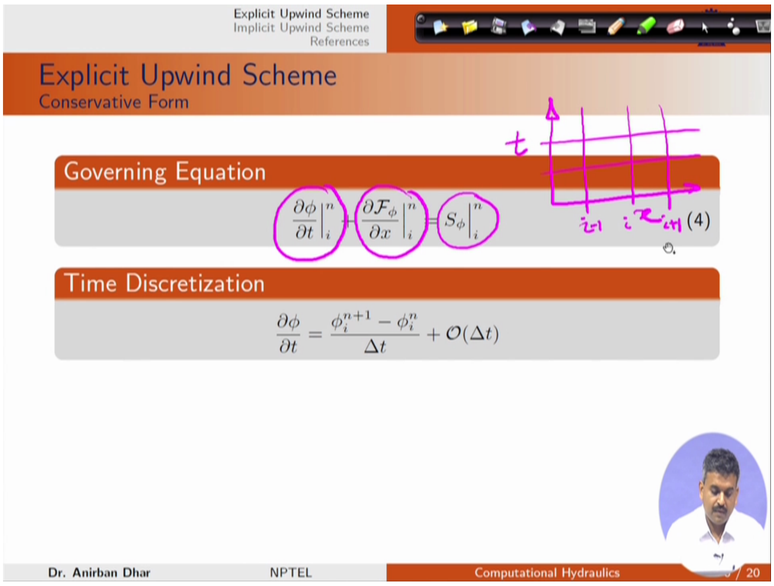
So what is this lambda? Lambda is equal to del Fby del phi. And we have this modified source sink term. This is Sphiminusdel phi. This is constant. Soactually we can write our Fphi del y del x as delphi. This phi into delphi by del x plus del F phi by del x where phi is constant value. So now this term is lambda and we can change this side for this term. And if we transfer this term on the right hand side and subtract from the original source sink term then we can get this modified one.

(Refer Slide Time 07:14)



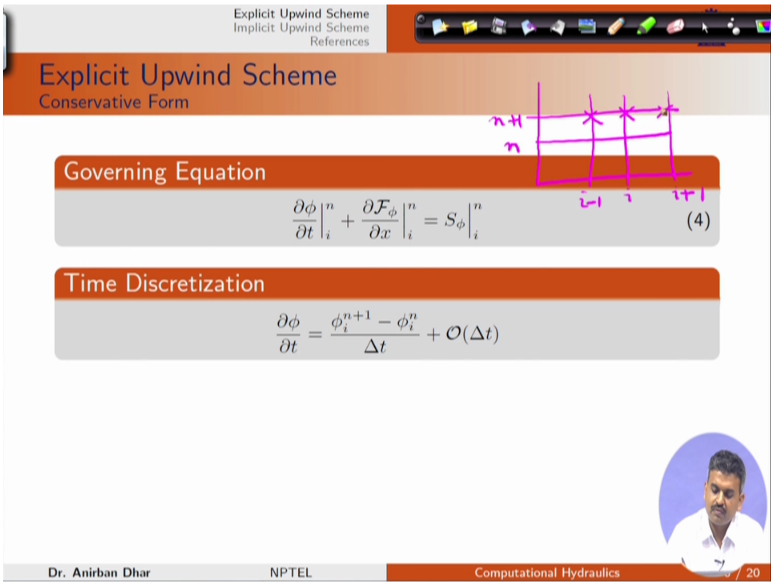
So let us utilize our explicit upwind scheme. What is this upwind? Depending on this value of lambda or sign of lambda, we generally change the scheme. If it is positive or negative we need to change our discretization in space. So we have our original conservative form equation. We are discretizing it at I n level. This is present time level. So if we consider in this case we have one space access and one time access other level for this one. In this case if we have I,Iminus1, I, Iplus 1.

(Refer Slide Time 08:36)



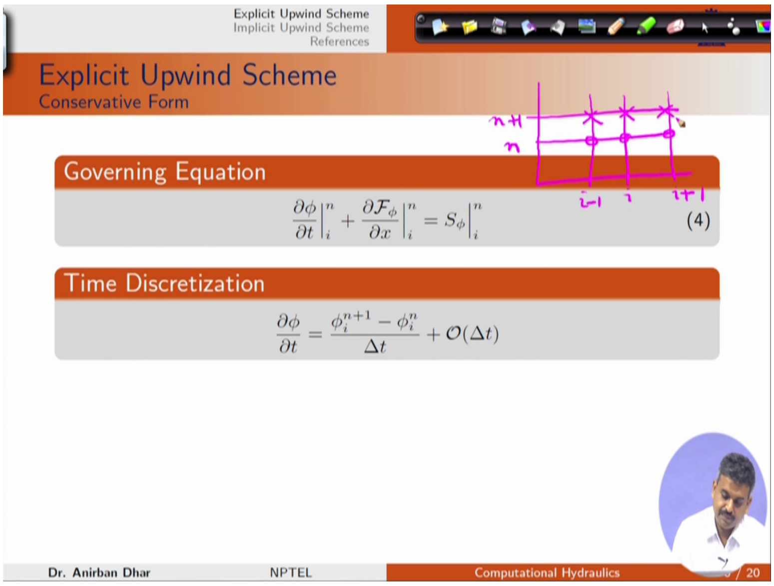
Then for timediscretization we have nth level n and n plus 1. For space discretization we have Iminus 1,I, Iplus 1. These values are actually unknown values.

(Refer Slide Time 09:01)



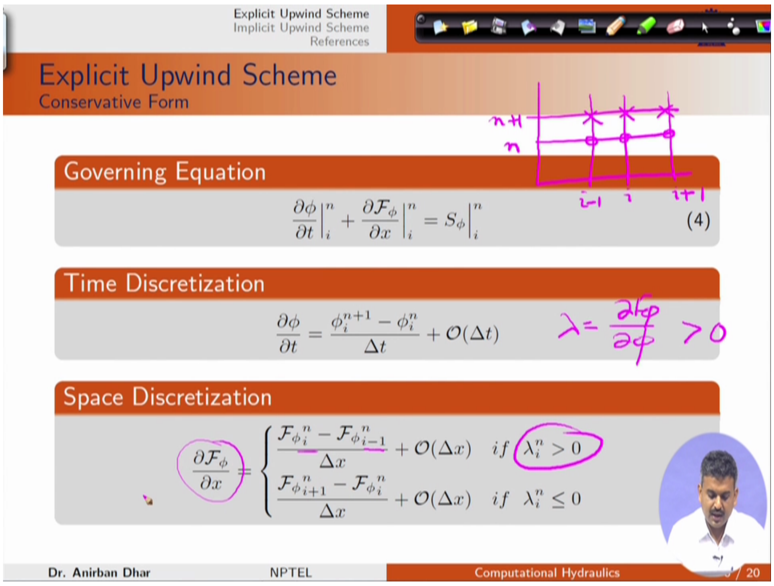
And these are our known values.

(Refer Slide Time 09:06)



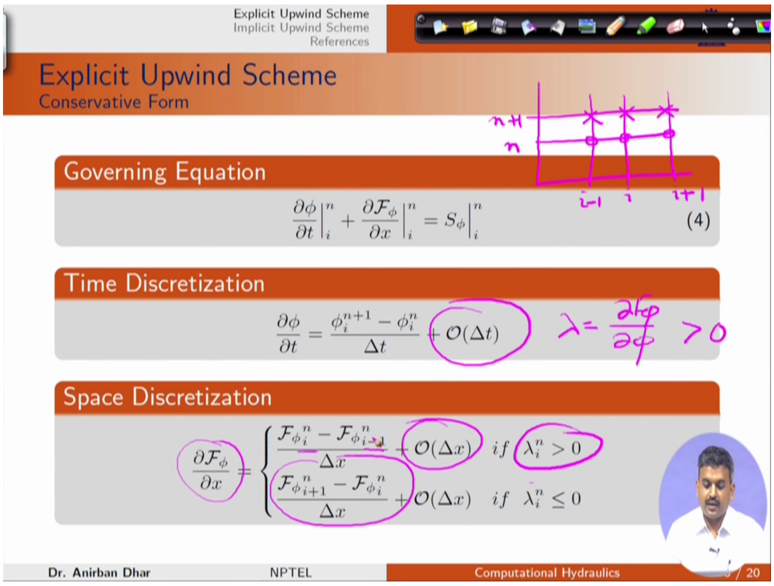
So with this information we can discretize our space derivative for flux. So interestingly in this case if lambda I n is positive, that means lambda equals to flux. So change in flux with respect tovariable phithat is positive, then we will be utilizing this derivative I, I minus 1 for delphi by del x.

(Refer Slide Time 09:52)



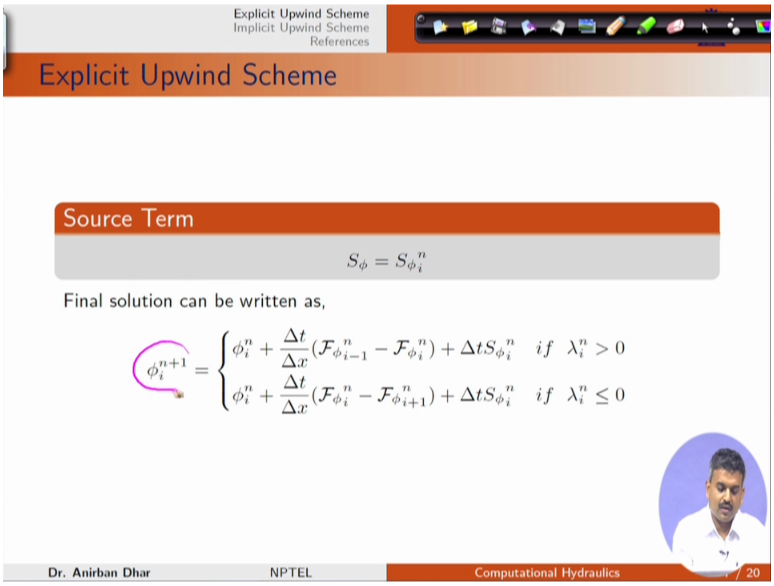
And if lambda is negative then we will be utilizing F Iplus 1 minus I value for this one. And in this case time discretization is of order delta t and space discretization is of order delta x. So overall accuracy for this one is delta x delta t.

(Refer Slide Time 10:25)



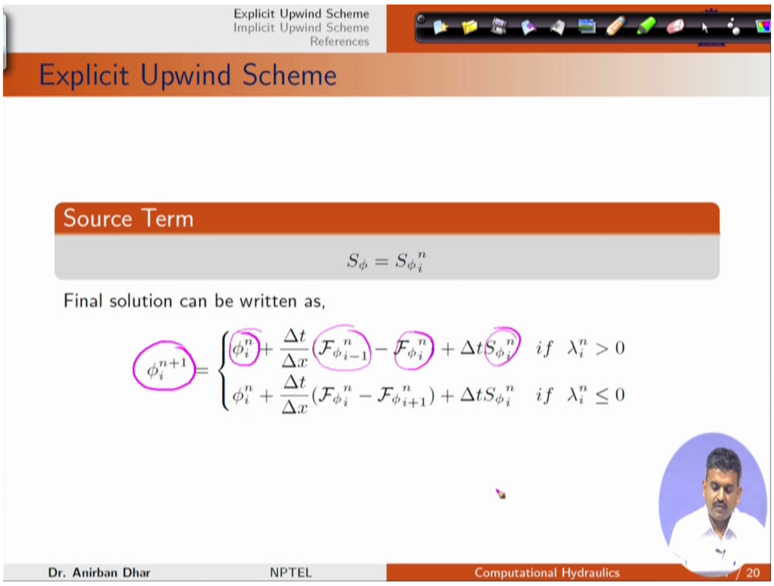
And for source sink term this is simple, we will be evaluating at I n level. Now we can write our final solution like this. If we change the sides for known variables on the right hand side. This is unknown level.

(Refer Slide Time 10:50)



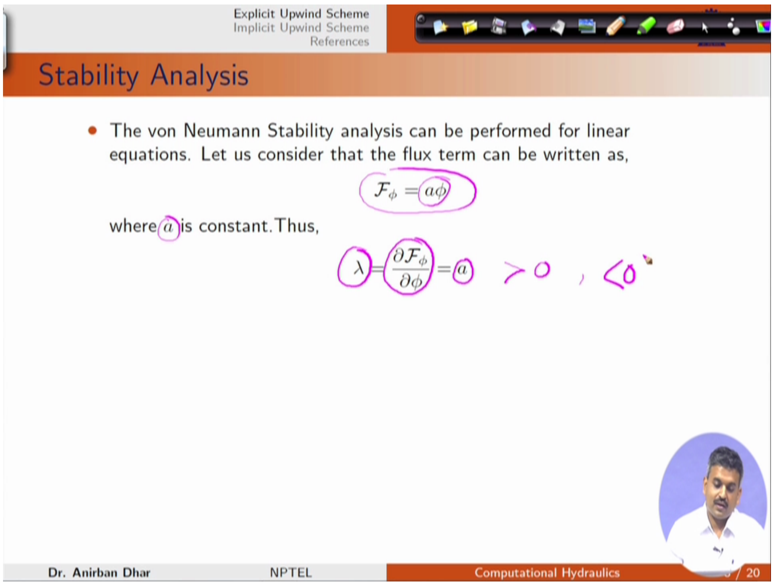
This is known level. This flux is also known. These values are known. This is also known. So we can write like this.

(Refer Slide Time 10:58)



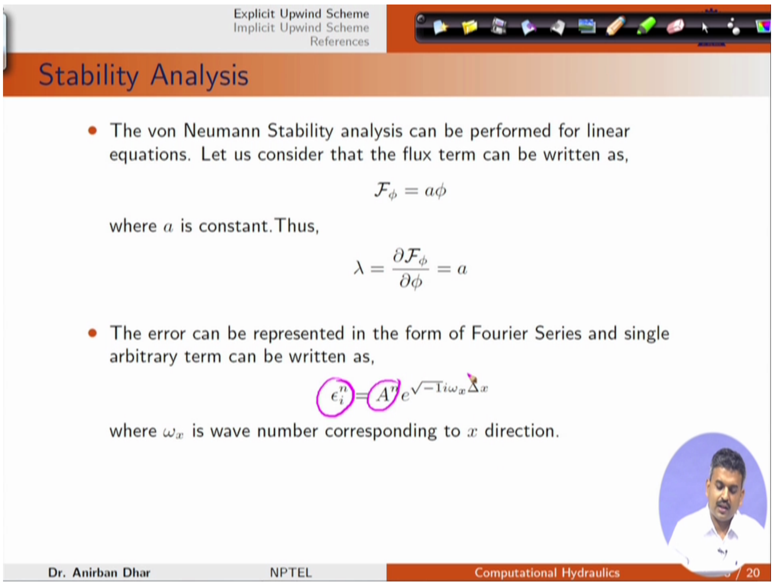
Now we need to analyze the stability of this particular scheme. But Von Neumann stability analysis can be performed for linear equations. Let us consider that the flux term can be written in terms of F phi where F phi equals to Aphi and A is some constant value. So lambda is delF phi by del phiwhich is A. Depending on the sign of the A, whether it is positive or negative, we can change the space discretization.

(Refer Slide Time 11:48)



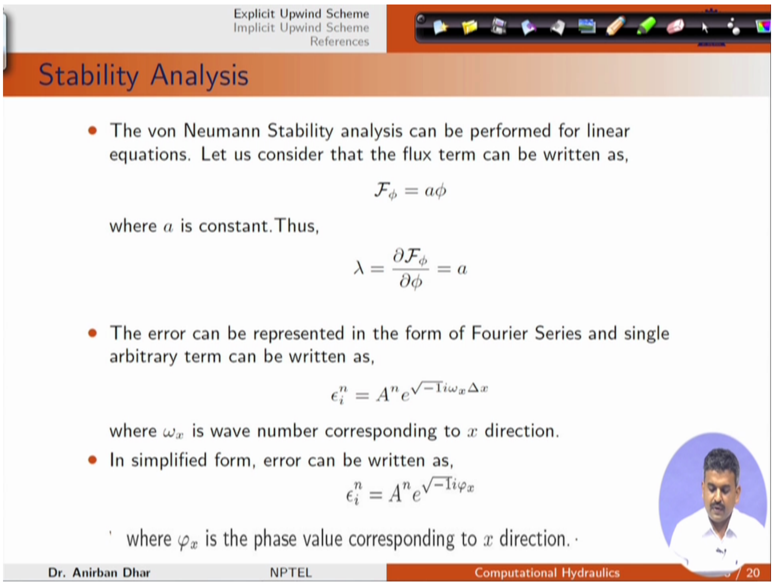
Now let us considerthat error equation again it can be error term. Epsilon I n can be written in the terms of amplitude and this face thing forx direction.

(Refer Slide Time 12:05)



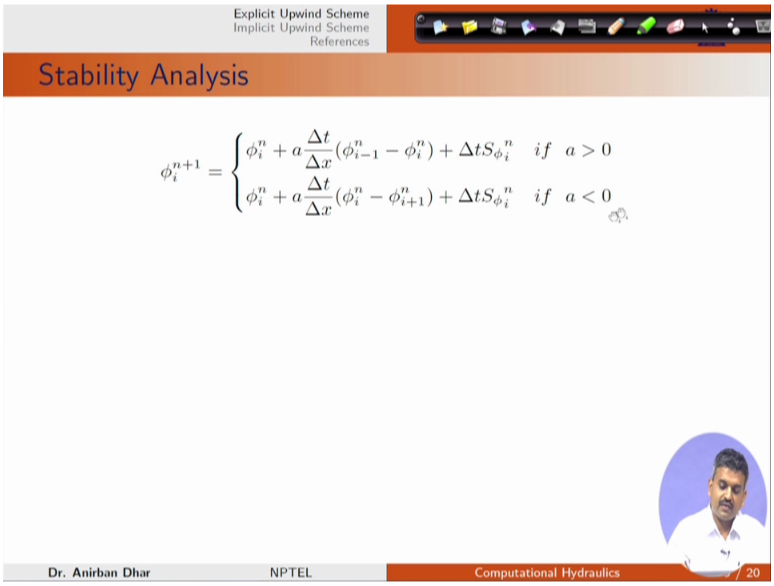
So we have this I n and A n I var phi x corresponding to that direction.

(Refer Slide Time 12:26)



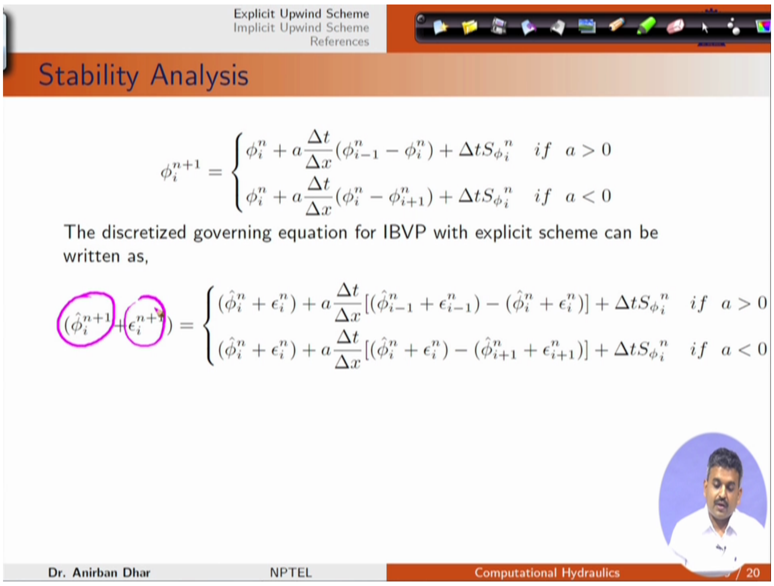
Now with this information we can proceed for stability analysis. Now we can bring that A term out of this bracket and we can write like this for A greater than zero and less than zero.

(Refer Slide Time 12:52)



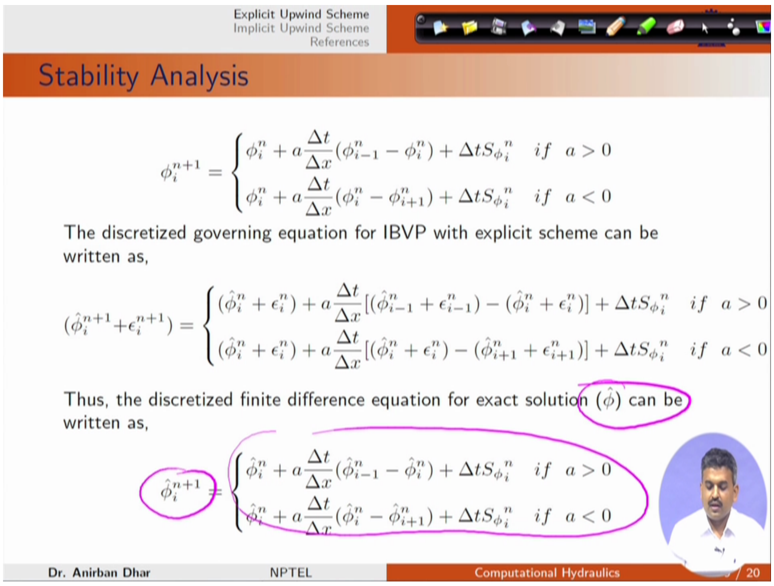
Now discretize governing equation for this initial boundary value problem. Butin this case boundary initial condition is a main thing because one side only it is defined. So it is like initial value problem. Now this is the solution which is coming from infinite precision computer and this is the amount of error involved in the particular step and particular discretization level.

(Refer Slide Time 13:41)



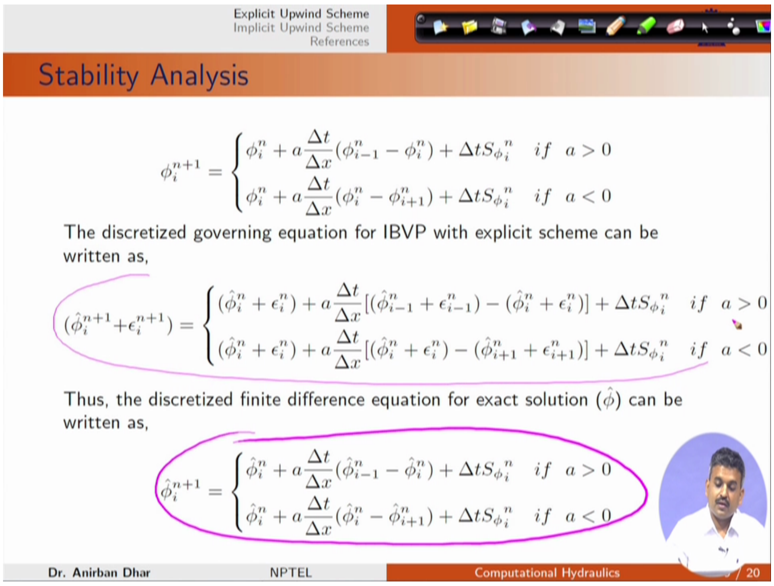
With this we can also write the discretized finite difference equation for exact solution. This is the exact solution think and exact solution should satisfy our original finite difference discretization. So we have written phi hat I n plus 1 and others are at known time level.

(Refer Slide Time 14:18)



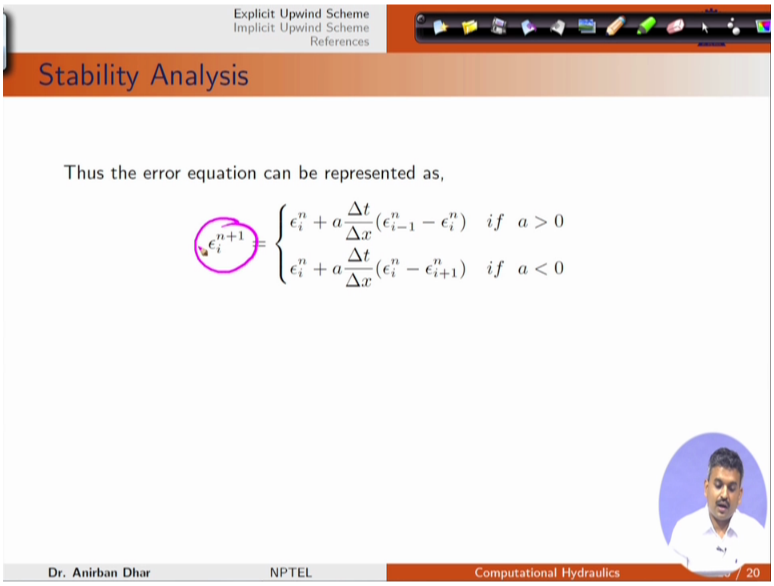
Now if we subtract our exact solution equation discretized form, from this one, we can get the corresponding error equation.

(Refer Slide Time 14:32)



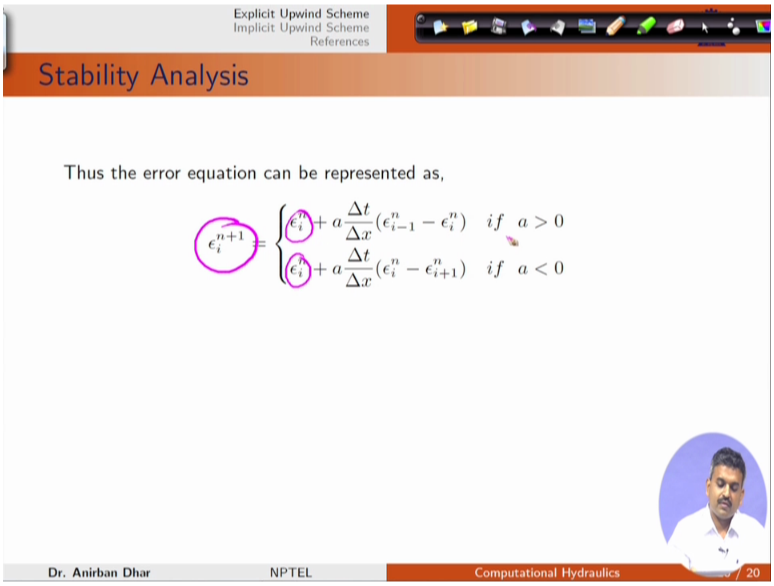
So error equation can be returned like this where this one is unknown.

(Refer Slide Time 14:43)



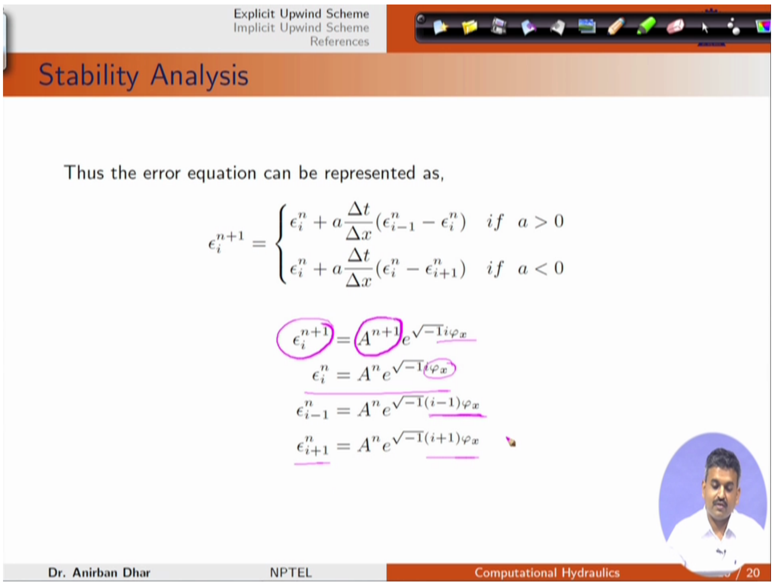
And these are known level things.

(Refer Slide Time 14:47)



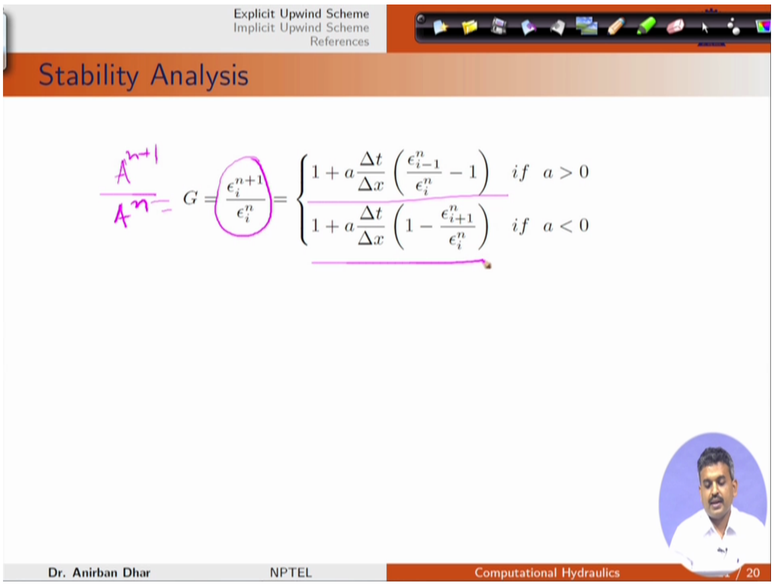
Now as per our definition we can write this epsilonI n plus 1 where only change in the amplitude, no change in this index for I. I n this is A and I varphi x minus 1, only change in index for I. I plus 1 only change in the index for I.

(Refer Slide Time 15:17)



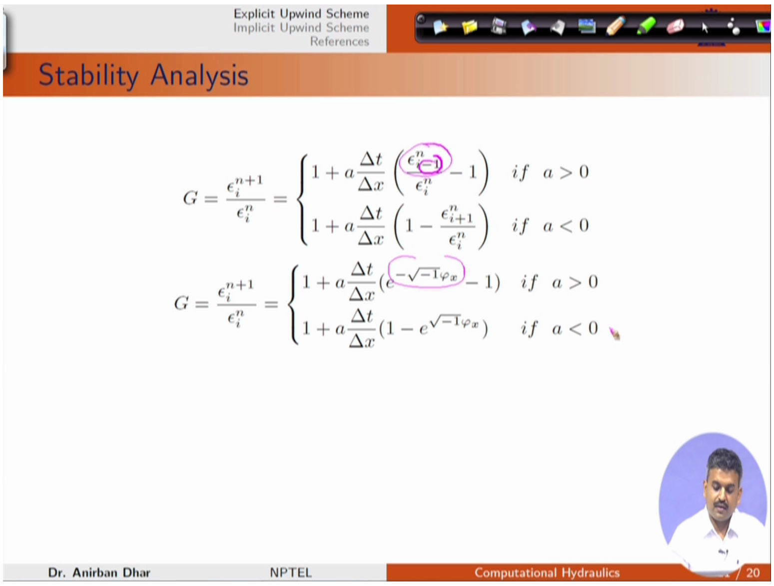
So with this expressions if we write our growth factor by dividing epsilon on both sides. This is epsilon I n plus 1 divided by epsilon I n and interestingly this is equal to A n plus 1 divided by A n. Now for A positive we have this form, A negative we have the second equation.

(Refer Slide Time 16:00)



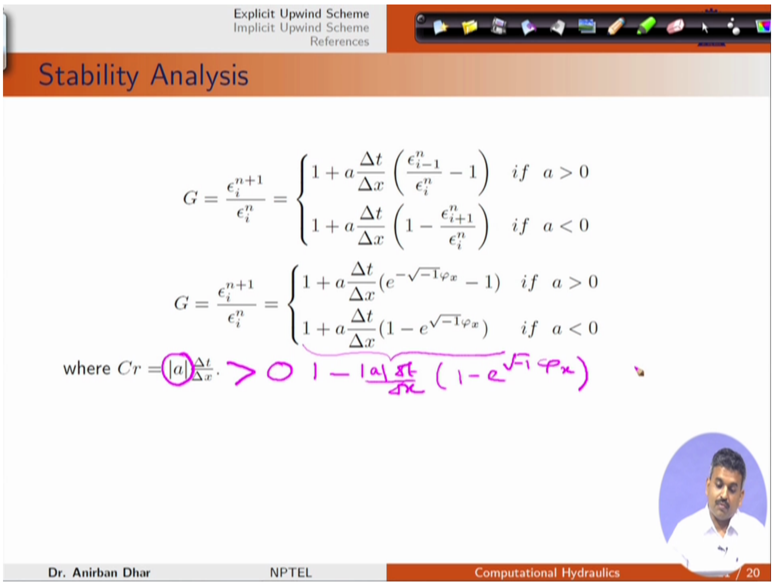
Now if you further simplify this one, we can write like this where this e to the power minus part is there. So minus I the minus this imaginary number into var phi x. And this is 1 plus 1 minus e to the power, imaginary number var phi x again for negative value.

(Refer Slide Time 16:35)



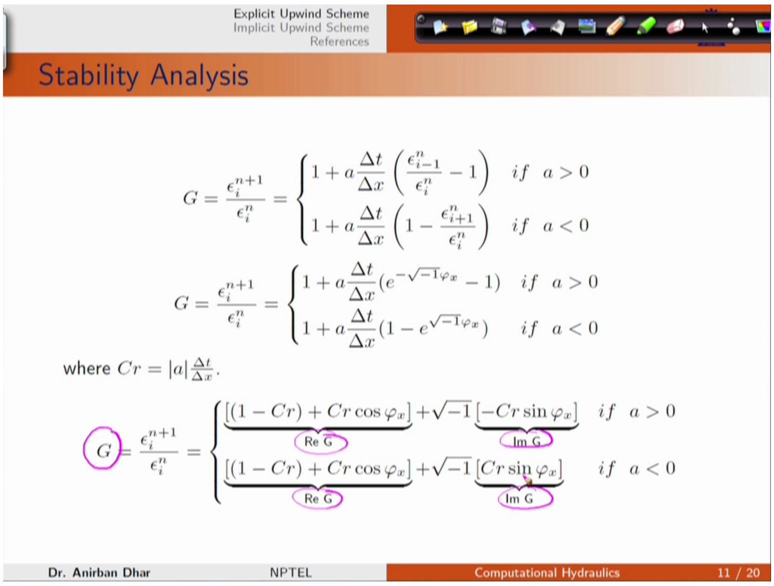
Now in this casewe can define a number which is Cr. And Cr is defined like this. Modulus of A into delta t divided by delta x. So always Cr is positive. Now the problem with this one is that in this case if A is negative we can write this equation as, 1 minus modulus of A into deltat divided by delta x 1 minus e to the power minus 1 var phi x. Only for negative value.

(Refer Slide Time 17:30)



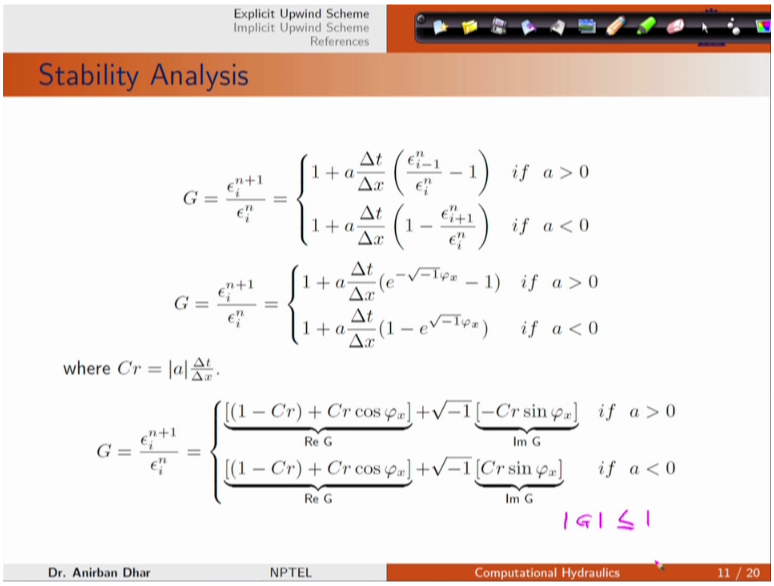
Now with this Cr we can proceed and we can define our amplification factor in terms of real and imaginary part of G in simplified form.

(Refer Slide Time 17:56)



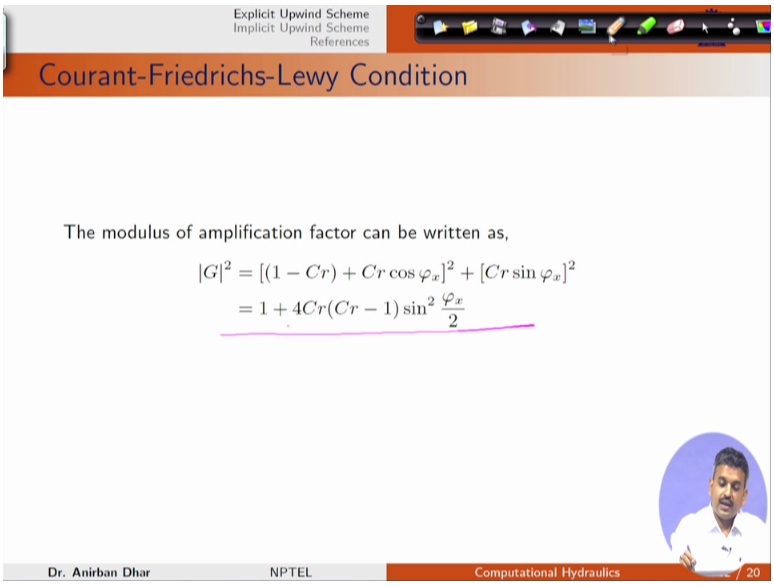
Now as per our Von Neumann stability criteria,this modulus of G should be less than equals to 1.

(Refer Slide Time 18:12)



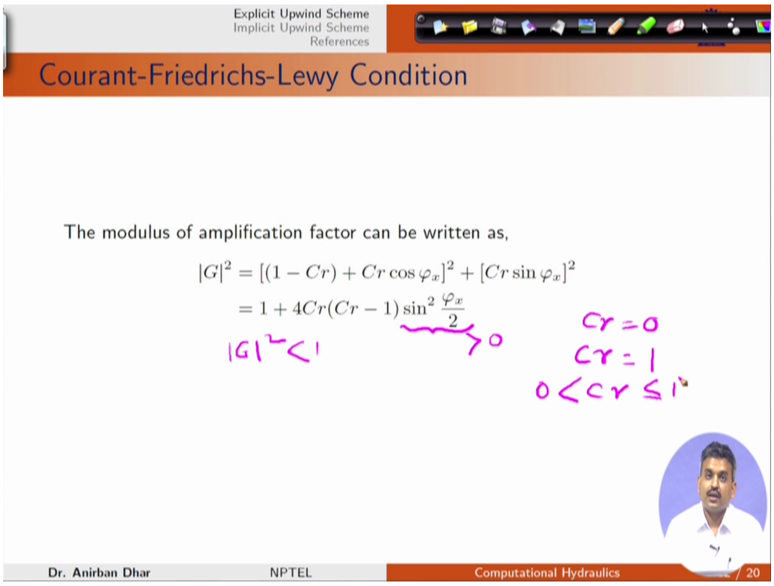
Now we need to evaluate the modulus of G. So if we evaluate the modulus of G in this case so basically we are taking square of real and imaginary part. Now this comes to this simplified form, 1 plus 4 Cr, Crminus 1 sin square phi x by 2.

(Refer Slide Time 18:52)



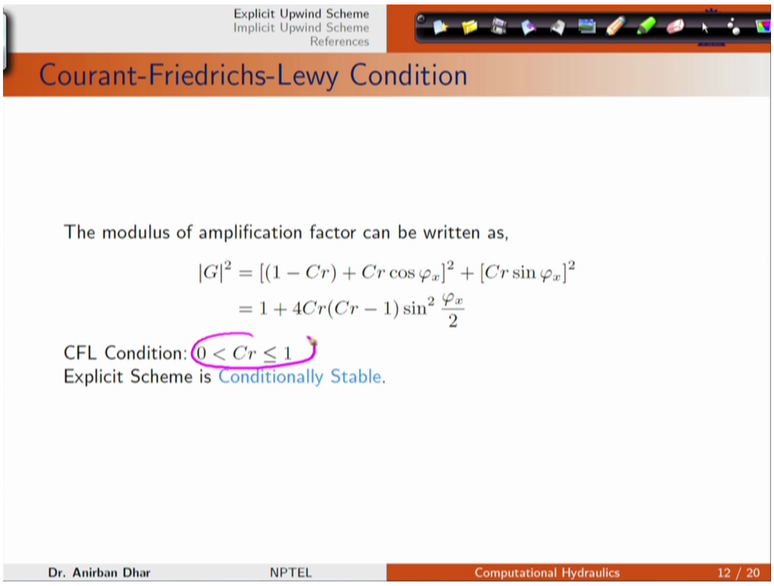
In this caseimportant point is that this quantity is always positive. Now as per our condition G Mod square should be less than equal to 1. If Cr equals to zero then we have 1. If Cr equals to 1, againthis is equals to 1. And for values less than1 and greater than zero. For Cr we can get G modulus square less than equal to 1. So we can say that for this condition where Cr is greater than zero, less than equal to 1. This scheme is stable. This is known as Courant Friedrich Lewy condition or CFL condition.

(Refer Slide Time 20:15)



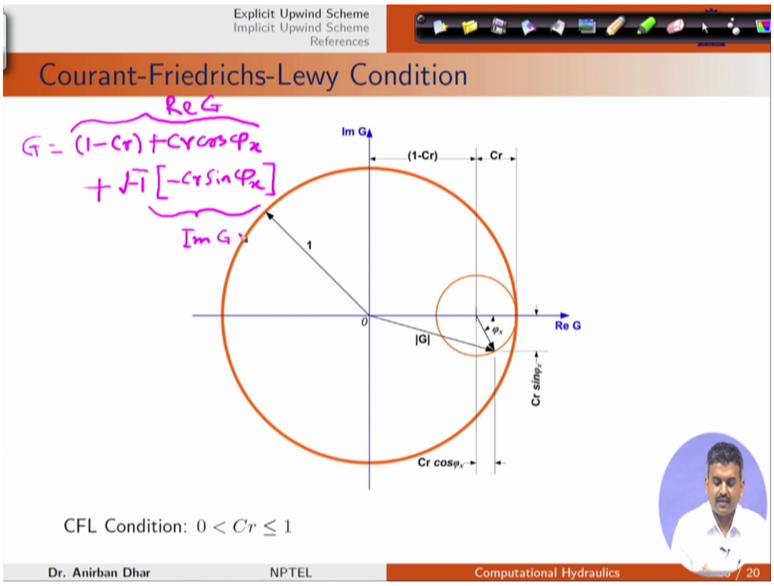
Thus we can say that explicit scheme is conditionally stable. And this is the actual condition.

(Refer Slide Time 20:33)



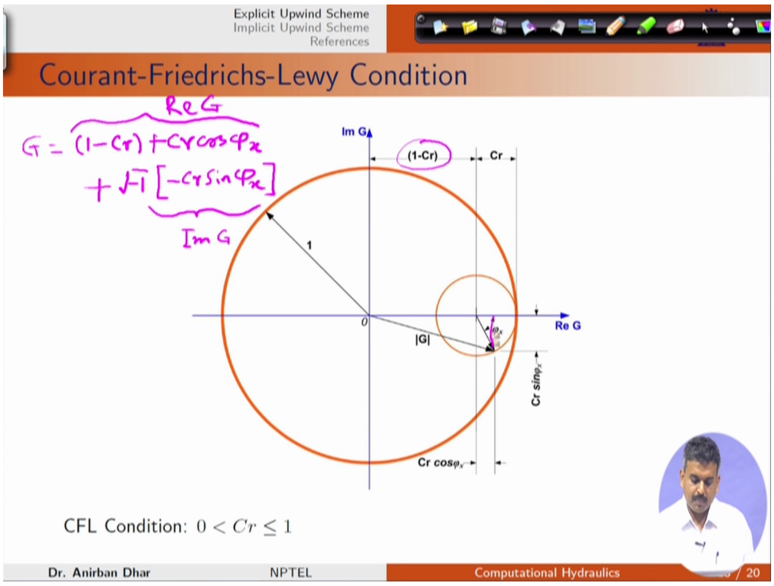
Now from geometrical point of view if we see, we can represent our G where G is equal to 1 minus Cr plus Cr cos var phi x plus square root minus 1 and within bracket minus Cr sin var phi x, this quantity. So this part is represented in the real axis or ReG. And this part is represented in the imaginary axis.

(Refer Slide Time 21:35)



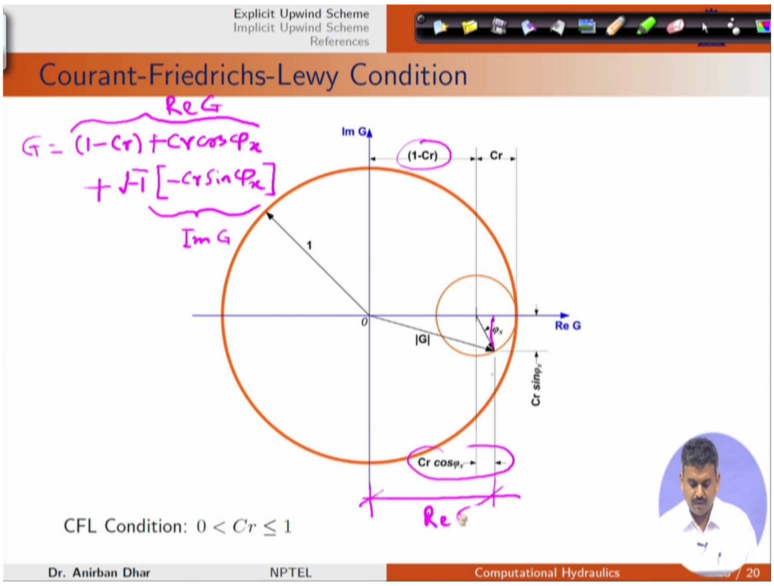
So this part is 1 minus Cr and this one is the projected one.

(Refer Slide Time 21:47)



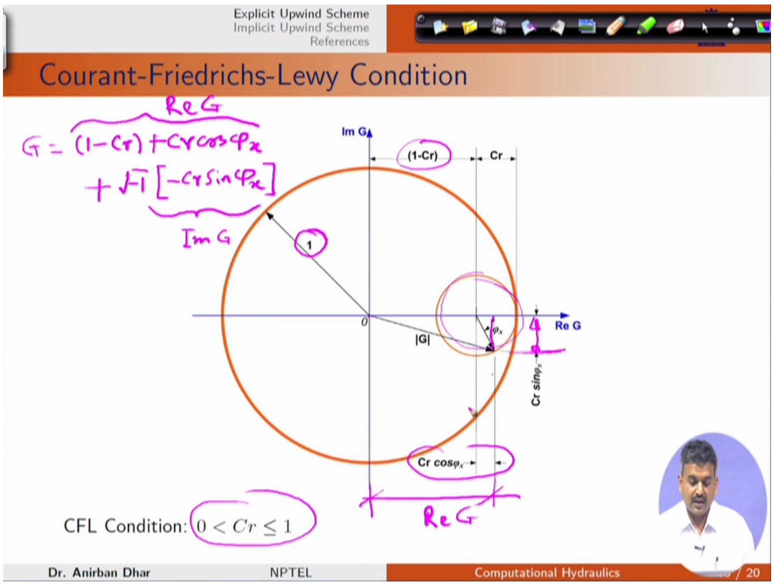
So this length is Cr into cos phi. So total length, this part is actually our real of G.

(Refer Slide Time 22:02)



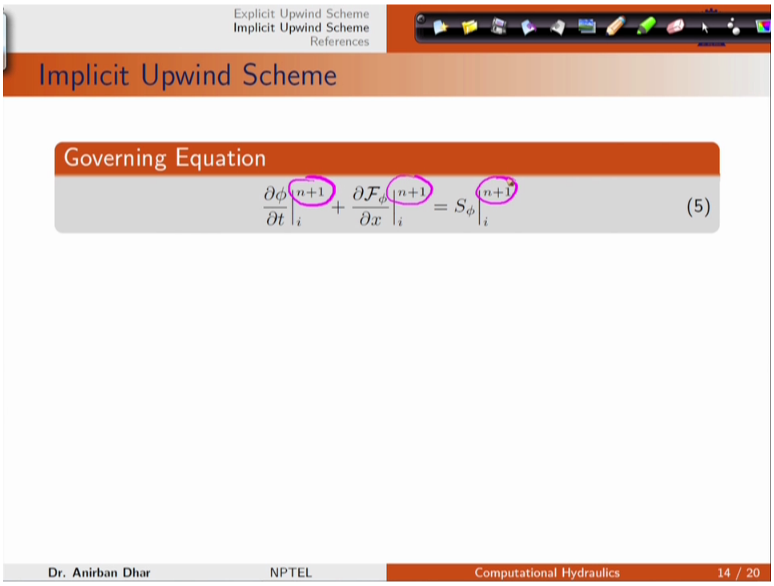
Now and in this case, this part is imaginary component of G. So this thing, this modulus of G should be within thiscircle with unit radius for stability. So we can say that in this case if Cr value is within 0 to 1, this condition will be satisfied. That means the circle will be within this bigger circle. So our stability criterion will be satisfied.

(Refer Slide Time 23:01)



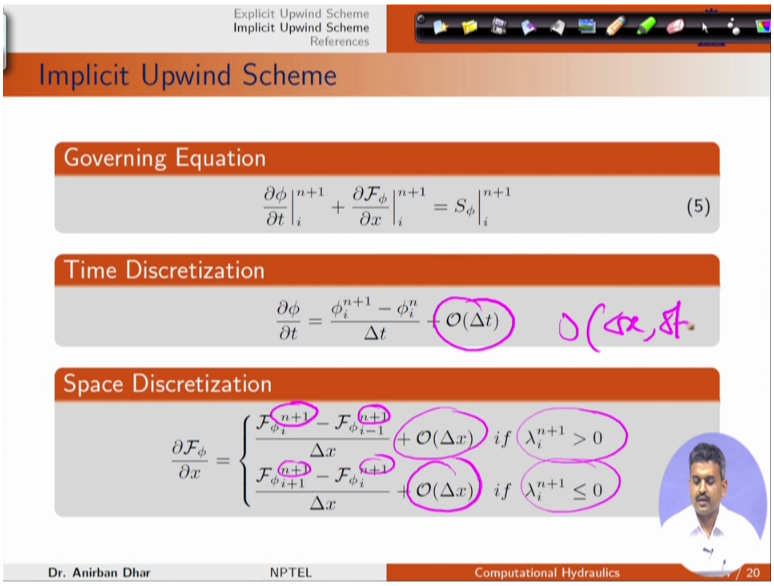
Now if we consider implicit upwind equation, we are discretizing it at n plus 1 level.

(Refer Slide Time 23:20)



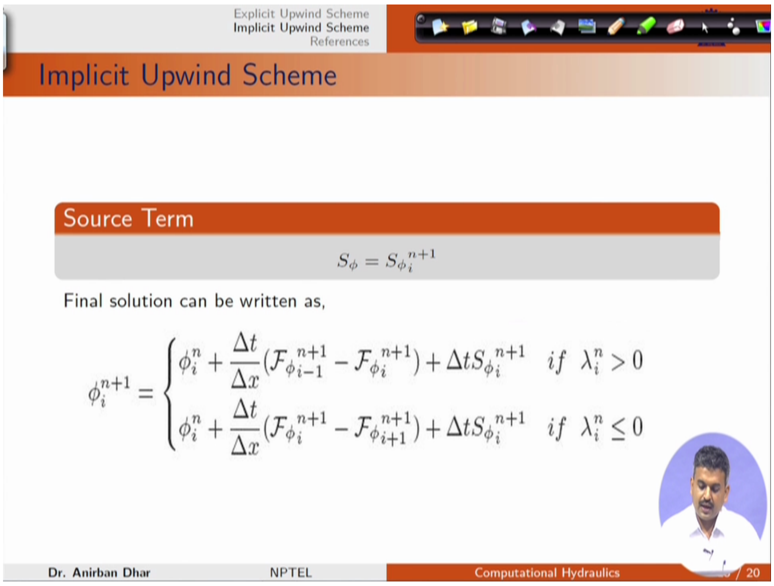
Nowif we consider our time discretization, this is our time discretization. We have truncation error of orderdel t. For space discretization again values are at nplus 1 level. For lambda I n plus 1 greater than zero, lambda I n plus 1 less than equal to zero. This is again of order delta x. Overall accuracy of the scheme is delta x delta t.

(Refer Slide Time 24:01)



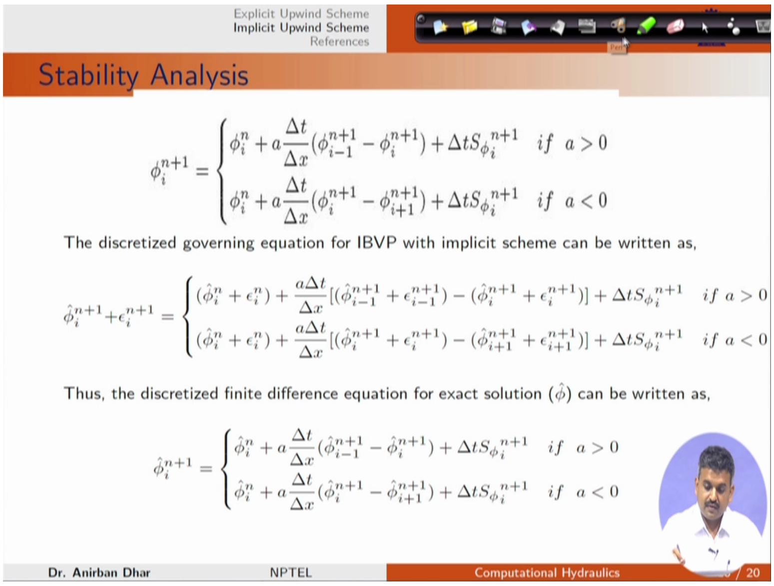
This is for source term and this source term is discretized at n plus 1 level. Now with this we can get the final solution which is in terms of values at nth level. And flux terms at nplus 1 level, source sink terms are source sink terms at n plus 1 level. This should be evaluated at nplus 1 level or at the nth level.

(Refer Slide Time 24:59)



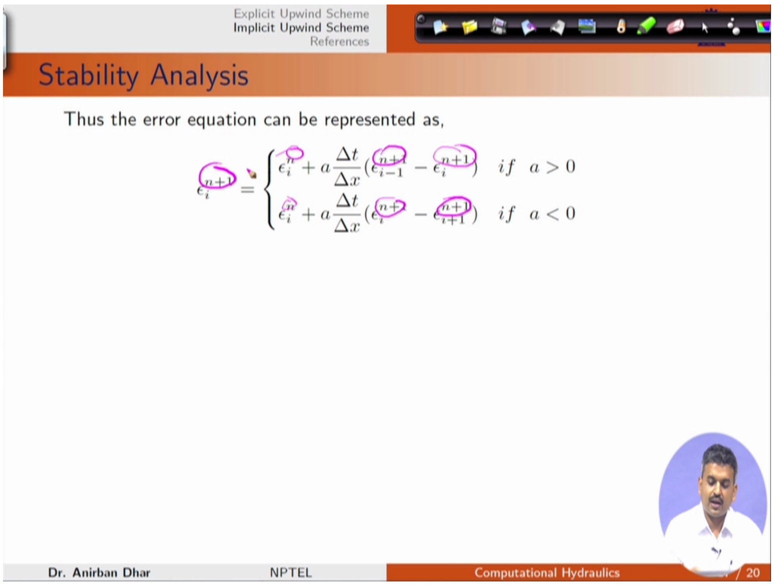
So stability analysis, again if we consider this, we need to see our final solution. We can again discretize the governing equationwith a implicit scheme. And in this casewe can write the exact discrete equation for the finite difference solutions. So in this casewe are getting the equation in terms of errors.

(Refer Slide Time 26:05)



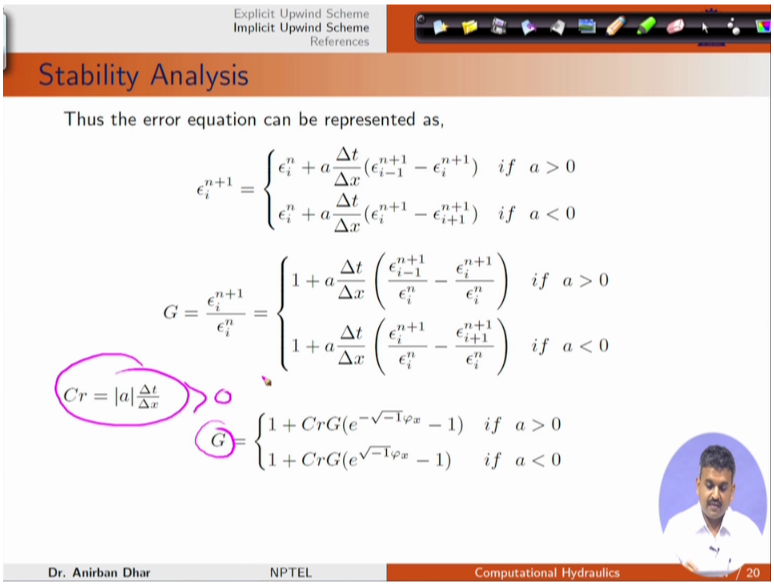
For epsilon I n plus 1 and epsilon I minus 1, n plus 1. All these are at nplus 1 levels. Only the first term is at nth level.

(Refer Slide Time 26:27)



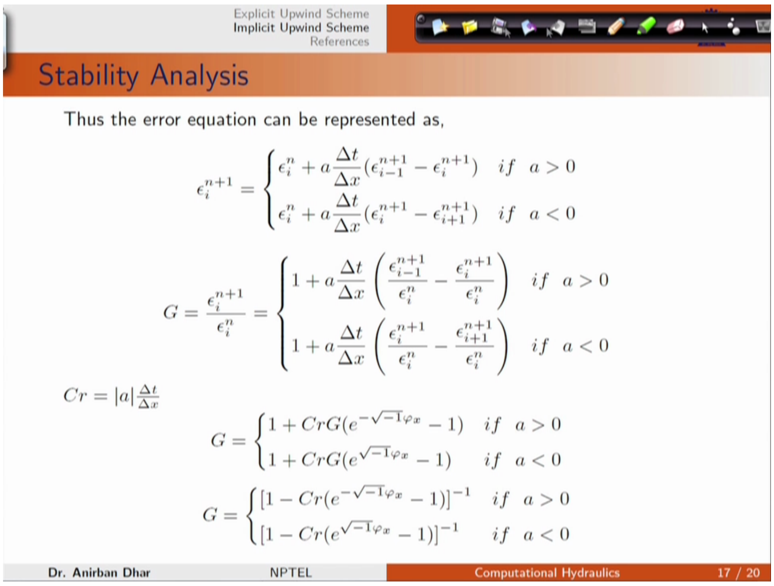
So we can write our (api) amplification factor by dividing it with epsilon I. Again defining the Courant number like this where this is always positive. So we can write G like this.

(Refer Slide Time 26:57)



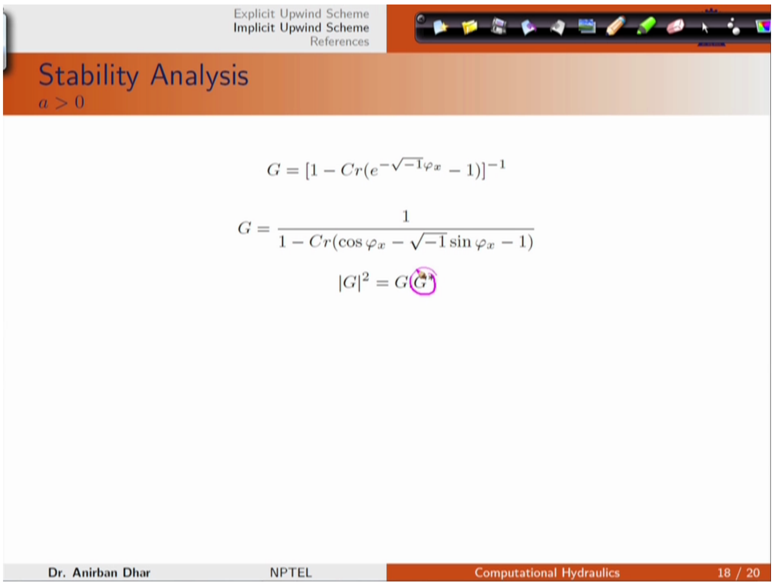
If we further simplify, we can write G in this format. Now we need to define the modulus for this G to evaluate the stability of the scheme.

(Refer Slide Time 27:12)



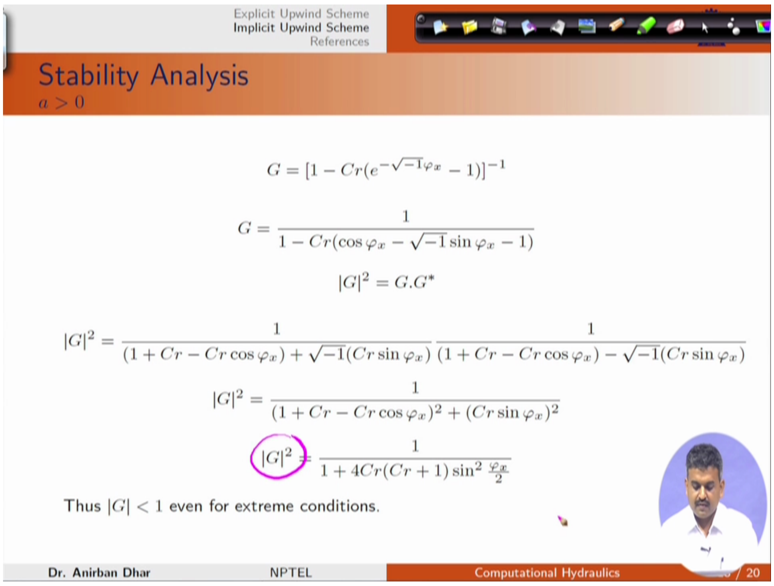
In this case considering the thing for A greater than zero, we can write like this. And with this we need to evaluate the G square. This is the conjugate complex number in this case.

(Refer Slide Time 27:42)



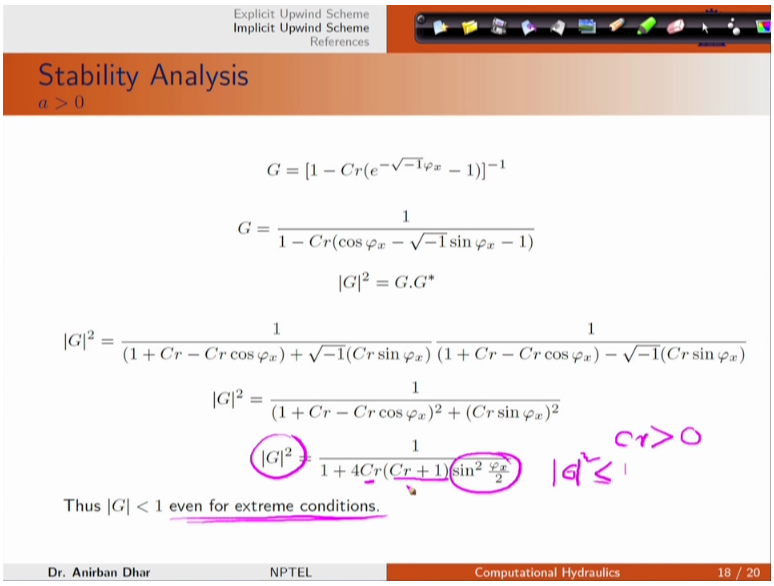
So with this if you proceed and write this thing for (mod) modulus of G square, we can get this one and further simplification will give this G square condition.

(Refer Slide Time 28:06)



In this case we can see that Cr is always positive. In sin square phi x by 2 is always positive, so Cr plus 1 is positive, CR is positive. So 1 by 1plus is something positive. This will give always this G Square or modulus of G square less than equal to 1. For extreme condition, this will be less than equal to 1.

(Refer Slide Time 28:42)



So we can see that for implicit scheme it is unconditionally stable. Now we have covered the finite difference thing. In the next lecture onwards we will be starting finite volume. Thank you.